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Research Article

Solving difference equations using fourier transform method

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ABSTRACT

This article mainly focuses on presenting a new accurate technique (Fourier Transform Method) for solving linear of m[th] order Difference Equations with constant coefficients. Also, a new lower triangular matrix was introduced to overcome problems related to finding the Fourier Transform of polynomials by rewriting standard-based polynomials through the fallen power polynomial base. Besides, five examples have been presented to illustrate the validity and accuracy of this method. The results reveal that the Fourier transform method is very effective and attractive in solving the difference equations.

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INTRODUCTION

Difference equations have motivated a variety amounts of research. They arise in dynamical systems, biology, electrical circuit analysis, economics, and models [1-4]. Many researchers have introduced different methods to solve these equations such as Berinde [5] solved the second-order difference equation by converting them into two first-order nonhomogeneous difference equations. In [6] Feldmann implemented the so-called discrete Laplace transformation method to solve systems of difference equations and linear difference equations with constant coefficients. Several methods for solving linear difference equations are Dirichlet series transform method [7], Chebyshev series [8], Differential transform method [9], Taylor polynomial [10], and other methods (see [11-15]).

In this article, we will employ Fourier Transform Method (FTM) to solve non homogeneous linear of m^{th}-order difference equation with constant coefficients, that

$$a_m\Delta^m U_n + a_{m-1}\Delta^{m-1}U_n + \cdots + a_1\Delta U_n + a_0 U_n = \zeta(n), \forall m = 1,2,\ldots \quad (1)$$

where ζ(n) is a given function, $a_m, a_{m-1}, \ldots a_1, a_0$ are constant coefficients, $\Delta^i U_n = \Delta^{i-1}\Delta U_n$ for all $i \geq 2$ and $\Delta U_n = U_{n+1} - U_n$. The article is organized as follows: In Section 2, new basic theorems about the Fourier transform method related to difference equations are introduced. In section 3, five examples are solved by using FTM. Finally, the conclusions are presented in section 4.

Preliminaries

Definition 2.1. [16] Let $U(t)$ be piecewise continuous function defined on the entire number line, then its Fourier transform in the angular frequency form is following:

$$\mathcal{F}[U(t)] = \int_{-\infty}^{\infty} U(t)e^{-i\omega t}dt = \hat{U}(\omega). \quad (2)$$

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Definition 2.2. [16] The inverse Fourier transform of $\widehat{U}(\omega)$ is given by:

$$\mathcal{F}^{-1}\left[\widehat{U}(\omega)\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\widehat{U}(\omega)e^{i\omega t}d\omega = U(t). \tag{3}$$

The difference equations are the problem of finding the given sequence over the relation of previous terms of a defined sequence on natural numbers. To make it possible to take the Fourier transform of the sequence we define a new function based on that sequence as follows:

Let U_n be any given sequence defined on natural numbers we define corresponding function

$$U(t) = \sum_{n=0}^{\infty} U_n S_n(t), \tag{4}$$

where $S_n(t) = \begin{cases} 1, & if\ n \le t < n+1 \\ 0 & , otherwise \end{cases}$.

The function $U(t)$ defined in this way with domain $[0,\infty)$ coincides with the sequence U_n in the natural numbers. In this case, the sequence U_n is the solution of the difference equation (1) if and only if the function $U(t)$ satisfies this equation.

Now we give the Fourier transform of this function and the same shifting properties as well.

$$\mathcal{F}[U(t)] = \widehat{U}(\omega) = \int_{-\infty}^{\infty} U(t)e^{-i\omega t}dt = \int_{-\infty}^{\infty} e^{-i\omega t}\sum_{n=0}^{\infty} U_n S_n(t)\, dt$$

$$= \sum_{n=0}^{\infty} U_n \int_{-\infty}^{\infty} S_n(t)e^{-i\omega t}dt = \sum_{n=0}^{\infty} U_n \int_{n}^{n+1} 1\, e^{-i\omega t}dt$$

$$= \sum_{n=0}^{\infty} U_n \frac{e^{-i\omega t}}{-i\omega}\Big|_{t=n}^{t=n+1} = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=0}^{\infty} U_n e^{-i\omega n},$$

so

$$\mathcal{F}[U(t)] = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=0}^{\infty} U_n e^{-i\omega n}. \tag{5}$$

