Journal of New Theory
ISSN: 2149-1402

49 (2024) 7-15
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

Parabolic Numbers: A New Perspective

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Article Info
Received: 01 Aug 2024
Accepted: 19 Oct 2024
Published: 31 Dec 2024
doi:10.53570/jnt.1526699
Research Article

Abstract — Thus far, many studies have been conducted on p-complex numbers. Depending on the sign of p, there are three cases: hyperbolic, dual, and elliptic. In the literature, dual numbers are called parabolic numbers, but they do not parameterize parabolas. Therefore, a number system that parameterizes parabolas is worth studying. This paper defines p as a function of the coordinate y and obtains a number system named parabolic numbers whose circles are parabolas. These parabolic numbers complete the set of number systems where circles are conic sections. Finally, this paper discusses the prospect of further research.

Keywords *Parabolic numbers, p-complex numbers, coordinate dependence*

Mathematics Subject Classification (2020) 15A66, 11H55

1. Introduction

The introduction of complex numbers in the form $z = x+iy$ with $i^2 = -1$ to generalize real numbers has had many critical applications from the fundamental theorem of algebra to advanced physics, such as the calculation of Feynman diagrams [1], needed in quantum field theory, and Bohmian interpretation of quantum mechanics [2] as well as the usual quantum mechanics [3]. Then, a generalization of complex numbers to p-complex numbers has been studied. For more details, see [4]. For p-complex numbers, p is defined via $i^2 = p$ where p can be negative, positive, or zero. These classes of number systems are called elliptic, hyperbolic, and dual, respectively. This nomenclature arises because, in these number systems, the circles, defined by the set of z where $|z|^2$ is constant, correspond to ellipses, hyperbolas, and two vertical lines. The dual numbers are also called parabolic numbers; however, they are quite distinct from our novel perspective on parabolic numbers in this study.

Elliptic numbers, to represent elliptical orbits in the central force problem of Newtonian gravity, have been studied in [5]. The case of hyperbolic orbits, however, has not yet been explored. It is feasible to extend the framework for elliptic numbers to hyperbolic numbers. What is missing is a new perspective on parabolic numbers, whose circles would correspond to parabolas. While dual numbers are also called parabolic, they do not parameterize a parabola, unlike the approach proposed in this study. Hence, the term is a new perspective. This paper introduces a number system based on hyperbolic numbers, where $p > 0$ is treated as a function of the y-coordinate. From the mathematical point of view, the results of the current study completes the list of number systems where circles are conic sections. It is hoped that the results provided here interest researchers.

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The study [6] considers l_p-complex numbers where the norm of an l_p-complex number z is given by $|z|_p \equiv (|x|^p + |y|^p)^{1/p}$ for a constant and positive number p. Here, for $p = 2$, it can be observed that the mentioned number system is the usual complex numbers. However, the number system [6] cannot parameterize parabolas since the exponent of $|x|$ and $|y|$ are the same, and their coefficients are positive, namely 1. Moreover, the distributive law does not hold unless $p = 2$ [6]. The number system defined in Section 2 named parabolic numbers has distributivity property. This is a clear advantage for parabolic numbers defined in this study.

Furthermore, in [7], the norm of a vector $r \in \mathbb{R}^3$ is defined as follows:

$$|r| \equiv \frac{|x|^{p_1}}{p_1} + \frac{|y|^{p_2}}{p_2} + \frac{|z|^{p_3}}{p_3}$$

for positive real numbers p_1, p_2, and p_3. Although this approach has one more dimension, it cannot describe parabolas. The reason is the same as that of the earlier work: The coefficients of $|x|$, $|y|$, and $|z|$ are positive. If, for instance, p_2 is made negative, then one has the correct sign. However, $|y|$ appears in the denominator with a positive power. Hence, a parabola can still not be parameterized even though negative p_i values are allowed.

The paper's organization is as follows: Section 2 defines generalized p-complex numbers with a coordinate dependence on p. Section 3 provides details about the properties of parabolic numbers. Section 4 offers a few ideas for applying parabolic numbers. Finally, Section 5 concludes the paper.

2. Generalized p-Complex Numbers

This section briefly mentions p-complex numbers and generalizes it by making p coordinate-dependent.

