Mathematical Modelling and Numerical Simulation with Applications, 2025, 5(2), 348–375

https://dergipark.org.tr/en/pub/mmnsa

ISSN Online: 2791-8564 / Open Access

https://doi.org/10.53391/mmnsa.1526946

RESEARCH PAPER

An EOQ model for the circularity of food waste to lessen greenhouse gas emission

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Abstract

Environmental protection initiatives are increasingly focusing on converting food and organic waste into renewable energy. In India, anaerobic digestion processes food waste and agricultural by-products into biogas, offering an eco-friendly alternative to fossil fuels for cooking, heating, and electricity. This approach aligns with the principles of a circular economy by minimizing resource waste, reducing environmental pollution, and promoting sustainable resource management, all of which contribute to a more resilient and efficient food system. This study explores an Economic Order Quantity (EOQ) model that incorporates the circularity of food waste. The EOQ model improves food waste systems by efficiently minimizing costs and lowering environmental impacts, including greenhouse gas emissions. The goal is to reduce waste, reduce emissions, and reduce ordering costs while maximizing profits. The degree of circularity in products influences consumer demand and unit profits, as consumers are increasingly aware of their environmental impact. In addition, we analyze how changes in system parameters affect optimal strategies, providing valuable insights for industry managers. This research helps determine the optimal product circularity index, thus minimizing food waste, increasing profits, and reducing environmental harm. We illustrate the performance of the integrated system using sensitivity analysis and visual tools, complemented by non-linear approaches to assess strategic impact.

Keywords: Food waste; circular economy; EOQ model; biogas; carbon emission; sustainability

AMS 2020 Classification: 90B05; 90B06

1 Introduction

In today's rapidly evolving world, sustainability has emerged as a critical global priority. Industries are under increasing pressure to find innovative solutions to address environmental challenges, particularly in the context of rising populations and growing consumption. These

➤ **Received:** 07.08.2024 ➤ **Revised:** 04.06.2025 ➤ **Accepted:** 12.06.2025 ➤ **Published:** 30.06.2025

trends exacerbate food security concerns and contribute to one of the most pressing global issues: food waste.

According to the Food and Agriculture Organization (FAO), approximately one-third of all food produced worldwide is wasted, resulting in significant environmental impacts. Decomposing food in landfills emits methane, a greenhouse gas 25 times more potent than carbon dioxide, which contributes substantially to global warming. Addressing this issue requires a shift from linear consumption models to circular economy (CE) practices, where waste is repurposed into valuable resources such as animal feed, compost, bioenergy, or new food products. Traditional inventory models, such as the Economic Order Quantity (EOQ) model, are designed to minimize inventory-related costs, including ordering and holding expenses. However, these models generally neglect the environmental externalities of waste, particularly the emissions associated with overproduction and disposal. Recent studies have begun to integrate sustainability considerations into EOQ frameworks, incorporating environmental costs such as emissions, recycling expenses, and waste management. In this context, managing food waste through circular principles and reducing greenhouse gas (GHG) emissions represent key priorities. Circularity introduces additional cost factors into the EOQ model, such as investment in recycling infrastructure and environmental impact mitigation, making the optimization problem more complex and relevant for modern supply chains.

Moraga et al. [1] emphasized CE as a framework to promote resource responsibility and support sustainable development goals. Klemeš et al. [2] highlighted a regenerative model that slows or closes resource loops, while Wan et al. [3] applied the 10-R principle using advanced mathematical functions to assess sustainability in European markets. These developments underscore the growing importance of CE-based modeling in supply chain design.

Rabta [4] introduced circularity in the EOQ model by analyzing how product reuse affects demand and pricing from the perspective of the retailer. However, their work did not extend to practical applications such as emission reduction or producer-focused decision-making. Since the production phase is critical in determining the circularity of a product, it is essential to explore how EOQ models can be adapted to reflect food waste recycling and environmental outcomes from the manufacturer's point of view. This leads to the following key research questions. How can the EOQ model be modified to incorporate the circularity of food waste while minimizing greenhouse gas emissions in a sustainable supply chain?

What is the impact of integrating circular economy practices into inventory decisions on food waste reduction and environmental performance?

Inventory model

Inventory management plays an essential role in minimizing food waste, especially in perishable supply chains, where timely ordering and efficient stock rotation are essential to prevent spoilage. Classical inventory models such as Economic Order Quantity (EOQ), Newsvendor, and Just-in-Time (JIT) have long been employed to reduce inventory costs. However, these models often fail to address the complexities of perishability and food deterioration unless appropriately modified. To address this, researchers have progressively extended traditional models to incorporate perishability and time-dependent deterioration. Foundational works such as those of Goyal [5] reviewed deteriorating inventory models since the 1990s, classifying them by shelf life and demand patterns, highlighting key extensions and recent developments. Hovelaque et al. [6] integrated carbon emissions into the EOQ framework, establishing a dual-objective model that balances retail profit and carbon footprint. Similarly, Taleizadeh et al. [7] incorporated environmental costs into EOQ with discounts and backordering, while Richer et al. [8] proposed a repair and disposal inventory

system using nonlinear integer programming. These efforts reflect a growing recognition of environmental externalities; however, they still focus largely on linear systems, with limited attention to regenerative resource flows.
Chakrabarti et al. [9] explained primarily the problem of restocking items that are going bad, which has a straight-line demand trend, regular restocking cycles, and shortages within each cycle. The study determines the optimal amount of reorders, the shortest time interval between reorders, and the shortest periods of shortage to minimize the average cost of the system. Donald [10] discussed Harris' square root formula, one of the most cited results in production and operations management, and examined the circumstances that led to its derivation and the reasons for its obscurity for many years. The transition of products from linear to non-linear deterioration rates addresses an important aspect of modern supply chain management. Inaniyan et al. [11] proposed a dual warehouse inventory system using both rented and owned facilities. Kunal et al. [12] examined the inventory system that reduces lost sales by investing in capital, minimizing lead times, and lowering ordering costs.

Circular economy integration

Despite this growing body of literature, the integration of circular economy (CE) principles, such as reuse, recycling, composting, and bioenergy recovery, into inventory models remains relatively underexplored. The CE paradigm shifts away from the traditional linear model ("take-make-dispose") and instead emphasizes closing the loop by maintaining product value throughout its lifecycle. This is particularly crucial in food systems, where the decomposition of organic waste produces methane, a greenhouse gas 25 times more potent than carbon dioxide. Several recent studies have begun to bridge this gap. The work "An EOQ Inventory Model for the Food Supply Chain Integrating Circular Economy Concepts" introduces a circularity index into the EOQ framework, allowing companies to determine optimal reuse and recycling strategies that reduce waste and maximize profitability. Another study, "An Economic Order Quantity Inventory Model for a Product with a Circular Economy Indicator," demonstrates how demand, cost, and price functions can be tied to circularity levels, helping firms evaluate the sustainability of different inventory decisions. Furthermore, "An EOQ Model for Circularity Index with Waste Minimization and Environmental Cost Consideration" highlights the significance of environmental cost modeling in sustainable inventory management practices. These models lay the groundwork for integrating CE thinking into operational research. However, most of them address general product flows or focus on manufacturing and durable goods, with limited applications to perishable food systems, where waste prevention is the most urgent. Although recent efforts have made progress in embedding circularity in EOQ models, there is a distinct lack of focus on food waste, which presents unique challenges such as perishability, short shelf life, and high GHG emissions during decomposition. In addition, few models simultaneously consider both economic optimization and emission reduction within a closed-loop food supply chain. This research seeks to fill this gap by proposing an EOQ model specifically designed for the circularity of food waste. By incorporating a circularity index, GHG emission metrics, and waste recovery costs, the model enables decision-makers to evaluate trade-offs between inventory efficiency and environmental performance. This study contributes to the growing body of literature on sustainable operations, offering a customized solution for green inventory management in food supply chains.

Emission reduction

Research on reducing emissions through food recycling identifies several effective pathways. Composting and anaerobic digestion are widely recognized as sustainable alternatives to landfill disposal, significantly reducing methane emissions. For example, Eriksson et al. [13] and De Laurentiis et al. [14] showed that these techniques not only curb greenhouse gas (GHG) emissions but also produce valuable coproducts such as renewable energy, soil amendments, and nutrient recovery. To quantify these environmental benefits, researchers have employed life cycle assessment (LCA) tools. Tonini et al. [15] reported that conversion of food waste to biogas through anaerobic digestion could reduce emissions by up to 50% compared to landfill disposal. Similarly, Papargyropoulou et al. [16] emphasized that applying a food waste hierarchy, which prioritizes reuse, redistribution, and recycling, can lead to significant emission savings at municipal and regional levels.nBeyond processing methods, the integration of food recycling strategies within the supply chain can amplify environmental and economic benefits. Thyberg and Tonjes [17] proposed the embedding of recycling centers in reverse logistics networks to minimize transportation emissions and enable closed-loop systems.

However, these studies tend to focus on post-consumer waste handling and collection efficiency, with limited attention to upstream supply chain decisions such as order quantities, shelf-life management, or inventory control. This gap is critical, as food systems account for a substantial share of global emissions. Emerging studies show the potential of CE-based inventory systems. Wani et al. [18] developed a model involving a waste supplier and a fiber bottle producer, focusing on the fact that green investments and CE technologies can increase profits by 8.076% while reducing emissions and resource consumption. Wani et al. [19] demonstrated that the use of municipal solid waste for the production of biodiesel could increase global production by 98.5%, highlighting the capacity of CE to convert waste into renewable energy. In another study, Wani et al. [20] used LCA to evaluate apple orchards, advocating for eco-friendly packaging and a supply chain redesign to reduce emissions. Nketiah et al. [21] evaluated the use of biogas from food waste in China's Jiangsu province, reporting notable reductions in coal use, carbon emissions, and potential for global warming. The study also found positive economic outcomes using return on investment (ROI), net present value (NPV), and cost savings analysis, further reinforcing the role of biogas in sustainable development.

Anaerobic digestion, a key CE strategy, produces biogas, a methane-rich renewable energy source, by breaking down organic material in oxygen-free conditions. This process not only reduces methane leaks from landfills but also generates energy, reduces odors and pathogens, and produces digestate, a nutrient-rich fertilizer. In the U.S., widespread adoption of biogas could reduce emissions equivalent to removing 800,000 to 11 million vehicles annually. In addition, it could generate more than 335,000 temporary and 23,000 permanent jobs while offering a sustainable outlet for livestock and food waste. Despite the growing body of literature on emission reduction and food waste recycling, few studies have developed inventory models that explicitly integrate climate emergency (CE) strategies and quantify the resulting emission reductions. Current models often treat waste recovery and emission control as downstream activities, rather than proactively integrating them into upstream inventory decisions like order quantity, reorder timing, and shelf-life optimization. This study addresses this gap by proposing an EOQ-based inventory model that combines food recycling options (e.g., composting, anaerobic digestion) and emission reduction metrics directly into the decision-making framework. The model incorporates a circularity index and an emission cost component, enabling joint optimization of profitability and environmental sustainability in food supply chains. In doing so, it advances the state-of-the-art in green inventory models and provides a novel tool to achieve cost-effective emission mitigation through circular food system practices.