The Basic Theorems

Theorem 2.1.1. Let $\widehat{U}(\omega)$ the Fourier transform of $U(t)$, $i.e.$ $\mathcal{F}[U(t)] = \widehat{U}(\omega)$, then

$$\mathcal{F}[U(t+k)] = e^{i\omega k}\widehat{U}(\omega) - \hat{S}_0\left(e^{i\omega k}U_0 + e^{i\omega(k-1)}U_1 + \cdots + e^{i\omega}U_{k-1}\right).$$

Proof. We assume that $U(t+k)$ is presented as follows

$$U(t+k) = \sum_{n=0}^{\infty} U_n S_n(t+k),$$

where $S_n(t+k) = \begin{cases} 1, & if\ n \le t+k < n+1 \\ & 0, otherwise \end{cases}$

$= \begin{cases} 1, & if\ n-k \le t < n-k+1 \\ & 0, otherwise \end{cases}$

Now using Fourier transform on both sides of Eq (6), we obtain

$$\mathcal{F}[U(t+k)] = \mathcal{F}\left[\sum_{n=0}^{\infty} U_n S_n(t+k)\right] = \sum_{n=0}^{\infty} U_n \mathcal{F}[S_n(t+k)]$$
$$= \sum_{n=k}^{\infty} U_n e^{i\omega k}\mathcal{F}[S_n(t)] = e^{i\omega k}\sum_{n=k}^{\infty} U_n e^{i\omega k}\,\hat{S}_0(\omega)e^{-i\omega n}$$
$$= e^{i\omega k}\hat{S}_0(\omega)\left(\sum_{n=0}^{\infty} U_n e^{-i\omega n} - U_0 - e^{-i\omega}U_1 - \cdots - e^{-i\omega(k-1)}U_{k-1}\right)$$
$$= e^{i\omega k}\widehat{U}(\omega) - \hat{S}_0\left(e^{i\omega k}U_0 + e^{i\omega(k-1)}U_1 + \cdots + e^{i\omega}U_{k-1}\right),$$

where

$$\mathcal{F}[S_n(t)] = \int_{n}^{n+1} e^{-i\omega t}dt = \frac{1-e^{-i\omega}}{i\omega}e^{-i\omega n}$$
$$= \mathcal{F}[S_0(t)]e^{-i\omega n} = \hat{S}_0(\omega)e^{-i\omega n}.$$

Theorem 2.1.2. If $U(t) = 1$ then $\mathcal{F}[U(t)] = \frac{1}{i\omega}$, for $Im(\omega) < 0$.

Proof. By using Eq (5) we get

$$\mathcal{F}[1] = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=k}^{\infty} e^{-i\omega n} = \frac{1-e^{-i\omega}}{i\omega}\frac{1}{1-e^{-i\omega}} = \frac{1}{i\omega},$$

for

$$\left|e^{-i\omega}\right| < 1 \Leftrightarrow \left|e^{-i(x+y)}\right| < 1 \Leftrightarrow \left|e^{-ix}\right|\left|e^{y}\right| < 1 \Leftrightarrow Im(\omega) < 0.$$

Theorem 2.1.3. If $U(t) = t$, then $\mathcal{F}[U(t)] = \frac{1}{i\omega(e^{i\omega}-1)}$, for $Im(\omega) < 0$.

Proof. By using Eq (5) we obtain

$$\mathcal{F}[t] = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=k}^{\infty} ne^{-i\omega n} = \frac{1-e^{-i\omega}}{i\omega}\frac{e^{-i\omega}}{(1-e^{-i\omega})^2}$$
$$= \frac{1}{i\omega(e^{i\omega}-1)},$$

for $Im(\omega) < 0$.

Theorem 2.1.4. Let $k \in \mathbb{N}$, $t \ge k$ and $U(t) = t^{\underline{k}} = t(t-1)\ldots(t-k+1)$ then $\mathcal{F}[U(t)] = \frac{k!}{i\omega(e^{i\omega}-1)^k}$, for $Im(\omega) < 0$.

Proof. We know that: $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}, |x| < 1$

By differentiating both sides with respect to x, we get:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{1}{(1-x)^2}, \sum_{n=2}^{\infty} n(n-1)x^{n-2} = \frac{2}{(1-x)^3},$$

And more generally

$$\sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)x^{n-k} = \frac{k!}{(1-x)^{k+1}}. \quad (7)$$

We can rewrite the equation (7) as follows

$$\sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)x^{n} = \frac{x^k k!}{(1-x)^{k+1}}, |x|<1. \quad (8)$$

Now by applying Fourier transform of our function and using Eq (5) we obtain

$$\mathcal{F}[t^{\underline{k}}] = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=k}^{\infty} n(n-1)(n-2)\dots(n-k+1)e^{-i\omega n}$$

$$= \frac{1-e^{-i\omega}}{i\omega}\frac{e^{-i\omega k}k!}{(1-e^{-i\omega})^{k+1}} = \frac{k!}{i\omega(e^{i\omega}-1)^k},$$

for $Im(\omega) < 0$.