2.1. p-Complex Numbers

In the literature, p-complex numbers ($\mathbb{C}_p$) are defined via $z = x + iy$ where $x, y \in \mathbb{R}$ and $i^2 = p$. For $p > 0$, these are referred to as hyperbolic numbers; for $p < 0$, as elliptic numbers; and for $p = 0$, as dual numbers. These numbers systems have been named as such because the constant norm of z, defined via $|z|^2 = zz^* = (x + iy)(x - iy) = x^2 - py^2$, corresponds to hyperbola ($p > 0$), ellipses ($p < 0$), and two vertical lines in the last case ($p = 0$). The concept of squared norm, defined by $|z|^2$, varies depending on the value of p: 1) It is Lorentzian for $p > 0$, which means it may assume any sign or be zero; 2) It is Euclidean for $p < 0$; and 3) It becomes a pseudo-norm-squared for $p = 0$, which is always nonnegative. p-complex numbers and their generalizations are widely studied in the literature [4, 8–15]. For an introduction to p-complex numbers, [16] may be a good reference. Additionally, there are hypercomplex numbers, where the non-real unit u satisfies $u^2 = \alpha + u\beta$ for some $\alpha, \beta \in \mathbb{R}$ [17], a generalization of p-complex numbers. In terms of hypercomplex numbers, [18] might be a valuable source. However, none of these systems include coordinate-dependent p.

As a direct application, the idea to define $i^2 = 1$, $i \notin \mathbb{R}$ is relevant in Einstein's special theory of relativity, where space-time has Lorentzian geometry instead of a Euclidean one. Hyperbolic numbers, defined as the set of numbers $\{z = x + iy : x, y \in \mathbb{R}, i^2 = 1, i \notin \mathbb{R}\}$, can model 2D Minkowski space-time. Because the norm-square of the distance vector between two points P_1 and P_2 denoted by $\vec{v} = (t, x)$ in 2D Minkowski space-time is given as $|\vec{v}|^2 = t^2 - x^2 = |t + ix|^2$ if the norm-square is zero, then the vector $\vec{v}$ is null or light-like; if it is positive, then the vector $\vec{v}$ is time-like, and if it is negative, then the vector $\vec{v}$ is space-like. The relation of hyperbolic numbers to the special theory of relativity is also noted in [15], which cites [19]. The book [20] might also be useful for readers who would like to learn more about the relation between hyperbolic numbers and the special theory of relativity. On the contrary, if $i \in \mathbb{R}$ is assumed, then the definition of a number z in the form $z = x + iy = x \pm y$ is reduced to a summation or subtraction operation on real numbers and would not, for example, yield the space-time structure in the special theory of relativity in 2D.

2.2. Coordinate Dependent p Value

When p is constant, the set of p-complex numbers whose norm is constant cannot represent a parabola. The squared norm of a p-complex number is quadratic in x, y when $p \neq 0$, and when $p = 0$, the unit circle is not a parabola.

The approach to defining a number system in which a circle is a parabola motivates the coordinate dependence of p. This topic should be investigated in the general setting where $p = p(x, y)$. However, the goal of this study is to define parabolic numbers. To this end, i and j are defined by $i^2 = 1$ and $j^2 = p = p(y) = \frac{1}{\alpha|y|}$, for $\alpha > 0$. Hence, $j = \frac{i}{\sqrt{\alpha|y|}}$. Here, i, the hyperbolic unit, is a square root of 1 but is not a real number; it is used to express the coordinate dependence of j. The number i will be useful in expressing a parabolic number in hyperbolic form, especially in the next section.

3. Properties of Parabolic Numbers

In this section, the algebraic operations on parabolic numbers are elaborated. A parabolic number z is expressed as $z = x + jy$, where the explicit form of j is utilized to represent z as follows:

$$z = x + jy = x + i\frac{y}{\sqrt{\alpha|y|}} = x + i\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$$

where $y = \mathrm{sgn}(y)|y|$ and $\sigma = \mathrm{sgn}(y)$. The value $\sigma_n = \mathrm{sgn}(y_n)$ is defined provided y has a subscript. Here, sgn is the sign function. In the remainder of this section, the following definition is used:

$$z_n \equiv x_n + i\frac{\sigma_n\sqrt{|y_n|}}{\sqrt{\alpha}}$$

and the symbol j is omitted.