Circular economy

The circular economy (CE) framework aims to close material loops and minimize waste through strategies such as reuse, recycling, and recovery. Within food systems, CE approaches focus on valorizing food waste through methods such as composting, anaerobic digestion, or repurposing waste for secondary uses. Seminal studies by Papargyropoulou et al. [16] underscore the potential of circularity to reduce food waste and mitigate associated environmental impacts. Despite growing interest, operational models that integrate CE principles into inventory management remain underexplored. Existing research focuses primarily on policy interventions, consumer behavior, and waste processing technologies, with limited attention to inventory control mechanisms. This research gap presents an opportunity to develop EOQ-based (Economic Order Quantity) models that align ordering decisions with waste valorization strategies, supporting circularity while reducing emissions. The core idea of CE in food systems is to establish a sustainable model where waste is not discarded but reused, recycled, or composted, ultimately reducing environmental burdens and increasing efficiency. This model encourages closing the resource use loop and optimizing sustainability at all stages of the supply chain. For example, Jamaludin et al. [22] examined the causes of food waste in Malaysia and explored alternative waste management strategies, including potential market applications. Their findings reveal that the average food waste per person is approximately 1.6 kg per week, with rice being the most frequently wasted item. Consumer behavior theories identify poor food handling practices and gender differences as key contributors to food waste. Talekar [23] provides a comprehensive review of food waste in Australia, analyzing primary sources, regulatory responses, and current management practices. The study also evaluates the use of food waste as a resource and highlights the challenges and future directions for the development of food waste biorefinery technologies. Authors in [24] developed a theoretical model to examine the relationship between the structure of the market and food waste. Their work could be expanded by investigating the impact of CE in differentiated production settings, such as Hotelling-style oligopolies, where retailers exert greater market power and recycling may not always yield optimal results.

Khan et al. [25] focus on optimizing production and circularity levels for manufacturers to reduce carbon emissions and promote environmental protection through strategic operational adjustments. Similarly, Thomas et al. [26] explored a two-tier supply chain in the plastic industry using a linear decreasing demand model tied to circularity levels. However, the linearity assumption may not adequately reflect complex real-world dynamics. Hence, there is a need for sustainable operational models that account for carbon emissions, inventory decisions, and linear and non-linear demand patterns shaped by circularity levels. In terms of waste valorization techniques, anaerobic digestion offers a compelling solution. This process involves the breakdown of organic matter by microorganisms in the absence of oxygen, producing biogas (primarily methane) suitable for the generation, heating, or production of biofuels. The remaining by-product, digestate, is nutrient-rich and can be used as fertilizer. Thus, anaerobic digestion not only reduces landfill waste and generates renewable energy but also provides a sustainable alternative to chemical fertilizers.

Food recycling

Food recycling has emerged as a practical and sustainable strategy within the larger framework of the circular economy (CE). Rather than allowing food waste to accumulate in landfills, recycling methods aim to transform waste into valuable secondary products such as animal feed, compost, biofertilizer, and bioenergy. This approach not only diverts organic waste from traditional disposal systems but also enables value recovery, contributing to resource efficiency and environmental

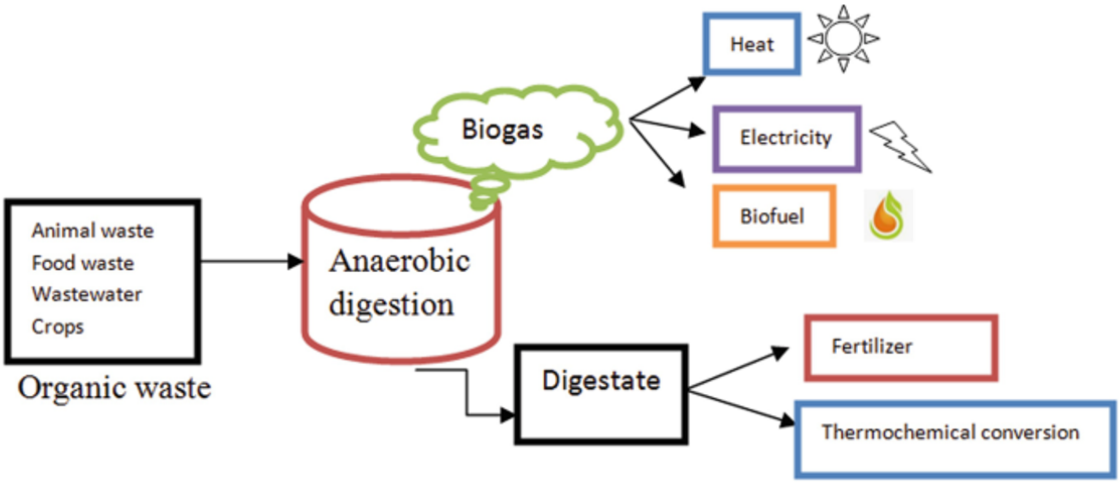

Figure 1. Anaerobic digestion of organic waste for energy and fertilizer recovery

sustainability.

Integrating food recycling into supply chain management (SCM) practices requires not only logistic coordination but also quantitative modeling to ensure its effectiveness. Although recent research has explored strategies for optimizing various stages of waste handling, such as collection, sorting, and redistribution, a significant gap remains in the development of models that explicitly link inventory control and ordering decisions with food recycling processes. These models must consider both economic and environmental performance metrics to be truly sustainable. The circularity of food waste highlights the importance of reducing, reusing, and recycling food materials to minimize environmental impacts, particularly greenhouse gas emissions. Organic waste in landfills generates methane, a highly potent greenhouse gas; reducing food waste is a critical strategy to mitigate emissions. Lu et al. [27] introduced a modified dynamic three-stage circular data envelope analysis (DEA) model to evaluate food production, consumption, and waste recycling in 27 European countries between 2008 and 2016. Their work highlights the inclusion of food waste recycling in the production phase as a CE strategy, offering empirical insights that can inform national food security policies. Similarly, Bernstad et al. [28] demonstrated that accessibility plays a vital role in promoting household participation in recycling initiatives.

On the technological side, Alam et al. [29] investigated the management of food waste at the Bangladesh University of Professionals (BUP) by generating biogas through dry anaerobic digestion under mesophilic conditions. The findings suggest that this method not only addresses the demands of energy but also offers environmental, economic, and social benefits to local communities. In a similar vein, Woon and Koks et al. [30] explored biogas production in Hong Kong, highlighting its viability as a renewable fuel source for vehicles. Kurniawan et al. [31] develop a framework linking agri-food waste valorization to the production of bioactive compounds and their integration into the circular economy model. Mehta et al. [32] highlight the critical role of institutional food waste from hospitals and post-secondary institutions in the design of strategies for reduction, reuse and recovery within this framework. Similarly, Pandey et al. [33] evaluated the potential of integrating nutrient recovery with anaerobic digestion to improve circular economy outcomes in the water–food–energy nexus, contributing to renewable energy production and global food security. From an operational management perspective, Sebatjane et al. [34] developed an integrated inventory model to determine optimal lot sizes, shipment frequencies, and cycle times, while also evaluating the benefits of investing in preservation technologies. Their work suggests that such investments can lead to cost reductions in inventory management. Rashid et al. [35] extended CE principles to the conversion of organic food waste into compost, using nonlinear statistical methods, including exponential, logarithmic, and logistic models. These approaches help to derive more accurate nonlinear demand functions and assess their impact on biogas production. Using Woon's [30] technique, our study implements cross-validation and non-linear modeling to create a solid numerical basis for incorporating recycling dynamics into inventory systems. We also examine the correlation between the cross-validation parameters and other assumptions related to the demand curves. We provide basic descriptions of the Circularity Index and the EOQ model in Section 2. Section 3 explains the assumptions, while Section 4 outlines the problem descriptions. Section 5 presents illustrations of non-linear models, including logarithmic, logistic, and exponential functions. Section 6 discusses the applications of these models. Section 7 focuses on sensitivity analysis and presents the related discussion. Section 8 analyzes the discussion and implementations in the food industry. Section 9 explores the managerial implications for the food industry. Finally, Section 10 presents the conclusions.

2 Circularity index with EOQ inventory model

The Economic Order Quantity (EOQ) model is a classical inventory management technique that typically assumes constant and predictable demand alongside non-linear cost structures. The application of EOQ or similar models can contribute to broader improvements in circularity within food systems. Rabta [4] was among the first to investigate how product circularity influences demand structures within an inventory system. However, this study did not incorporate the carbon emissions generated by inventory operations. Later, Woon and Koksin [30] applied a circular linear demand model to the plastics industry, but acknowledged that such linear demand patterns may not adequately represent all real-world scenarios. This highlights the need to include carbon emission factors in sustainable inventory models that can handle both linear and non-linear demand structures. Circular economy (CE) techniques have many advantages, but organizations sometimes struggle to implement them because of the high costs and complicated technology involved. Moreover, Rabta [4] did not explore the full spectrum of demand and per-unit profit combinations in EOQ models, especially the impacts of non-linear demand and profit patterns. Our approach builds on these insights by incorporating a product-specific circularity index and modifying the traditional EOQ assumptions accordingly. We focus on a single-product, single-location inventory system where the manager orders a fixed quantity Q each period to satisfy demand. The product is available in a standard or circular version, with circularity represented by a parameter θ ranging from 0 to 1 $[0 \leq \theta \leq 1]$. Importantly, both the demand rate and the gross profit per unit depend on the level of circularity of the product. For clarity, the following notations in Table 1 are introduced:

3 Assumption

- The rate of demand varies according to the product's circularity index.
- For a given parameter θ, the demand rate is deterministic, well-established, and remains steady.
- The fixed holding and ordering costs are not related to each other.
- The demand and gross profit per unit are not linearly related to the circularity of the product. Therefore, the fraction of produced goods that undergo remanufacturing determines the demand and the gross profit per unit.
- The manufacturer maintains a consistent production and sales strategy, focusing on a single product to fulfill customer demand, with operations continuing over an unlimited time horizon. In addition, the production rate consistently exceeds the customer demand rate.
- Carbon emissions from various manufacturing activities are considered, including setup operations, warehouse activities, and the production process, which are recognized as the main sources of emissions. The manufacturer faces penalties from government authorities based on

Table 1. Parameter values and their descriptions

Parameters	Descriptions
θ	Product circularity index
Q	Economic Order of Quantity (units/year)
H	Holding cost (units/year), where $h \neq 0$
G	Emissions resulting from storage operations of carbon (units/year)
α	Base demand
β	Circularity impact on demand
γ	Decay rate parameter
λ	Base profit
μ	Circularity profit impact
T	Carbon tax (unit of weight)
e	Carbon emission (unit of weight)
K	Fixed setup cost (dollars/setup)
$P(\theta)$	Demand function (units/unit of time)
$Q(\theta)$	Profit function (unit of time)
EOQ Model	Economic Order Quantity Model

the total carbon emissions produced.