Theorem 2.1.5. Let $a \in \mathbb{C}$, $a \neq 0$ and $U(t) = a^t$, then $\mathcal{F}[U(t)] = \frac{e^{i\omega}-1}{i\omega(e^{i\omega}-a)}$, for $Im(\omega) < \log\left|\frac{1}{a}\right|$.

Proof. By using Eq (5) we get

$$\mathcal{F}[a^t] = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=0}^{\infty} a^n e^{-i\omega n} = \frac{1-e^{-i\omega}}{i\omega}\sum_{n=0}^{\infty}(ae^{-i\omega})^n$$
$$= \frac{1-e^{-i\omega}}{i\omega}\frac{1}{1-ae^{-i\omega}} = \frac{e^{i\omega}-1}{i\omega(e^{i\omega}-a)},$$

for $|ae^{-i\omega}| < 1 \Leftrightarrow Im(\omega) < \log\left|\frac{1}{a}\right|$.

Lemma 2.1.6. Let $a \in \mathbb{C}$, $a \neq 0$ and $U(t) = ta^t$, then $\mathcal{F}[U(t)] = \frac{a(e^{i\omega}-1)}{i\omega(e^{i\omega}-a)^2}$, for $Im(\omega) < \log\left|\frac{1}{a}\right|$.

Corollary 2.1.7. $\mathcal{F}\left[\frac{1}{a(a-1)}\left(ta^t - \frac{a}{a-1}(a^t-1)\right)\right] = \frac{1}{i\omega(e^{i\omega}-a)^2}$.

Theorem 2.1.8. If $\widehat{U}(\omega) = \frac{1}{i\omega(e^{i\omega}-a)}$, then $\mathcal{F}^{-1}[\widehat{U}(\omega)] = U(t) = \frac{a^t-1}{a-1}$, for $Im(\omega) < \log\left|\frac{1}{a}\right|, a \neq 0,1$.

Proof. By definition of inverse Fourier transform, we get

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega(e^{i\omega}-a)}\right] = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega t}}{i\omega(e^{i\omega}-a)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-1)}}{i\omega(1-ae^{-i\omega})}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{i\omega(t-1)}}{i\omega}\sum_{n=0}^{\infty}(ae^{-i\omega})^n\, d\omega = \sum_{n=0}^{\infty}\frac{1}{2\pi}a^n\int_{-\infty}^{\infty}\frac{e^{i\omega(t-1-n)}}{i\omega}d\omega$$

$$= \sum_{n=0}^{\infty} a^n\mathcal{F}^{-1}\left[\frac{1}{i\omega}\right]_{t=t-1-n} = \sum_{n=0}^{\infty} a^n \begin{cases}1, & if\ 0 \le t < \infty \\ 0, & otherwise\end{cases} |t = t-1-n$$

$$= \sum_{n=0}^{\infty}\begin{cases}a^n, & if\ n+1 \le t < \infty \\ 0, & otherwise\end{cases}$$

$$= \begin{cases}1, & if\ 1 \le t < \infty \\ 0, & otherwise\end{cases} + \begin{cases}a, & if\ 2 \le t < \infty \\ 0, & otherwise\end{cases} + \begin{cases}a^2, & if\ 3 \le t < \infty \\ 0, & otherwise\end{cases}$$

$$+ \dots + \begin{cases}a^n, & if\ n+1 \le t < \infty \\ 0, & otherwise\end{cases} + \dots$$

$$= \begin{cases}1, & if\ 1 \le t < 2 \\ 1+a, & if\ 2 \le t < 3 \\ 1+a+a^2 & , if\ 3 \le t < 4 \\ \vdots & \\ 1+a+\dots+a^{n-1}, & if\ n \le t < n+1 \\ \vdots & \end{cases}$$

$$= \sum_{n=0}^{\infty}\frac{a^n-1}{a-1}S_n(t) = \frac{a^t-1}{a-1},$$

for $\left|ae^{-i\omega}\right| < 1 \Leftrightarrow Im(\omega) < \log\left|\frac{1}{a}\right|, a \neq 0,1$.