3.1. Addition

The sum of two parabolic numbers is defined as follows:

$$\begin{aligned} z_1 \oplus z_2 &= \left(x_1 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \oplus \left(x_2 + i\frac{\sigma_2\sqrt{|y_n|}}{\sqrt{\alpha}}\right) \\ &\equiv x_1 + x_2 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}} + i\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} \\ &= x_3 + i\frac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} \end{aligned}$$

If $z_3 = z_1 \oplus z_2$, the following are obtained:

$$x_3 \equiv x_1 + x_2$$

and

$$\sigma_3\sqrt{|y_3|} \equiv \sigma_1\sqrt{|y_1|} + \sigma_2\sqrt{|y_2|}$$

It can be observed that the addition operation is closed on $\mathbb{R}^2$ and commutative and associative.

3.2. Multiplication

The multiplication of two parabolic numbers is defined as follows:

$$\begin{aligned} z_1 \otimes z_2 &= \left(x_1 + i\frac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \otimes \left(x_2 + i\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}}\right) \\ &= x_1x_2 + \frac{\sigma_1\sigma_2}{\alpha}\sqrt{|y_1|}\sqrt{|y_2|} + \frac{i}{\sqrt{\alpha}}(x_1\sigma_2\sqrt{|y_2|} + x_2\sigma_1\sqrt{|y_1|}) \end{aligned}$$

If $z_3 = z_1 \otimes z_2$, then the following are obtained:

$$x_3 \equiv x_1x_2 + \frac{\sigma_1\sigma_2}{\alpha}\sqrt{|y_1|}\sqrt{|y_2|} \quad \text{and} \quad \sigma_3\sqrt{|y_3|} \equiv x_1\sigma_2\sqrt{|y_2|} + x_2\sigma_1\sqrt{|y_1|}$$

It can be observed that the multiplication operation is closed on $\mathbb{R}^2$. It is obvious that the multiplication is commutative; however, more care is needed to show associativity. The expression $(z_1 \otimes z_2) \otimes z_3$ is as follows:

$$\begin{aligned} (z_1 \otimes z_2) \otimes z_3 = {} & x_1x_2x_3 + \tfrac{1}{\alpha}(x_1\sigma_2\sigma_3\sqrt{|y_2|}\sqrt{|y_3|} + x_2\sigma_1\sigma_3\sqrt{|y_1|}\sqrt{|y_3|} + x_3\sigma_1\sigma_2\sqrt{|y_1|}\sqrt{|y_2|}) \\ & + \tfrac{i}{\sqrt{\alpha}}(x_1x_2\sigma_3\sqrt{|y_3|} + x_1x_3\sigma_2\sqrt{|y_2|} + x_2x_3\sigma_1\sqrt{|y_1|}) + \tfrac{i}{\alpha^{3/2}}\sigma_1\sigma_2\sigma_3\sqrt{|y_1|}\sqrt{|y_2|}\sqrt{|y_3|} \end{aligned} \tag{3.1}$$

Using the commutativity of multiplication, $(z_1 \otimes z_2) \otimes z_3 = z_3 \otimes (z_1 \otimes z_2)$ can be written. When the indices in (3.1) are mapped via $(1, 2, 3) \mapsto (2, 3, 1)$ and it is observed that the expression is invariant, the associativity of multiplication is proven.

3.3. Complex Conjugation and Division

The complex conjugate of a parabolic number is defined by $z^* \equiv x - i\sigma\sqrt{|y|/\alpha}$, for all $z = x + i\sigma\sqrt{|y|/\alpha}$. If the norm of z is non-zero, the multiplicative inverse of z is defined as $z^{-1} \equiv \frac{z^*}{|z|^2}$, although the norm may be negative. In other words, if $|z| \neq 0$, then $\frac{1}{z} \equiv \frac{z^*}{|z|^2}$ is defined.