4 Problem description

Given the assumptions stated, this section presents the total profit of the manufacturer per unit time for a single product within the circular economy framework. Before performing a detailed analysis, it is essential to define the customer demand and gross profit structures per unit, both of which are influenced by the circularity index of the product, denoted by θ. The relationships among circularity, demand, and per-unit profit are modeled using nonlinear functions. Specifically, an exponential inverse demand function is employed, following the formulation from [23], $D(\theta) = \alpha + \beta e^{-\gamma\theta} and \quad P(\theta) = \lambda + \mu e^{-\gamma\theta}$, where *theta* ≥ 0 is the circularity index (a higher value indicates greater circularity)? $\alpha, \beta, \gamma, \lambda, \mu \in \mathbb{R}$ are parameters with real value, $\alpha > 0$ and $\lambda > 0$ ensure that the baseline demand and profit are positive, $\gamma > 0$ control the decay rate (sensitivity to circularity), β and μ determine the strength and direction of the impact of circularity on demand and profit, respectively. These parameters reflect the sensitivity of demand and profit to the circularity level of the product. Nonlinear models, such as the exponential form used here, are essential, since linear models do not capture the complex effects of circularity. Higher circularity influences not only demand but also production costs and prices, thereby affecting consumer behavior, material choices, production technologies, and environmental policies such as carbon taxes.

To reduce carbon emissions in production and inventory under circular economy practices, firms must adopt models that reflect the nonlinear relationships between demand and per-unit profit. Capturing these dynamics is crucial for optimizing the circularity index and maximizing profit within sustainability constraints.

To address this research gap, this study proposes a sustainable order quantity model that integrates production policies, circularity levels, carbon emissions, and the non-linear impact of circularity on demand and profitability. The model uses advanced demand functions, such as the logarithmic relationship introduced by Rabta [4], to better represent consumer responses to circularity. This enhances both the realism and the practical relevance of the framework in modeling sustainable operations $D(\theta) = P_0 + alog(1 + \gamma\theta)$, and the logistic form $P(\theta) = P_0 + \frac{a}{1+e^{-\delta(\theta-\theta_0)}}$, where P_0, θ, γ, and are constants, with $P_0, \theta, \delta, > 0, a \geq 0$, and $\theta_0 \in [0, 1]$.. Furthermore, other possible non-linearities could be examined to assess the impact of the circularity level on both demand and per-unit gross profit. The manufacturer first creates and subsequently sells a quantity Q, each generating a gross profit per unit of $P(\theta)$ over one business cycle. Hence, the total gross profit from the Q units is Q*$P(\theta)$. Furthermore, the manufacturer incurs the setup costs denoted by K and completely pays the costs $\frac{1}{2}HQ(1-D(\theta))$. In addition, the manufacturer is responsible for a carbon tax imposed by the authorities, calculated based on emissions from setup operations, warehouse activities, and production processes. $T\{e+fP(\theta)T+\frac{G}{2}Q(1-P(\theta))+(1-P(\theta))^2\}$ are introduced by Md. Al-Amin Khan [25]. Finally, the overall manufacturer's profit is given by

$$D(\theta)=\alpha+\beta e^{-\gamma\theta}, \quad and \quad P(\theta)=\lambda+\mu e^{-\gamma\theta}, \tag{1}$$

$$\begin{aligned}\phi(Q,\theta) &= D(\theta)P(\theta)-\frac{D(\theta)K}{Q}-\frac{1}{2}HQ(1-D(\theta))\\ &\quad -T[\frac{eD(\theta)}{Q}+fD(\theta)+\frac{G}{2}Q(1-D(\theta))+(1-D(\theta))^2],\end{aligned}$$

$$\phi(Q,\theta)=D(\theta)P(\theta)-TfD(\theta)-\frac{D(\theta)(K+Te)}{Q}-\frac{1}{2}(H+GT)Q(1-D(\theta))-QT(1-D(\theta))^2. \tag{2}$$

The objective now is to find the ideal quantity to produce (Q*) within a business cycle and the optimal circular leveling ($\theta^* \in [0,1]$) for these produced units, enabling the manufacturer to achieve the highest profit per unit of time. Consequently, the production process presents the following optimization challenge:
Problem A:

$$max \quad \phi(Q,\theta), \quad where \quad \sigma=\{(Q,\theta): 0<Q<\infty \quad and \quad 0\le\theta\le 1\}, \quad (Q,\theta)\epsilon\sigma. \tag{3}$$

Solution techniques

The demand patterns, gross profit per unit, and production quantities allow the optimal solution to be categorized into three separate cases:
Case 1: When considering both demand and per-unit gross profit increase as a function of θ, or if one of them increases while the other variable stays invariant for θ, the optimal solution for Problem A (3) is $\theta^*=1$, and

$$Q=\sqrt{\frac{2D(1)(K+Te)}{(H+GT)(1-D(1))(1-D(1))^2}}.$$

Case 2: Considering demand and per-unit gross profit decrease as a function of θ, or if one decreases whereas the other variable stays invariant for θ, the optimal solution for the manufacturer is either θ^*= 0, and

$$Q=\sqrt{\frac{2D(0)(K+Te)}{(H+GT)(1-D(0))(1-D(0))^2}}.$$

Case 3: The ideal solution is found when one function for demand or per-unit gross profit increases and the other decreases to θ. Problem A (3) may differ from Cases 1 and 2. The Lagrange multiplier technique finds the optimal solution in this situation.

$$
\begin{aligned}
L(Q,\theta) = D(\theta)P(\theta) - TfD(\theta) - \frac{D(\theta)(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(1-D(\theta)) \\
- QT(1-D\theta))^2 - \mu_1(\theta-1) + \mu_2\theta.
\end{aligned}
\tag{4}
$$

To find the Karush-Kuhn-Tucker conditions necessary for an optimal solution are: Equation twice partially differential for θ, Q to find the Hessian matrix

$$
\frac{\partial^2 L}{\partial Q^2} = -\left(\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right), \tag{5}
$$

$$
\begin{aligned}
\frac{\partial L}{\partial \theta} = (\alpha+\beta e^{-\gamma\theta})(\gamma\mu e^{-\gamma\theta}) + \gamma\beta\mu(e^{-\gamma\theta})(\lambda+\mu e^{-\gamma\theta}) - Tf(\beta\gamma e^{-\gamma\theta}) \\
- \frac{\gamma\beta e^{-\gamma\theta}(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(1+\beta\gamma e^{-\gamma\theta}) - QT\left[1-2\left[(\alpha+\beta e^{-\gamma\theta})(\beta\gamma e^{-\gamma\theta})\right]\right],
\end{aligned}
\tag{6}
$$

$$
\begin{aligned}
\frac{\partial^2 L}{\partial \theta^2} &= (\alpha+\beta e^{-\gamma\theta})(\gamma^2\mu e^{-\gamma\theta}) + \gamma^2\beta\mu(e^{-\gamma\theta})^2 \\
&- (\lambda+\mu e^{-\gamma\theta})(\gamma^{2\beta e^{-\gamma\theta}}) + \gamma^2\mu\beta(e^{-\gamma\theta})^2 - Tf(\beta\gamma^2 e^{-\gamma\theta}) - \frac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q} \\
&- \frac{1}{2}(H+GT)Q(1-\beta\gamma^2 e^{-\gamma\theta}) - QT[2[(\alpha+\beta e^{-\gamma\theta})(\beta\gamma^2 e^{-\gamma\theta}) + \gamma^2\beta^2(e^{-\gamma\theta})^2]],
\end{aligned}
$$

$$
\frac{\partial^2 L}{\partial Q \partial L} = \frac{\partial^2 L}{\partial L \partial Q} = -\begin{cases}
\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) \\
+\left(\frac{(H+GT)}{2}\right)(1+\beta\gamma e^{-\gamma\theta}) \\
+2T\left(\alpha+\beta e^{-\gamma\theta}\right)\beta\gamma e^{-\gamma\theta},
\end{cases}
$$

$$
H = \begin{vmatrix}
\frac{\partial^2 \pi}{\partial \theta^2} & \frac{\partial^2 \pi}{\partial Q \partial \theta} \\
\frac{\partial^2 \pi}{\partial \theta \partial Q} & \frac{\partial^2 \pi}{\partial Q^2}
\end{vmatrix},
$$

$$\det H > 0 \Leftrightarrow \begin{bmatrix} (\alpha+\beta e^{-\gamma\theta})(\gamma^2\mu e^{-\gamma\theta}) + \gamma^2\beta\mu(e^{-\gamma\theta})^2 \\ -(\lambda+\mu e^{-\gamma\theta})(\gamma^2\beta e^{-\gamma\theta}) + \gamma^2\mu\beta(e^{-\gamma\theta})^2 \\ -Tf(\beta\gamma^2 e^{-\gamma\theta}) - \dfrac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q} \\ -\dfrac{1}{2}(H+GT)Q(1-\beta\gamma^2 e^{-\gamma\theta}) \\ -QT\left[2\left((\alpha+\beta e^{-\gamma\theta})(\beta\gamma^2 e^{-\gamma\theta}) + \gamma^2\beta^2(e^{-\gamma\theta})^2\right)\right] \\ -\left(\dfrac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right) \end{bmatrix} - \begin{pmatrix} \left(\dfrac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\dfrac{H+GT}{2}\right)(1+\beta\gamma e^{-\gamma\theta}) \\ +2T(\alpha+\beta e^{-\gamma\theta})\beta\gamma e^{-\gamma\theta} \end{pmatrix}^2,$$

$$\det H > 0 \Leftrightarrow \begin{bmatrix} (\alpha+\beta e^{-\gamma\theta})(\gamma^2\mu e^{-\gamma\theta}) + \gamma^2\beta\mu(e^{-\gamma\theta})^2 \\ -(\lambda+\mu e^{-\gamma\theta})(\gamma^2\beta e^{-\gamma\theta}) + \gamma^2\mu\beta(e^{-\gamma\theta})^2 \\ -Tf(\beta\gamma^2 e^{-\gamma\theta}) - \dfrac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q} \\ -\dfrac{1}{2}(H+GT)Q(1-\beta\gamma^2 e^{-\gamma\theta}) \\ -QT\left[2\left((\alpha+\beta e^{-\gamma\theta})(\beta\gamma^2 e^{-\gamma\theta}) + \gamma^2\beta^2(e^{-\gamma\theta})^2\right)\right] \\ -\left(\dfrac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right) \end{bmatrix} > - \begin{pmatrix} \left(\dfrac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\dfrac{H+GT}{2}\right)(1+\beta\gamma e^{-\gamma\theta}) \\ +2T(\alpha+\beta e^{-\gamma\theta})\beta\gamma e^{-\gamma\theta} \end{pmatrix}^2,$$

$$\mu_1(\theta-1) = 0, \tag{7}$$

$$\mu_2 = 0, \tag{8}$$

$$\mu_1, \mu_2 \geq 0. \tag{9}$$

The complementary conditions (7) and (8) yield three potential solutions for the optimization Problem A (3).
* Either θ= 0, and μ_1= 0

$$Q = \sqrt{\frac{2D(0)(K+Te)}{(H+GT)(1-D(0))(1-D(0))^2}},$$

$$\mu_2 = D(0)P(0)' + D(0)'P(0) - TfD(0)' - \frac{D(0)'(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(1-D(0)) - QT(1-D'(0))^2.$$

* Or θ= 1, and μ_2= 0

$$Q = \sqrt{\frac{2D(1)(K+Te)}{(H+GT)(1-D(1))(1-D(1))^2}},$$

$$\mu_1 = D(1)P(1)' + D(1)'P(1) - TfD(1)' - \frac{D(1)'(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(1-D(1)) - QT(1-D'(1))^2.$$