Theorem 2.1.9. If $a \neq 0$ $\widehat{U}(\omega) = \frac{1}{i\omega(e^{2i\omega}+a^2)}$, then

$$\mathcal{F}^{-1}[\widehat{U}(\omega)] = U(t) = \frac{1}{1+a^2}\left(1-|a|^t\left(cos\left(\frac{\pi t}{2}\right)+\frac{1}{|a|}sin\left(\frac{\pi t}{2}\right)\right)\right),$$

$$for\ Im(\omega) < \log\left|\frac{1}{a}\right|.$$

Proof. By definition of inverse Fourier transform, we get

$$\mathcal{F}^{-1}\left[\frac{1}{i\omega(e^{2i\omega}+a^2)}\right] = \frac{1}{2ai}\mathcal{F}^{-1}\left[\frac{1}{i\omega(e^{i\omega}-ai)} - \frac{1}{i\omega(e^{i\omega}+ai)}\right]$$

$$= \frac{1}{2ai}\left(\frac{(ai)^t-1}{ai-1}+\frac{(-ai)^t-1}{ai+1}\right) = \frac{|a|^t}{2ai}\left(\frac{(i)^t}{ai-1}+\frac{(-i)^t}{ai+1}\right)+\frac{1}{2ai}\left(\frac{1}{1-ai}+\frac{1}{1+ai}\right),$$

$$= -\frac{|a|^t}{2ai}\frac{\left((1+ai)e^{\frac{i\pi t}{2}}+(ai-1)e^{-\frac{i\pi t}{2}}\right)}{1+a^2}+\frac{1}{1+a^2},$$

$$= \frac{1}{1+a^2}\left(1-|a|^t\left(cos\left(\frac{\pi t}{2}\right)+\frac{1}{|a|}sin\left(\frac{\pi t}{2}\right)\right)\right),$$

for $\left|ae^{-i\omega}\right| < 1 \Leftrightarrow Im(\omega) < \log\left|\frac{1}{a}\right|, a \neq 0,1$.

Fourier Transforms of Polynomials

We have seen above Fourier transform of falling power polynomial

$$t^{\underline{k}} = t(t-1)\dots(t-k+1)$$

can be formulated like $\widehat{U}(\omega) = \frac{k!}{i\omega(e^{i\omega}-1)^k}$. Now if we want to evaluate Fourier transform of any polynomial we might face complex expressions, and it is better to avoid that complexity, if we write a polynomial in the form of a polynomial of falling powers, we can find the Fourier transform more easily and without any confusion. For example instead of $U(t) = t^2$ we can rewrite this function as $U(t) = t(t\text{-}1) + t$ and by taking Fourier transform we get:

$$\mathcal{F}[t^2] = \mathcal{F}[t(t-1)+t] = \frac{2!}{i\omega(e^{i\omega}-1)^2}+\frac{1}{i\omega(e^{i\omega}-1)}.$$

In this case, the problem arises of expressing a polynomial in terms of a polynomial of falling powers.

We want to find a_k such that

$$t^k = a_1 t^{\underline{1}} + a_2 t^{\underline{2}} + \cdots + a_k t^{\underline{k}}, \tag{9}$$

here some powers of t in terms falling powers polynomial:

$$t = t$$
$$t^2 = t + t(t-1)$$
$$t^3 = t + 3t(t-1) + t(t-1)(t-2)$$
$$t^4 = t + 7t(t-1) + 6t(t-1)(t-2) + t(t-1)(t-2)(t-3)$$

If we continue like this, we get a triangle of corresponding coefficients:

$$S = (S_{i,j})_{i,j\geq 1} = \begin{pmatrix} 1 & & & & & & \\ 1 & 1 & & & & & \\ 1 & 3 & 1 & & & & \\ 1 & 7 & 6 & 1 & & & \\ 1 & 15 & 25 & 10 & 1 & & \\ 1 & 31 & 90 & 65 & 15 & 1 & \\ \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \ddots \end{pmatrix}$$

Coefficient matrix S is lower triangular matrix that k^{th} row elements gives coefficients of falling power polynomials in the expansion of polynomial t^k. We can formulate this matrix by relation

$$S_{ij} = \begin{cases} 1, if\ i \geq 1, j = 1 \\ 1, if\ i = j \\ 0, if\ i < j \\ S_{i-1,j-1} + j \cdot S_{i-1,j}, if\ i \geq 3, j > 1 \end{cases}.$$

Applications of Fourier Transform Method

In this section, some examples are presented to show the efficiency and validity of the proposed method.