3.4. Distributive Property

The distributive property for three parabolic numbers is the equality $z_1 \otimes (z_2 \oplus z_3) = (z_1 \otimes z_2) \oplus (z_1 \otimes z_3)$. For $n \in \{1, 2, 3\}$, let $z_n = x_n + i\frac{\sigma_n\sqrt{|y_n|}}{\sqrt{\alpha}}$. Then, $z_2 \oplus z_3$ is calculated as follows:

$$z_2 \oplus z_3 = (x_2 + x_3) + i\left(\frac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \frac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right)$$

Hence,

$$\begin{aligned} z_1 \otimes (z_2 \oplus z_3) &= \left(x_1 + i\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \otimes \left[(x_2 + x_3) + i\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right)\right] \\ &= x_1(x_2 + x_3) + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right) \\ &\quad + i\left[x_1\left(\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + \tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}}\right) + (x_2 + x_3)\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right] \\ &= x_1x_2 + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + i\left(x_1\tfrac{\sigma_2\sqrt{|y_2|}}{\sqrt{\alpha}} + x_2\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \\ &\quad + x_1x_3 + \tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} + i\left(x_1\tfrac{\sigma_3\sqrt{|y_3|}}{\sqrt{\alpha}} + x_3\tfrac{\sigma_1\sqrt{|y_1|}}{\sqrt{\alpha}}\right) \\ &= (z_1 \otimes z_2) \oplus (z_1 \otimes z_3) \end{aligned}$$

This proves the distributive property on parabolic numbers. Therefore, the equality $(z_2 \oplus z_3) \otimes z_1 = (z_2 \otimes z_1) \oplus (z_3 \otimes z_1)$ is also valid due to commutativity of the multiplication on parabolic numbers.

3.5. Euler's Formula for Parabolic Numbers

In this subsection, Euler's formula is generalized to parabolic numbers. For the case of p-complex numbers, see [16]. For a parabolic number z, the expression e^z is calculated. If z is written in hyperbolic representation, then $z = x + jy = x + i\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$ is valid. Hence, $e^z = e^{x+i\beta}$ where $\beta = \frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}$. Because the real and imaginary part of $x + i\beta$ commutes, we have $e^z = e^x e^{i\beta}$. The expression $e^{i\beta}$ is calculated as follows:

$$e^{i\beta} = \sum_{n=0}^{\infty} \frac{(i\beta)^n}{n!} = \sum_{n=0}^{\infty} \frac{\beta^{2n}}{(2n)!} + i\sum_{n=0}^{\infty} \frac{\beta^{2n+1}}{(2n+1)!} = \cosh(\beta) + i\sinh(\beta)$$

Hence, the following is obtained:

$$e^z = e^{x+jy} = e^x\left[\cosh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right) + i\sinh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right)\right]$$

A simplification comes from the fact that $\sigma \in \{-1, 0, 1\}$, cosh is an even function and sinh is an odd function:

$$e^z = e^{x+jy} = e^x\left[\cosh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right) + i\sigma\sinh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right)\right]$$

The previous expression is in hyperbolic representation. Its parabolic representation can be obtained, as well. For that purpose, define $a + jb = \cosh(\beta) + i\sinh(\beta)$. It is observed that $a = \cosh(\beta)$ and $\sinh(\beta) = \frac{\sigma_b\sqrt{|b|}}{\sqrt{\alpha}}$. When the last equality is solved for b, $b = \alpha\sigma_b\sinh^2(\beta)$. From the expression $\sinh(\beta) = \frac{\sigma_b\sqrt{|b|}}{\sqrt{\alpha}}$, $\sigma_b = \sigma$. Hence,

$$\begin{aligned}
e^{jy} &= \cosh(\beta) + j\alpha\sigma\sinh^2(\beta)\\
&= \cosh\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right) + j\alpha\sigma\sinh^2\left(\frac{\sigma\sqrt{|y|}}{\sqrt{\alpha}}\right)\\
&= \cosh\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right) + j\alpha\sigma\sinh^2\left(\frac{\sqrt{|y|}}{\sqrt{\alpha}}\right)
\end{aligned}$$

3.6. Flatness of the Parabolic Number Manifold

Using the norm of $|z|^2 = x^2 - \frac{y^2}{\alpha|y|}$, for $y > 0$, (the case $y < 0$ is straightforward), the metric is defined via the line element:

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \frac{\mathrm{d}y^2}{\alpha y} \tag{3.2}$$