* Either μ_1 = 0, and μ_2 = 0 from Eq. (5) where $Q = \sqrt{\frac{2D(\theta)(K+Te)}{(H+GT)(1-D(\theta))(1-D(\theta))^2}}$ and from

$$(D(\theta)P(\theta)' + D(\theta)'P(\theta) - TfD(\theta)'))^2 = \frac{2D'(\theta)(H+GT)(1-D(\theta))}{D(\theta)} - \frac{(K+Te)D(\theta)}{1-D(\theta)} - \frac{2D(\theta)(K+Te)}{(H+GT)(1-D(\theta))}t(1-D(\theta))^2, \tag{10}$$

$$\phi(\theta) = (D(\theta)P(\theta)' + D(\theta)'P(\theta) - TfD(\theta)'))^2 - \frac{2D'(\theta)(H+GT)(1-D(\theta))}{D(\theta)} - \frac{(K+Te)D(\theta)}{1-D(\theta)} - \frac{2D(\theta)(K+Te)}{(H+GT)(1-D(\theta))}T(1-D(\theta))^2. \tag{11}$$

Eq. (11) can be easily solved using several numerical approaches for circular leveling (θ). Because the function $\phi(\theta)$ is not linear, and can be used to find many possible circular indices in the range [0, 1]. All potential solutions must adhere to the necessary feasibility constraints, such as the positivity of Q, $\theta \in [0,1]$, and Eq. (9). We evaluated the second-order derivative criterion to determine whether a potential solution qualifies as a local optimizer. Furthermore, we determine the global optimizer when several local optimizers are present by evaluating the manufacturer's profit across the local optima. The KKT criteria guarantee an ideal solution when the manufacturer's profit (Q, θ) exhibits concavity. The concavity of ϕ(Q, θ) is completely dependent on the configurations of both D(θ) and P(θ). Consequently, the concavity of $\phi(Q,\theta)$ is now examined in three potential combinations of the previously specified forms of D(θ) and P(θ).

Non-linear and exponential relationship

Consider

$$D(\theta) = (\alpha + \beta e^{-\gamma\theta}), \quad \text{and} \quad P(\theta) = (P_0 + be^{\delta(\theta-1)}),$$

where $\alpha > 0, P_0 > 0, \beta > 0$, and $b < 0$. In this case,

$$\phi(Q,\theta) = D(\theta)P(\theta) - \frac{TfD(\theta)}{Q} - \frac{1}{2}(H+GT)Q(1-D(\theta)) - QT(1-D(\theta))^2 - \mu_1(\theta-1) + \mu_2,$$

$$\phi(Q,\theta) = (\alpha+\beta e^{-\gamma\theta})\left(P_0 + be^{\delta(\theta-1)}\right) - Tf(\alpha+\beta e^{-\gamma\theta}) - \left(\frac{(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q}\right)$$
$$-\frac{1}{2}(H+GT)Q\left(1-(\alpha+\beta e^{-\gamma\theta})\right) - QT\left(1-(\alpha+\beta e^{-\gamma\theta})^2\right) - \mu_1(\theta-1) + \mu_2\theta.$$

Partially differential to twice θ, Q to find the Hessian matrix

$$\frac{\partial^2\phi}{\partial Q^2} = -\left(\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right), \tag{12}$$

$$\frac{\partial^2 L}{\partial Q\partial\theta} = \frac{\partial^2 L}{\partial\theta\partial Q} = -\Bigg[\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\frac{(H+GT)}{2}\right)\left(1+\beta\gamma e^{-\gamma\theta}\right)$$
$$-2T\left(1-(\alpha+\beta e^{-\gamma\theta})\right)\beta\gamma e^{-\gamma\theta}\Bigg],$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \left(\alpha+\beta e^{-\gamma\theta}\right)\left(\delta^2 be^{\delta\theta}\right) - \delta b\gamma\beta e^{\delta-\gamma\theta} + (P_0 + be^{\delta(\theta-1)})(\gamma^2\beta e^{-\gamma\theta})$$
$$+(\beta\gamma\delta be^{-\gamma\theta})e^{\delta(\theta-1)} - Tf\gamma^2\beta e^{-\gamma\theta} - \frac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q}$$
$$+\frac{(H+GT)}{2}Q(\gamma^2\beta e^{-\gamma\theta}) - 2QT\left((1-(\alpha+\beta e^{-\gamma\theta}))(\gamma^2\beta e^{-\gamma\theta}) - \gamma^2\beta^2(e^{-\gamma\theta})^2\right),$$

$$H = \begin{vmatrix} \frac{\partial^2\pi}{\partial\theta^2} & \frac{\partial^2\pi}{\partial Q\partial\theta} \\ \frac{\partial^2\pi}{\partial\theta\partial Q} & \frac{\partial^2\pi}{\partial Q^2} \end{vmatrix},$$

$$\det H > 0 \Leftrightarrow \left(\left(\alpha+\beta e^{-\gamma\theta}\right)\left(\delta^2 be^{\delta\theta}\right)\right) - \left(\delta b\gamma\beta e^{\delta-\gamma\theta}\right) + \left((P_0+be^{\delta(\theta-1)})(\gamma^2\beta e^{-\gamma\theta})\right)$$
$$+\left((\beta\gamma\delta be^{-\gamma\theta})e^{\delta(\theta-1)}\right) - \left(Tf\gamma^2\beta e^{-\gamma\theta}\right) - \left(\frac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q}\right)$$
$$+\left(\frac{(H+GT)}{2}Q(\gamma^2\beta e^{-\gamma\theta})\right)$$
$$-\left(2QT\left((1-(\alpha+\beta e^{-\gamma\theta}))(\gamma^2\beta e^{-\gamma\theta})\right) - \left(\gamma^2\beta^2(e^{-\gamma\theta})^2\right)\right) \times \left(-\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right)$$
$$-\left[\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\frac{(H+GT)}{2}\right)\left(1+\beta\gamma e^{-\gamma\theta}\right) - 2T(1-(\alpha+\beta e^{-\gamma\theta}))\beta\gamma e^{-\gamma\theta}\right]^2, \tag{13}$$

$$\det H > 0 \Leftrightarrow \Bigg[\left(\alpha + \beta e^{-\gamma\theta}\right)\left(\delta^2 b e^{\delta\theta}\right) - \delta b \gamma \beta e^{\delta-\gamma\theta} + \left(P_0 + b e^{\delta(\theta-1)}\right)\left(\gamma^2 \beta e^{-\gamma\theta}\right) + \beta\gamma\delta b e^{-\gamma\theta} e^{\delta(\theta-1)} - T f \gamma^2 \beta e^{-\gamma\theta} - \frac{\gamma^2 \beta e^{-\gamma\theta}(K+Te)}{Q} + \frac{(H+GT)}{2} Q \left(\gamma^2 \beta e^{-\gamma\theta}\right) - 2QT\Big((1-(\alpha+\beta e^{-\gamma\theta}))\left(\gamma^2\beta e^{-\gamma\theta}\right) - \gamma^2\beta^2 (e^{-\gamma\theta})^2\Big)\Bigg] \times \left(-\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right) > -\Bigg[\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\frac{(H+GT)}{2}\right)\left(1+\beta\gamma e^{-\gamma\theta}\right) - 2T(1-(\alpha+\beta e^{-\gamma\theta}))\beta\gamma e^{-\gamma\theta}\Bigg]^2. \tag{14}$$

As H is semidefinite negative, the objective function is concave. The curve shown in Figure 2 slopes downward, indicating concavity. Additionally, Figure 3 shows the position of the global optimal solution for the profit. Consequently, the manufacturer makes the highest profit when the products have a higher level of circularity. In this situation, one of the potential solutions is both feasible and optimal. The various scenarios for this solution include:
i) Either $\theta = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta)(K+Te)}{(H+GT)(1-(\alpha+\beta))(1-(\alpha+\beta))^2}},$$

$\mu_1 = 0$, and

$$\mu_2 = (\alpha+\beta)(\delta b) + (P_0+b)(-\gamma\beta) + Tf(\gamma\beta) + \frac{(\gamma\beta)(K+Te)}{Q} + \frac{1}{2}(H+GT)Q(\gamma\beta) + 2QT(1-(\alpha+\beta e^{-\gamma\theta}))(\gamma\beta).$$

ii) Or $\theta = 1$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma}))(1-(\alpha+\beta e^{-\gamma}))^2}}.$$

If $\mu_2 = 0$, and

$$\mu_1 = (\alpha+\beta e^{-\gamma})(\delta b e^{\delta}) - (P_0 + b e^{-\gamma})(\gamma\beta e^{-\gamma}) + Tf(\gamma\beta e^{-\gamma}) - \frac{(\gamma\beta e^{-\gamma})(K+Te)}{Q} + \frac{1}{2}(H+GT)Q(\gamma\beta e^{-\gamma}) + 2QT(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{\gamma}).$$

iii) $\mu_1 = \mu_2 = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))(1-(\alpha+\beta e^{-\gamma\theta}))^2}},$$

$$\begin{aligned}&\left(\left(\alpha+\beta e^{-\gamma\theta}\right)\left(\delta b e^{\delta}\theta\right)-\left(P_0+be^{-\gamma(\theta-1)}\right)\left(\gamma\beta e^{-\gamma\theta}\right)+Tf\left(\gamma\beta e^{-\gamma\theta}\right)\right)^2\\&=\frac{\left(\gamma\beta e^{-\gamma\theta}\right)(K+Te)\left[(H+Gt)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)^2\right]}{2\left(\alpha+\beta e^{-\gamma\theta}\right)(K+Te)}\\&+\left(\frac{2\left(\alpha+\beta e^{-\gamma\theta}\right)(K+Te)}{(H+GT)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)^2}\right)\cdot\frac{1}{2}(H+GT)\left(\gamma\beta e^{-\gamma\theta}\right)\\&+2T\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(\gamma\beta e^{\gamma\theta}\right),\end{aligned}\tag{15}$$

is expressed as $\phi_1(\theta) = 0$ with

$$\begin{aligned}\phi_1(\theta)=&\left(\left(\alpha+\beta e^{-\gamma\theta}\right)\left(\delta b e^{\delta}\theta\right)-\left(P_0+be^{-\gamma(\theta-1)}\right)\left(\gamma\beta e^{-\gamma\theta}\right)+Tf\left(\gamma\beta e^{-\gamma\theta}\right)\right)^2\\&-\frac{\left(\gamma\beta e^{-\gamma\theta}\right)(K+Te)\left[(H+GT)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)^2\right]}{2\left(\alpha+\beta e^{-\gamma\theta}\right)(K+Te)}\\&+\left(\frac{2\left(\alpha+\beta e^{-\gamma\theta}\right)(K+Te)}{(H+GT)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)^2}\right)\cdot\frac{1}{2}(H+GT)\left(\gamma\beta e^{-\gamma\theta}\right)\\&+2T\left(1-\left(\alpha+\beta e^{-\gamma\theta}\right)\right)\left(\gamma\beta e^{\gamma\theta}\right).\end{aligned}\tag{16}$$

The formulation of $\phi_1(\theta)$ is non-linear in θ, enabling the identification of several solutions for θ (not all of which need to be feasible) for Eq. (15).