The line element at a point (x, y) is defined as the infinitesimal distance between the points (x, y) and $(x + \mathrm{d}x, y + \mathrm{d}y)$. Hence, the Lorentzian norm-square of the number $\mathrm{d}x + j\mathrm{d}y$ is evaluated at the point (x, y). This fact justifies the line element defined in (3.2). When $\xi = 2\sqrt{\frac{y}{\alpha}}$ is defined, the line element can be written as follows:

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}\xi^2 \tag{3.3}$$

Thus, the Riemann tensor vanishes, and the manifold of parabolic numbers is trivially flat. Moreover, the parabolic number set is isomorphic to 2D Minkowski space-time. This is expected since there is a one-to-one map between parabolic and hyperbolic numbers. To observe this, a map defined by $y \mapsto \xi = 2\sqrt{\frac{y}{\alpha}}$ such that $y > 0$ is one-to-one. The case $y < 0$ is similar, and for $y = 0$, define $\xi = 0$, where the line element in (3.3) is that of hyperbolic numbers.

If $p = p(y)$ only depends on the y variable, then the line element can be transformed via $\mathrm{d}s^2 =$

$\mathrm{d}x^2 - p(y)\mathrm{d}y^2$ into the form:

$$\mathrm{d}s^2 = \mathrm{d}x^2 - \mathrm{d}t^2$$

where $t = \int \mathrm{d}y\sqrt{p(y)}$, which again results in a flat manifold. However, if $p = p(x,y)$, which is not investigated in this study, then there may be curvature in the manifold, which is not the case for parabolic numbers. For example, consider the case $p(x,y) = \sin^2(x)$. Then, the manifold's Ricci scalar, the only degree of freedom in 2D, is $R = 2$. It is another problem whether $p(x,y) = \sin(x)^2$ defines a consistent number system. Consequently, the number manifold may be non-flat depending on $p(x,y)$.

4. A Few Applications

A circle in parabolic numbers is a parabola given by:

$$|z|^2 = x^2 - \frac{|y|}{\alpha} = \frac{A}{\alpha}$$

where $A \in \mathbb{R}$. Hence,

$$|y| = \alpha x^2 - A$$

For an illustration, see Figure 1 drawn with Mathematica 13.3. From Figure 1, note that when $A = 1$, there is no $y \in \mathbb{R}$ such that $|y| = x^2 - 1$, for $|x| < 1$. Hence, the domain of the parabola as a function of x is $\mathbb{R} - (-1, 1)$.

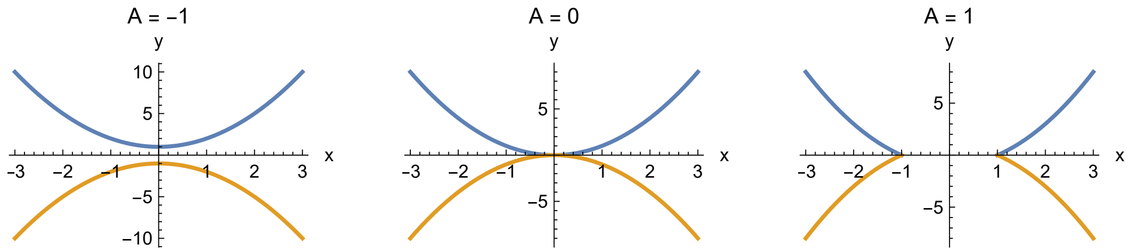

Figure 1. Some circles in parabolic numbers where $\alpha = 1$ and $A \in \{-1, 0, 1\}$

Any parabola can be expressed in this form through rotation, translation, and scaling. In the central force problem of Newtonian gravity, there are three types of trajectories: 1) Elliptic, 2) Hyperbolic, and 3) Parabolic. In [5], elliptical complex numbers where $p < 0$ and p is constant are used to model elliptic trajectories in the central force problem of Newtonian gravity. The case of hyperbolic trajectories can be approached using a similar method. Only the hyperbolic numbers where $p > 0$ are needed instead of elliptic numbers. However, it has not been studied yet. With the parabolic numbers introduced in this paper, parabolic trajectories can finally be parameterized. Another application involves projectile motion. Without air friction, the trajectory of a projectile is a parabola. Moreover, the trajectory of a charged particle under a constant electric field is also a parabola if the particle has a velocity component perpendicular to the electric field. An example of this is as follows: Consider a trajectory such as $|z|^2 = 0$. This results in $|y| = \alpha x^2$ and thus $y = -\alpha x^2$ such that $y \leq 0$. The equations of motion for an electron under constant electric field are:

$$m\ddot{x} = 0 \quad \text{and} \quad m\ddot{y} = qE$$

where $E > 0$ is the electric field and $q < 0$ is the charge of the electron. When these differential equations are integrated, the following two results are obtained:

$$x(t) = v_{0x}t + x_0$$

and

$$y(t) = \frac{1}{2}\frac{qE}{m}t^2 + v_{0y}t + y_0$$

Using the relation $y = -\alpha x^2$ and by expressing t as a function of x, the value α can be obtained as follows:

$$\alpha = -\frac{1}{2}\frac{qE}{mv_{0x}^2}$$

where $\alpha > 0$ since $q < 0$. This information determines the path's shape. Similarly, by applying the initial conditions and using specific values for q, m, and E, the position of an electron as a function of time can be determined. To illustrate, the values of the initial conditions, along with q, m, and E, can be chosen such that the numerical value of α equals $1/2$ in the corresponding units. Figure 2 drawn with Mathematica 13.3 illustrates the electron's trajectory under a constant electric field, which is shown via arrows.

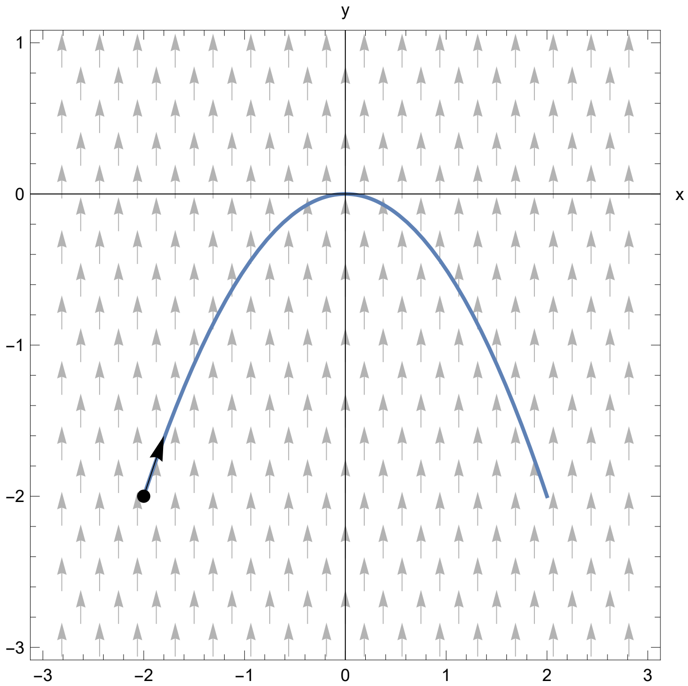

Figure 2. Trajectory of an electron under constant electric field where the numerical value of α is $\frac{1}{2}$

5. Conclusion

Elliptic numbers parameterize ellipses, and hyperbolic numbers parameterize hyperbola. However, there has not been a number system that parameterizes parabola. Through the number system introduced in this paper, parabolic numbers, a type of hyperbolic number where the imaginary unit has a specific coordinate dependence and is distinct from dual numbers, parabolas can be parameterized. The paper is the first study in the available literature considering the coordinate dependence of p. The choices of $p = p(x, y)$ in the more general setting and respective consistency relations are left to future studies.

A few other ideas that may be considered in future studies can be summarized as follows: 1) Whether a sign changing and coordinate dependent p can be consistently defined; 2) What would be the curvature the manifold on which coordinate-dependent p-complex numbers are defined; and 3) Whether it could be generalized to fours dimensions, such as modifying the quaternion algebra, where p_1, p_2, and p_3 are coordinate dependent (for more details on generalized quaternions, see [21]). The study [7] introduces the three-complex numbers system in which p_1, p_2, and p_3 are positive. In this approach, coordinate-dependent p_1, p_2, and p_3 can also be studied.

Author Contributions

The author read and approved the final version of the paper.

Conflicts of Interest

The author declares no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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