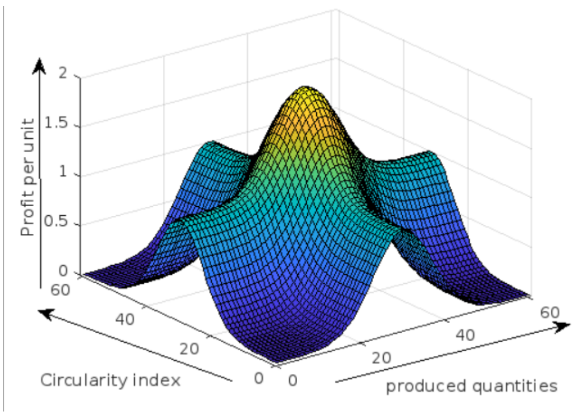

Figure 2. Curve shows concavity

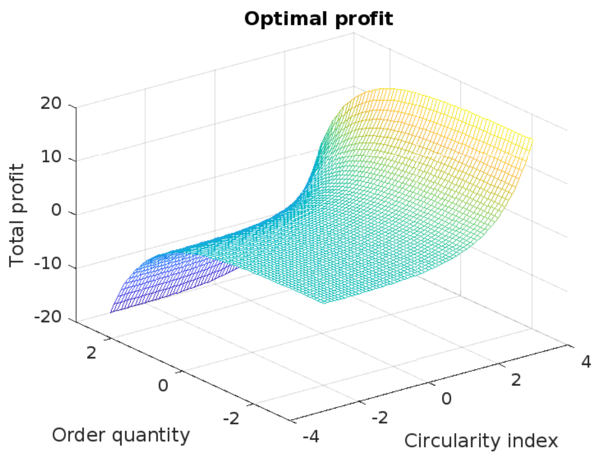

Figure 3. Optimal profit

Non-linear and logistic relationship

Let's consider the logistic per-unit gross profit and demand defined as follows:

$$D(\theta) = (\alpha + \beta e^{-\gamma\theta}), \text{ and } P(\theta) = P_0 + \frac{b}{1 + e^{-\delta(\theta-\theta_0)}},$$

where $\alpha, P_0, \delta, \beta$ are all greater than zero, $b < 0$ and $\theta_0 \in [0, 1]$. In this case,

$$\begin{aligned}\phi(Q,\theta) = {} & (\alpha + \beta e^{-\gamma\theta}) \left(P_0 + \frac{b}{1 + e^{-\delta(\theta-\theta_0)}}\right) Tf(\alpha + \beta e^{-\gamma\theta}) \\ & - \frac{(\alpha + \beta e^{-\gamma\theta})(K + Te)}{Q} - \frac{1}{2}(H + GT)\, Q(1 - (\alpha + \beta e{-}\gamma\theta)) \\ & - QT(1 - (\alpha + \beta e{-}\gamma\theta))^2 - \mu_1(\theta - 1) + \mu_2\theta,\end{aligned}$$

partially twice differentiate for Q, θ

$$\frac{\partial^2\phi}{\partial Q^2} = -\left(\frac{2(\alpha + \beta e^{-\gamma\theta})(K + Te)}{Q^3}\right), \tag{17}$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial Q\partial\theta} = \frac{\partial^2\phi}{\partial\theta\partial Q} = -\Bigg[& \left(\frac{(\gamma\beta e^{-\gamma\theta})(K + Te)}{Q^2}\right) + \left(\frac{(H + GT)}{2}\right) Q \left(\beta\gamma e^{-\gamma\theta}\right) \\ & - 2T\left(1 - (\alpha + \beta e^{-\gamma\theta})\right)\beta\gamma e^{-\gamma\theta}\Bigg],\end{aligned}$$

$$\begin{aligned}\frac{\partial^2\phi}{\partial\theta^2} = {} & (\alpha + \beta e^{-\gamma\theta})\left(\frac{b}{\delta^2 e^{-\delta}(\theta - \theta_0)}\right) + \left(\frac{b\beta\gamma e^{\gamma\theta}}{\delta e^{-\delta(\theta-\theta_0)}}\right) + \left(P_0 + \frac{b}{1 + e^{-\delta(\theta-\theta_0)}}\right)\left(\gamma^2\beta e^{-\gamma\theta}\right) \\ & - \left(\gamma\beta e^{-\gamma\theta}\right)\left(\frac{b}{\delta e^{-\delta(\theta-\theta_0)}}\right) - (\gamma^2 tf\beta e^{-\gamma\theta}) - \frac{(\gamma^2\beta e^{-\gamma\theta})}{Q} + \frac{(\gamma^2\beta e^{-\gamma\theta}Q(H + GT))}{2} \\ & + 2QT\left((1 - (\alpha + \beta e^{-\gamma\theta})\right)\left(\gamma^2\beta e^{-\gamma\theta - 2QT\gamma^2\beta^2(e^{-\gamma\theta})}\right).\end{aligned}$$

To guarantee concavity, the Hessian matrix is required to be negative semi-definite. Furthermore, det $H > 0 \Leftrightarrow$

$$\begin{aligned}&\Bigg(\left(\alpha + \beta e^{-\gamma\theta}\right)\left(\frac{a}{\delta^2 e^{-\delta(\theta-\theta_0)}}\right) + \left(\frac{a\beta\gamma e^{\gamma\theta}}{\delta e^{-\delta(\theta-\theta_0)}}\right) + \left(P_0 + \frac{a}{1 + e^{-\delta(\theta-\theta_0)}}\right)\left(\gamma^2\beta e^{-\gamma\theta}\right) \\ & - \left(\gamma\beta e^{-\gamma\theta}\right)\left(\frac{a}{\delta e^{-\delta(\theta-\theta_0)}}\right) - \gamma^2 Tf\beta e^{-\gamma\theta} - \frac{\gamma^2\beta e^{-\gamma\theta}}{Q} + \frac{\gamma^2\beta e^{-\gamma\theta}Q(H + GT)}{2} \\ & + 2QT\left(1 - \left(\alpha + \beta e^{-\gamma\theta}\right)\right)\left(\gamma^2\beta e^{-\gamma\theta} - 2QT\gamma^2\beta^2 e^{-\gamma\theta}\right)\Bigg)\left(-\frac{2\left(\alpha + \beta e^{-\gamma\theta}\right)(K + Te)}{Q^3}\right) \\ & > \left(-\left[\left(\frac{\left(\gamma\beta e^{-\gamma\theta}\right)(K + Te)}{Q^2}\right) + \left(\frac{(H + GT)}{2}\right) Q\left(\beta\gamma e^{-\gamma\theta}\right) - 2T\left(1 - \left(\alpha + \beta e^{-\gamma\theta}\right)\right)\beta\gamma e^{-\gamma\theta}\right]\right)^2.\end{aligned}$$

As H is semidefinite negative, the objective function is concave.
i) Either $\theta = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta)(K+Te)}{(H+GT)(1-(\alpha+\beta))(1-(\alpha+\beta))^2}},$$

$\mu_1 = 0$, and

$$\mu_2 = \frac{b(\alpha+\beta)}{\delta e^{\delta\theta_0}} + \left(P_0 + \frac{b}{1+e^{\delta\theta_0}}\right)(\gamma\beta) + Tf(\gamma\beta) + \frac{(\gamma\beta)(K+Te)}{Q}$$
$$-\frac{1}{2}(H+GT)Q(\gamma\beta) - 2QT(1-(\alpha+\beta))(\gamma\beta).$$

This method is possible and optimum if and only if $\mu_2 \geq 0$.
ii) Or $\theta = 1$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma}))(1-(\alpha+\beta e^{-\gamma}))^2}},$$

$\mu_2 = 0$, and

$$\mu_1 = (\alpha+\beta e^{-\gamma})\frac{b}{(\delta e^{\delta}(1-\theta_0))} - (P_0 + \frac{b}{1+e^{-\delta(1-\theta_0)}}(\gamma\beta e^{-\gamma}\theta) + Tf(\gamma\beta e^{-\gamma})$$
$$+\frac{(\gamma\beta e^{-\gamma})(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(\gamma\beta e^{-\gamma}) - 2QT(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{\gamma}).$$

This method is possible and optimum if and only if, $\mu_1 \geq 0$.
iii) $\mu_1 = \mu_2 = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))(1-(\alpha+\beta e^{-\gamma\theta}))^2}}.$$

Then, we have

$$\begin{aligned}
&\left((\alpha+\beta e^{-\gamma})\frac{b}{(\delta e^{\delta}(1-\theta_0))}\right) - \left((P_0 + \frac{b}{1+e^{-\delta(1-\theta_0)}}(\gamma\beta e^{-\gamma}\theta) + Tf(\gamma\beta e^{-\gamma})^2\right)^2 \\
&= \left(\frac{(\gamma\beta e^{-\gamma})(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))(1-(\alpha+\beta e^{-\gamma\theta}))^2}{(2\left(\alpha+\beta e^{-\gamma\theta}\right))}\right) \\
&-\left(\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))(1-(\alpha+\beta e^{-\gamma\theta}))^2}\right)\{\frac{1}{2}(H+GT)Q(\gamma\beta e^{-\gamma})\} \\
&-2T(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{\gamma}).
\end{aligned} \tag{18}$$

The above equation is expressed as $\phi_2(\theta) = 0$ with

$$\phi_2(\theta) = \left((\alpha+\beta e^{-\gamma})\cdot\frac{b}{\delta e^{\delta}(1-\theta_0)}\right) - \left(\left(P_0+\frac{b}{1+e^{-\delta(1-\theta_0)}}\right)(\gamma\beta e^{-\gamma}\theta) + Tf(\gamma\beta e^{-\gamma})^2\right)^2$$
$$-\left(\frac{(\gamma\beta e^{-\gamma})(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))^2}{2(\alpha+\beta e^{-\gamma\theta})}\right)$$
$$-\left(\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))^2}\right)\cdot\left(\frac{1}{2}(H+GT)Q(\gamma\beta e^{-\gamma}) - 2T(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{\gamma})\right). \tag{19}$$

The formulation of $\phi_2(\theta)$ is non-linear in θ, enabling the identification of several solutions for θ (not all of which need to be feasible) for Eq. (18).

Exponential inverse relationship and logarithmic relationship

Let us consider,

$$D(\theta) = (\alpha+\beta e^{-\gamma\theta}), \quad \text{and} \quad P(\theta) = P_0 + alog(1+\gamma\theta),$$

$$\phi(Q,\theta) = (\alpha+\beta e^{-\gamma\theta})(P_0+alog(1+\gamma\theta)) - Tf(\alpha+\beta e^{-\gamma\theta})$$
$$-\frac{(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q} - \frac{1}{2}(H+GT)Q(1-(\alpha+\beta e{-}\gamma\theta))$$
$$-QT(1-(\alpha+\beta e{-}\gamma\theta))^2 - \mu_1(\theta-1) + \mu_2\theta,$$

partially twice differentiate concerning Q, θ

$$\frac{\partial^2\phi}{\partial Q^2} = -\left(\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{Q^3}\right), \tag{20}$$

$$\frac{\partial^2\phi}{\partial Q\partial\theta} = \frac{\partial^2\phi}{\partial\theta\partial Q} = -\left[\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\frac{(H+GT)}{2}\right)Q\left(\beta\gamma e^{-\gamma\theta}\right)\right.$$
$$\left. -2T\left(1-(\alpha+\beta e^{-\gamma\theta})\right)\beta\gamma e^{-\gamma\theta}\right],$$

$$\frac{\partial^2\phi}{\partial\theta^2} = \left(-\frac{(1+\gamma\theta)(a\gamma\beta e^{-\gamma\theta}+)(a\gamma^2)(\alpha+\beta e^{-\gamma\theta})}{(1+\gamma\theta)^2}\right) - \left(\frac{a\gamma^2\beta e^{-\gamma\theta}}{(1+\gamma\theta)}\right) - \left(P_0+alog(1+\gamma\theta)(\gamma^2\beta e^{-\gamma\theta})\right)$$
$$-\left(tf(\gamma^2\beta e^{-\gamma\theta})\right) - \left(\frac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q}\right) + \left(\frac{\gamma^2\beta e^{-\delta\theta}(H+GT)Q}{2}\right)$$
$$-(2QT\gamma^2\beta e^{-\gamma\theta})\left(1-(\alpha+\beta e^{-\gamma\theta})\right) - \gamma^2\beta^2 2QT(e^{-\gamma\theta})^2, \tag{21}$$

Table 2. Impact of parameters on optimal policies and profit with non-linear demand

Parameter	Value	Exponential		Logistic		Logarithmic	
		Q	ϕ_1	Q	ϕ_2	Q	ϕ_3
	68600	22457	122652.30	22564	125462.20	22546	122451.60
R	68800	22458	122.10	22564	125462.20	22545	122455.20
	69000	22459	121526.20	22566	125463.20	22545	122455.30
	59600	21546	122565.00	22532	125442.20	22533	124421.20
α	59800	21548	121566.30	22522	125442.20	22433	124451.30
	60000	21550	12266.40	22522	125442.30	22433	124462.20
	8300	21236	112365.00	22456	125365.20	22566	124562.30
β	8400	21235	112364.10	22456	125365.20	22566	124562.00
	8500	21237	112365.10	22456	125365.20	22566	124562.00
	1.6	20175	113221.20	22445	125365.20	20325	124455.20
P_0	1.8	20176	113222.10	22436	125365.20	20325	124453.20
	2	20177	113222.20	22436	125365.20	20325	124453.30
	-0.26	20145	115462.00	22423	124562.30	20145	124445.30
θ	-0.25	20145	114562.00	22424	124562.30	20145	124445.30
	-0.24	20145	114562.00	22424	124563.20	20145	124445.30
	144	19856	114446.00	22568	125365.20	20142	124332.20
K	162	19866	114452.10	22568	125365.20	20142	124332.20
	180	19866	114462.10	22568	125365.20	20142	124332.20
	0.16	19652	114235.20	22546	125345.20	20140	124222.00
H	0.18	19654	114235.20	22546	125345.20	20140	124222.00
	0.2	19655	114235.30	22546	125345.20	20140	124222.00
	0.168	12457	113112.10	23546	124563.20	20140	124122.00
G	0.189	12465	113112.10	22445	124530.20	20140	124121.00
	0.210	12477	113112.10	23654	125634.20	20140	124121.00
	0.1	12365	112452.10	23654	124469.20	20140	124110.20
e	0.07	12364	112452.00	23655	124563.20	20140	124110.20
	0.13	12364	112452.00	23656	124563.20	20140	124110.20
	0.024	11253	112452.00	21546	124563.20	20110	124110.00
f	0.027	11365	112452.00	21547	124563.20	20110	124110.00
	0.030	11356	112452.00	21547	124563.20	20110	124110.00
	1.2	11452	112452.00	21365	124456.30	20145	124120.00
t	1.35	11452	112452.00	21366	124456.30	20150	124142.00
	1.5	11454	112452.00	21365	124456.30	20150	124144.00
	8	11256	112354.00	21365	12444.30	20110	125422.30
δ	9	11256	112355.10	21456	12445.30	20110	125432.20
	10	11256	112365.30	21466	12445.30	20111	124232.20
	0.48	11256	112254.10	21456	12453.30	20012	124120.20
θ_0	0.54	11256	112254.10	21456	12453.30	20012	124121.20
	0.6	11256	112254.10	21456	12455.30	20012	124121.20

$$\det H > 0 \Leftrightarrow$$

$$\begin{aligned}
&\left(-\frac{(1+\gamma\theta)(a\gamma\beta e^{-\gamma\theta}+)(a\gamma^2)(\alpha+\beta e^{-\gamma\theta})}{(1+\gamma\theta)^2}\right) - \left(\frac{a\gamma^2\beta e^{-\gamma\theta}}{(1+\gamma\theta)}\right) - \left(P_0 + alog(1+\gamma\theta)(\gamma^2\beta e^{-\gamma\theta})\right) \\
&- \left(Tf(\gamma^2\beta e^{-\gamma\theta})\right) - \left(\frac{\gamma^2\beta e^{-\gamma\theta}(K+Te)}{Q}\right) \\
&+ \left(\frac{\gamma^2\beta e^{-\delta\theta}(H+GT)Q}{2} - (2Qt\gamma^2\beta e^{-\gamma\theta})\left(1-(\alpha+\beta e^{-\gamma\theta})\right) - \gamma^2\beta^2 2QT(e^{-\gamma\theta})^2\right) \\
&> \left(-\left\{\left(\frac{(\gamma\beta e^{-\gamma\theta})(K+Te)}{Q^2}\right) + \left(\frac{(h+GT)}{2}\right) Q\left(\beta\gamma e^{-\gamma\theta}\right) - 2T\left((1-(\alpha+\beta e^{-\gamma\theta})\right)\beta\gamma e^{-\gamma\theta}\right\}\right).
\end{aligned} \tag{22}$$

As H is negative semi-definite, the objective function is concave.
i) Either $\theta = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta)(K+Te)}{(H+GT)(1-(\alpha+\beta))(1-(\alpha+\beta))^2}},$$

$\mu_1 = 0$, and

$$\mu_2 = a\gamma(\alpha+\beta) - (P_0)(\gamma\beta) + Tf\gamma\beta + \frac{\gamma\beta(K+Te)}{Q} - \frac{(H+GT)Q(\gamma\beta)}{2} + 2QT(1-(\alpha+\beta))(\gamma\beta).$$

This method is possible and optimum if and only if $\mu_2 \geq 0$.
ii) Or $\theta = 1$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma}))(1-(\alpha+\beta e^{-\gamma}))^2}},$$

$\mu_2 = 0$, and

$$\mu_1 = (\alpha+\beta e^{-\gamma})\frac{a\gamma}{1+\gamma} - (P_0 + alog(1+\gamma))\gamma\beta e^{-\gamma} + Tf\gamma\beta e^{-\gamma} + \frac{\gamma\beta e^{-\gamma(K+Te)}}{Q} - \frac{(H+GT)Q(\gamma\beta e^{-\gamma})}{2} + 2QT(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{-\gamma}).$$

This method is possible and optimum if and only if $\mu_1 \geq 0$.
iii) $\mu_1 = \mu_2 = 0$,

$$Q = \sqrt{\frac{2(\alpha+\beta e^{-\gamma\theta})(K+Te)}{(H+GT)(1-(\alpha+\beta e^{-\gamma\theta}))(1-(\alpha+\beta e^{-\gamma\theta}))^2}},$$

$$\left(\left(\frac{a\gamma[\alpha+\beta e^{-\gamma\theta}]}{1+\gamma\theta}\right) - \left(P_0 + alog(1+\gamma\theta)\gamma\beta e^{-\gamma\theta}\right) + tf\gamma\beta e^{-\gamma\theta}\right)^2 = \left(\frac{\gamma\beta e^{-\gamma\theta}(K+Te)}{Q}\right) - \left(\frac{(H+GT)Q(\gamma\beta e^{-\gamma})}{2} + 2QT(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{-\gamma})\right). \tag{23}$$

The above equation is expressed as $\phi_3(\theta) = 0$ with

$$\phi_3(\theta) = \left(\left(\frac{a\gamma[\alpha+\beta e^{-\gamma\theta}]}{1+\gamma\theta}\right) - \left(P_0 + alog(1+\gamma\theta)\gamma\beta e^{-\gamma\theta}\right) + Tf\gamma\beta e^{-\gamma\theta}\right)^2 - \left(\frac{\gamma\beta e^{-\gamma\theta}(K+Te)}{Q}\right) - \left(\frac{(H+GT)Q(\gamma\beta e^{-\gamma})}{2} + 2QT(1-(\alpha+\beta e^{-\gamma}))(\gamma\beta e^{-\gamma})\right). \tag{24}$$

The formulation of $\phi_3(\theta)$ is non-linear in θ, enabling the identification of several solutions for θ (not all of which need to be feasible) for Eq. (23).

5 Illustrations

Non-linear and exponential relationships

The model takes into account unit profit functions and nonlinear demand. Adopting a non-linearity assumption makes the model more accurate and shows different behaviors than the linear case, such as higher profit variance across different levels of circularity and maybe several local extremes. There are many potential combinations between the profit functions and the demand functions. As an example, profit levels can show more fluctuation across different circularity levels, and many are extremely close to each other. The unit gross profit of a product is determined by its nonlinear logistic relationship, which is based on the circularity index $D(\theta) = (\alpha + \beta e^{-\gamma\theta})$ and $P(\theta) = (P_0 + be^{\delta(\theta-1)})$ and the nonlinear demand. The data presented below are sourced from Rabta [4]: Define δ = 10, a= 8500, and b = -0.25. θ = 0, Q^*= 22463.20, and ϕ^*=121520.30 is the optimal solution.
Comparison of two simple solutions θ = 1 with Q = 24256.20 for a profit of ϕ^* = 12255.30. The demand from clients rises significantly with this combination, whereas the gross profit per unit steadily declines. The results show that as demand increases and per-unit profit decreases exponentially θ, producing non-circular products becomes more advantageous than only making circular ones.

Non-linear relationship and logistic relationship

The Economic Order Quantity (EOQ) model is widely used to determine the optimal order quantity that minimizes inventory expenses, including order and storage costs. However, in the context of your results, the EOQ model needs to be adapted to account for nonlinear profit and demand functions, as well as the environmental impact, specifically greenhouse gas emissions associated with different levels of circularity. The demand for the product is represented by a logarithmic function, $D(\theta) = (\alpha + \beta e^{-\gamma\theta})$, and $P(\theta) = P_0 + \frac{b}{1+e^{-\delta(\theta-\theta_0)}}$. Let a = 8500, b = - 0.25 and α = 5. The resulting overall profit is Q^* = 22426.20 with a gross profit of ϕ^* = 16154.28. When α=10, the value of Q increases to 35462.54, while the gross profit remains ϕ^* = 16154.28. Think of the same parameters as previously, but with a = 12000 (a demand increase of up to 20%). The best solution is now defined as: θ^*= 1, Q^* = 22484, and ϕ^* = 16723.30. Taking into account the influence of the circularity index and addressing the issue as a traditional EOQ, we determine: θ = 0, Q^* = 10232.20, and ϕ^* = 117,921, resulting in a reduction of 37% profits. Therefore, the highest overall profit value is preferred to maximize the total profit.

Non-linear relationship and logarithmic relationship

In this context, the demand is described by a nonlinear logarithmic function, while the unit profit function takes the form of a nonlinear exponential $D(\theta) = (\alpha + \beta e^{-\gamma\theta})$, and $P(\theta) = P_0 + a\,log(1 + \gamma\theta)$. Let a = 8500, b = - 0.25 and α = 5. The resulting overall profit is Q^* = 24426.20 with a gross profit of ϕ^* = 122545.28. When α=10, the value of Q increases to 26462.54, while the gross profit remains ϕ^* = 123154.28. Since the logarithmic demand function in θ causes the demand to increase slowly with increasing θ values. Still, the linear per-unit gross profit θ falls dramatically with increasing θ values, the optimal solution for the manufacturer does not involve a very high or very low circularity level for manufactured goods. The objective function lacks concavity, and the stationary points of the Lagrangian are determined by solving a nonlinear equation. With parameters set as after confirming the second-order optimal condition, two local maximizers are found, with one representing the global optimum, which produces a profit 1% higher than when the least favorable scenario occurs at indicating a saddle point where profit drops by 5.6% compared

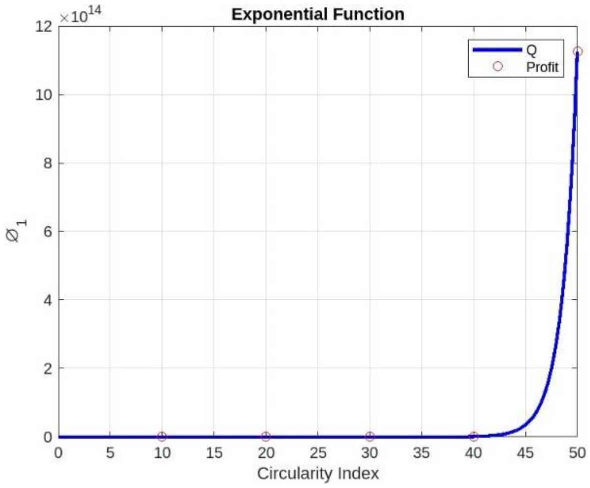

Figure 4. Behavior of ϕ_1

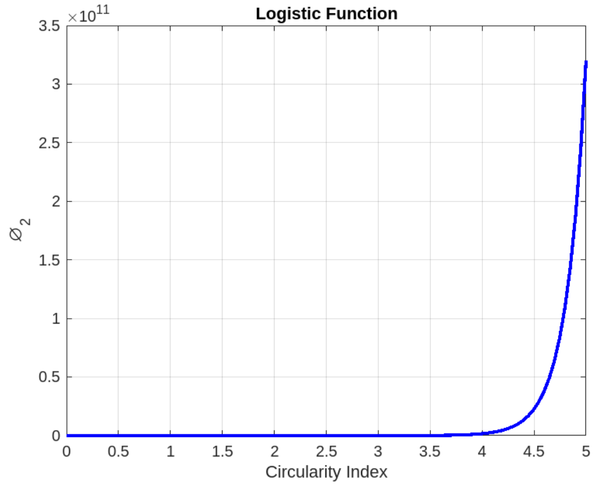

Figure 5. Behavior of ϕ_2

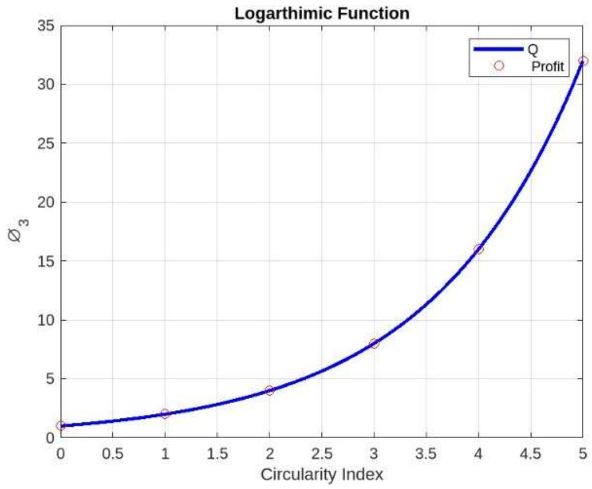

Figure 6. Behavior of ϕ_3

to the optimal value. This analysis indicates that as the threshold approaches, manufacturers should focus on producing items with a circularity level below this threshold; conversely, to optimize profit, prioritize high-circularity items if the threshold is near. The results indicate that the traditional EOQ model and theoretical implementation assume constant demand and linear relationships, which are not sufficient to capture the complexities introduced by non-linear demand and circularity. By modifying the EOQ model to incorporate these complex relationships, manufacturers can optimize both economic profit and environmental sustainability.

6 Applications

An EOQ-based model integrated with the principles of circular economy can revolutionize food waste management by enhancing efficiency, reducing environmental impact, and promoting the sustainable use of resources in various sectors. Food manufacturers can also utilize EOQ to effectively manage by-products, transforming waste into secondary products such as animal feed, bioenergy, compost, or biodegradable packaging, enhancing a closed-loop system that maximizes resource efficiency. Governments and policymakers can also use EOQ models to develop regulations that incentivize businesses to adopt circular food systems, such as tax benefits for waste reduction or subsidies for sustainable packaging innovations. By shifting from a linear "produce-consume-waste" model to a circular system that ensures surplus food is repurposed, redistributed, or recycled rather than discarded, EOQ strategies can significantly contribute

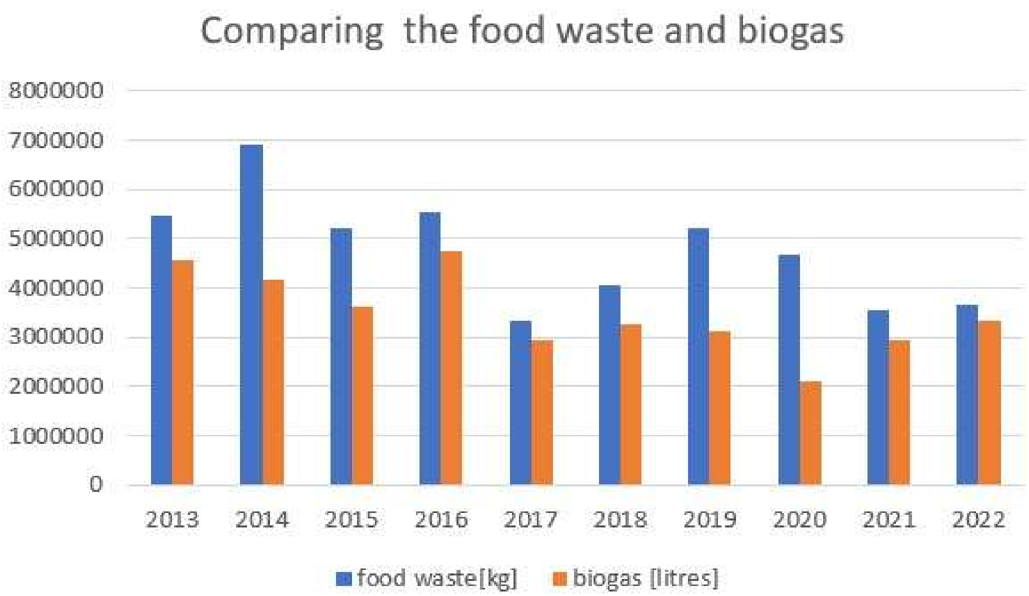

Figure 7. Comparing food waste and biogas

to sustainability by optimizing resource utilization, reducing environmental harm, lowering operational costs, promoting long-term food security, and fostering a more responsible and efficient global food system.

India generates nearly 50 million tonnes of food waste annually, with major losses occurring in the agricultural, wholesale, and retail stages due to inadequate storage and logistics. Biogas initiatives are expanding, with cities like Indore, Pune, and Bhubaneswar processing up to 2,000 tonnes/day of food waste in energy through municipal biogas plants. India's biogas potential exceeds 125 billion cubic meters, supported by the MNRE's SATAT scheme to promote the nationwide adoption of compressed biogas from organic waste.

7 Sensitivity analysis

A detailed sensitivity analysis demonstrates how adopting circular economy measures in inventory management can support company-level decision-making by balancing profitability with environmental sustainability. The findings suggest that businesses can adjust their order quantities and circularity levels without significantly affecting overall profit margins. For example, under a logistic demand model and linear profit function, increasing the circularity index parameter from K=144 to K=160 results in a marginal profit increase of 114,446.00 to 114,452.10. This indicates that integrating more circular practices, such as reusing materials or reducing waste, can be implemented without incurring additional costs. Furthermore, mathematical modeling reveals that the optimal degree of circularity is not static but varies depending on factors such as demand growth, cost structures, and environmental parameters. Therefore, in certain cases, adopting a more circular approach can be achieved without incurring additional costs. Figure 4 demonstrates the behavior of the exponential function ϕ_1 and its two feasible solutions in Table 2. Consequently, the manufacturer achieves the global maximum total profit per unit of time for both feasible circular indices. Figure 5 shows the graphical representation of the feasible solutions for the logistic functions of equation ϕ_2 in Table 2. Figure 6 presents the graphical representation of the feasible solutions for the functions of the logistic equation ϕ_3 in Table 2. Further research should extend these concepts to inventory, production, and supply chain models.

8 Discussion and implementations in food industries

Theoretically, integrating the Economic Order Quantity (EOQ) model and the principles of circular economy into food waste management provides a strategic framework to minimize both operational costs and environmental impacts. By optimizing inventory levels and aligning procurement with actual consumption patterns, EOQ reduces overordering and spoilage, directly cutting down food waste and associated greenhouse gas emissions. A 10-year analysis of food waste and corresponding biogas production in Tamil Nadu, based on data from food and agriculture organizations, is illustrated in Figure 7 to highlight the state's potential for sustainable waste-to-energy conversion. Practically, this approach supports the development of closed-loop systems in which organic waste is recovered through anaerobic digestion to produce biogas, as demonstrated by companies such as Arla Foods and Bioenergy DevCo. These systems convert waste into renewable energy and organic fertilizers, promoting sustainable agriculture and energy independence. In addition, waste segregation, redistribution, and composting practices, supported by smart technologies and green logistics, further improve resource efficiency and reduce emissions. The implementation of the model also involves the evaluation of regional waste sources, the evaluation of current management practices, and the alignment with national policies to design cost-effective and environmentally sound strategies. In general, combining EOQ with circular food waste management offers a solid path to reducing emissions, conserving resources, and supporting sustainable development goals.

9 Managerial implications for the food industry

Food waste recycling presents valuable managerial opportunities for the food industry by reducing operational costs, improving sustainability, and improving brand reputation. Managers should focus on optimizing forecasting, inventory, and production processes to minimize waste generation while exploring scalable recycling solutions such as biogas, compost, and bioplastics. The food industry can improve profits by adopting sustainable waste minimization practices without increasing green investment, using recycling factors such as μ_1, μ_2 and $\phi(Q, \theta)$ to convert organic waste into valuable resources such as biogas. Optimizing forecasting and production processes reduces overproduction, reduces disposal costs, and supports efficient resource use. As environmental regulations tighten, the integration of reduce-reuse-recycle technologies and improving waste minimization efficiency (δ) becomes essential to maintain profitability and environmental compliance. Integrating circular supply chain models and working in partnership with waste management and recycling partners can transform organic waste into profitable resources. In addition, staying ahead of regulatory requirements and investing in green technologies can help ensure compliance and generate long-term financial returns. In general, treating food waste as a resource supports strategic, environmental, and economic goals, positioning businesses for sustainable growth.

10 Conclusion

This article introduces an EOQ inventory model under a circular economy framework, where product circularity varies by a defined index. Both the order quantity and the circularity level are treated as decision variables that influence demand, cost, and price. This study demonstrates that a circular and sustainable economy offers an effective strategy to reduce food waste and greenhouse gas emissions while improving business profitability. By integrating recycling technologies and optimizing order quantities through an EOQ-based model, firms can manage food waste efficiently and economically. From 2013 to 2022, Tamil Nadu data show that higher carbon emissions negatively impact circularity by increasing operational costs through taxation. This discourages investment in sustainable waste-to-energy practices despite their long-term benefits. However, adopting advanced recycling methods such as anaerobic digestion, composting, and waste-to-energy can significantly mitigate environmental impacts. The proposed model not only reduces logistical and operational costs but also supports sustainability goals through emissions control and resource recovery. The EOQ-based model proves to be economically viable and environmentally sustainable for the management of food waste. Sensitivity analysis reveals that as carbon emissions per unit increase, circular practices tend to decline due to the added cost of carbon taxes. This environmental burden discourages investment in circularity. To sustain profitability, manufacturers must improve consumer awareness of circular features, for example, by labeling products as eco-friendly, which is both economically viable and environmentally sustainable for management.

Declarations

Use of AI tools

The authors declare that they have not used Artificial Intelligence (AI) tools in the creation of this article.

Data availability statement

The data used in this study can be accessed by clicking the following link: https://www.fao.org/statistics/data-collection/general/en

Ethical approval (optional)

The authors state that this research complies with ethical standards. This research does not involve either human participants or animals.

Consent for publication

Not applicable

Conflicts of interest

The authors declare that they have no conflict of interest.

Funding

No funding was received for this research.

Author's contributions

V.S.: Conceptualization, Methodology, Resources, Formal analysis, Writing - Original Draft, Visualization. K.K.: Conceptualization, Methodology, Software, Validation, Formal Analysis, Writing – Review & Editing. The authors have read and agreed to the published version of the manuscript.

Acknowledgements

The authors appreciate the editor and anonymous reviewers for their careful evaluation of the manuscript. Their thoughtful feedback and recommendations significantly contributed to improving the clarity and quality of this research.

References

[1] Moraga, G., Huysveld, S., Mathieux, F., Blengini, G.A., Alaerts, L., Van Acker, K. et al. Circular economy indicators: What do they measure? *Resources, Conservation and Recycling*, 146, 452-461, (2019). [CrossRef]

[2] Klemeš, J.J., Fan, Y.V. and Jiang, P. Plastics: friends or foes? The circularity and plastic waste footprint. *Energy Sources, Part A: Recovery, Utilization, and Environmental Effects*, 43(13), 1549-1565, (2021). [CrossRef]

[3] Wan, Y., Kober, T. and Densing, M. Nonlinear inverse demand curves in electricity market modeling. *Energy Economics*, 107, 105809, (2022). [CrossRef]

[4] Rabta, B. An Economic Order Quantity inventory model for a product with a circular economy indicator. *Computer & Industrial Engineering*, 140, 106215, (2020). [CrossRef]

[5] Goyal, S.K. and Giri, B.C. Recent trends in modeling of deteriorating inventory. *European Journal of Operational Research*, 134(1), 1–16, (2001). [CrossRef]

[6] Hovelaque, V. and Bironneau, L. The carbon-constrained EOQ model with carbon emission dependent demand. *International Journal of Production Economics*, 164, 285-291, (2015). [CrossRef]

[7] Taleizadeh, A.A. and Pentico, D.W. An economic order quantity model with partial backordering and all-units discount. *International Journal of Production Economics*, 155, 172-184, (2014).[CrossRef]

[8] Richter, K. and Dobos, I. Analysis of the EOQ repair and waste disposal problem with integer setup numbers. *International Journal of Production Economics*, 59(1-3), 463-467, (1999). [CrossRef]

[9] Chakrabarti, T. and Chaudhuri, K.S. An EOQ model for deteriorating items with a linear trend in demand and shortages in all cycles. *International Journal of Production Economics*, 49(3), 205-213, (1997). [CrossRef]

[10] Erlenkotter, D. Ford Whitman Harris's economical lot size model. *International Journal of Production Economics*, 155, 12-15, (2014). [CrossRef]

[11] Inaniyan, R. and Kumar, G. A comprehensive analysis of linear and non-linear deterioration rates in dual-warehouse inventory systems. *Decision Analytics Journal*, 10, 100396, (2024).[CrossRef]

[12] Soni, H.N. and Shah, K. Continuous review inventory model with lost sale reduction and ordering cost dependent on lead time for the mixtures of distributions. *Yugoslav Journal of Operations Research*, 31(2), 221-233, (2021). [CrossRef]

[13] Eriksson, M., Strid, I. and Hansson, P.A. Carbon footprint of food waste management options in the waste hierarchy–a Swedish case study. *Journal of Cleaner Production*, 93, 115-125, (2015).[CrossRef]

[14] De Laurentiis, V., Corrado, S. and Sala, S. Quantifying household waste of fresh fruit and vegetables in the EU. *Waste Management*, 77, 238-251, (2018). [CrossRef]

[15] Tonini, D., Albizzati, P.F. and Astrup, T.F. Environmental impacts of food waste: Learnings and challenges from a case study on UK. *Waste Management*, 76, 744-766, (2018). [CrossRef]

[16] Papargyropoulou, E., Lozano, R., Steinberger, J.K., Wright, N. and bin Ujang, Z. The food waste hierarchy as a framework for the management of food surplus and food waste. *Journal of Cleaner Production*, 76, 106-115, (2014). [CrossRef]

[17] Thyberg, K.L. and Tonjes, D.J. Drivers of food waste and their implications for sustainable

policy development. *Resources, Conservation and Recycling*, 106, 110-123, (2016). [CrossRef]

[18] Wani, N.A. and Mishra, U. Lifecycle assessment and electro-spinning technique for a sustainable fiber bottle production system with controllable waste and wastewater treatment. *Journal of Cleaner Production*, 443, 141026, (2024).[CrossRef]

[19] Wani, N.A. and Mishra, U. A sustainable municipal solid waste supply chain management with biodiesel energy production using microwave technology. *Environment, Development and Sustainability*, 26, 12863-12900, (2024). [CrossRef]

[20] Wani, N.A. and Mishra, U. An integrated circular economic model with controllable carbon emission and deterioration from an apple orchard. *Journal of Cleaner Production*, 374, 133962, (2022).[CrossRef]

[21] Nketiah, E., Song, H., Adjei, M., Adu-Gyamfi, G., Obuobi, B. and Cudjoe, D. Assessment of energy generation potential and mitigating greenhouse gas emissions from biogas from food waste: Insights from Jiangsu Province. *Applied Energy*, 371, 123717, (2024).[CrossRef]

[22] Jamaludin, H., Elmaky, H.S.E. and Sulaiman, S. The future of food waste: Application of circular economy. *Energy Nexus*, 7, 100098, (2022). [CrossRef]

[23] Talekar, S., Ekanayake, K., Holland, B. and Barrow, C. Food waste biorefinery towards circular economy in Australia. *Bioresource Technology*, 388, 129761, (2023).[CrossRef]

[24] Correani, L., Morganti, P., Silvestri, C. and Ruggieri, A. Food waste, circular economy, and policy with oligopolistic retailers. *Journal of Cleaner Production*, 407, 137092, (2023). [CrossRef]

[25] Khan, M.A.A., Cárdenas-Barrón, L.E., Treviño-Garza, G. and Céspedes-Mota. Optimal circular economy index policy in a production system with carbon emissions. *Expert Systems with Applications*, 212, 118684, (2023). [CrossRef]

[26] Thomas, A. and Mishra, U. A sustainable circular economic supply chain system with waste minimization using 3D printing and emissions reduction in plastic reforming industry. *Journal of Cleaner Production*, 131128, (2022).[CrossRef]

[27] Lu, L.C., Chiu, S.Y., Chiu, Y.H. and Chang, T.H. Three-stage circular efficiency evaluation of agricultural food production, food consumption, and food waste recycling in EU countries. *Journal of Cleaner Production*, 343, 130870, (2022).[CrossRef]

[28] Bernstad, A., la Cour Jansen, J. and Aspegren, A. Door-stepping as a strategy for improved food waste recycling behaviour–Evaluation of a full-scale experiment. *Resources, Conservation and Recycling*, 73, 94-103, (2013).[CrossRef]

[29] Alam, M., Sultan, M.B., Mehnaz, M., Fahim, C.S.U., Hossain, S. and Anik, A.H. Production of biogas from food waste in laboratory scale dry anaerobic digester under mesophilic condition. *Energy Nexus*, 7, 100126, (2022). [CrossRef]

[30] Woon, K.S. and Lo, I.M.C. A proposed framework of food waste collection and recycling for renewable biogas fuel production in Hong Kong. *Waste Management*, 47(A), 3-10, (2016). [CrossRef]

[31] Kurniawan, T.A., Othman, M.H.D., Liang, X., Goh, H.H. and Chew, K.W. From liquid waste to mineral fertilizer: Recovery, recycle and reuse of high-value macro-nutrients from landfill leachate to contribute to circular economy, food security, and carbon neutrality. *Process Safety and Environmental Protection*, 170, 791-807, (2023). [CrossRef]

[32] Mehta, R. and Oh, C. Institutional food waste and the circular economy: Is it time to revisit produce waste in global food supply chains? *Global Food Security*, 43, 100819, (2024). [CrossRef]

[33] Pandey, A.K., Thakur, S., Mehra, R., Kaler, R.S.S., Paul, M. and Kumar, A. Transforming Agri-food waste: Innovative pathways toward a zero-waste circular economy. *Food Chemistry: X*, 28, 102604, (2025). [CrossRef]

[34] Sebatjane, M. The impact of preservation technology investments on lot-sizing and shipment strategies in a three-echelon food supply chain involving growing and deteriorating items. *Operations Research Perspectives*, 9, 100241, (2022).[CrossRef]

[35] Rashid, M.I. and Shahzad, K. Food waste recycling for compost production and its economic and environmental assessment as circular economy indicators of solid waste management. *Journal of Cleaner Production*, 317, 128467, (2021).[CrossRef]

Mathematical Modelling and Numerical Simulation with Applications (MMNSA)
(https://dergipark.org.tr/en/pub/mmnsa)

How to cite this article: Vennila, S. & Karthikeyan, K. (2025). An EOQ model for the circularity of food waste to lessen greenhouse gas emission. *Mathematical Modelling and Numerical Simulation with Applications*, 5(2), 348-375. https://doi.org/10.53391/mmnsa.1526946