Example 3.1. Consider the following linear of 2^{nd} order difference equation

$$\Delta^2 U_n + 6\Delta U_n + 5U_n = 3n + 7, \tag{10}$$

with $U(0) = U(1) = 0$

We can rewrite equation (10) as follows

$$U(t+2) + 4U((t+1) = 3t + 7. \tag{11}$$

By applying Fourier transform of (11), gives

$$e^{2i\omega}\widehat{U} + 4e^{i\omega}\widehat{U} = \frac{3}{i\omega(e^{i\omega}-1)} + \frac{7}{i\omega}.$$

This can be reduced to

$$\widehat{U} = \frac{3}{i\omega(e^{i\omega}-1)(e^{i\omega}+4)e^{i\omega}} + \frac{7}{i\omega(e^{i\omega}+4)e^{i\omega}}.$$

$\widehat{U}$ can be represented as a partial fraction expansion as follows

$$\begin{aligned}\widehat{U} &= \frac{3}{20}\left[\frac{4}{i\omega(e^{i\omega}-1)} + \frac{1}{i\omega(e^{i\omega}+4)} - \frac{5}{i\omega e^{i\omega}}\right] \\ &\quad + \frac{7}{4}\left[\frac{1}{i\omega e^{i\omega}} - \frac{1}{i\omega(e^{i\omega}+4)}\right], \\ &= \frac{12}{20}\frac{1}{i\omega(e^{i\omega}-1)} - \frac{8}{5}\frac{1}{i\omega(e^{i\omega}+4)} + \frac{1}{i\omega e^{i\omega}}.\end{aligned} \tag{12}$$

By applying inverse Fourier transform of (12), gives

$$\begin{aligned}U(t) &= \frac{3}{5}t - \frac{8}{5}\left[\frac{(-4)^t - 1}{-5}\right] + H(t-1) \\ &= \begin{cases} 0, & t = 0 \\ \frac{3}{5}t - \frac{8}{25}(-4)^t + \frac{17}{25}, & 1 \leq t < \infty \end{cases}.\end{aligned}$$

Where $H(t)$ is Heaviside step function.

Example 3.2. Consider the following linear of 1^{st} order difference equation

$$\Delta U_n = n^2 + n - 2, \tag{13}$$

with the condition $U(0) = 0$.

We can rewrite equation (13) as follows

$$U(t+1) - U((t) = t^2 + t - 2, \tag{14}$$

by the formula (9), we can write equation (14) as follows

$$U(t+1) - U((t) = t^{\underline{2}} - t + 2t - 2, \tag{15}$$

By applying Fourier transform of (15), gives

$$e^{i\omega}\widehat{U} - \widehat{U} = \frac{2}{i\omega(e^{i\omega}-1)^2} + \frac{2}{i\omega(e^{i\omega}-1)} - \frac{2}{i\omega}.$$

This can be reduced to

$$\widehat{U} = 2\left[\frac{1}{i\omega(e^{i\omega}-1)^3} + \frac{1}{i\omega(e^{i\omega}-1)^2} - \frac{1}{i\omega(e^{i\omega}-1)}\right]. \tag{16}$$

By applying inverse Fourier transform of (16), gives

$$U(t) = \frac{2}{3!}\big(t(t-1)(t-2)\big) + \frac{2}{2!}\big(t(t-1)\big) - \frac{2}{1!}(t) = \frac{1}{3}(t^3 - 7t).$$

Example 3.3. Consider the following linear of 3^{rd} order difference equation

$$\Delta^3 U_n + 2\Delta^2 U_n + 5\Delta U_n = 2^n, \tag{17}$$

with the conditions $U(0) = 0$, $U(1) = 1$, $U(2) = 1$.

We can rewrite equation (17) as follows

$$U(t+3) - U(t+2) + 4U(t+1) - 4U(t) = 2^t. \quad (18)$$

By applying Fourier transform of (18), gives

$$e^{3i\omega}\widehat{U} - \hat{S}_0(e^{2i\omega} + e^{i\omega}) - \left(e^{2i\omega}\widehat{U} - \hat{S}_0 e^{i\omega}\right) + 4e^{i\omega}\widehat{U} - 4\widehat{U} = \frac{1}{i\omega(e^{i\omega}-2)},$$

So

$$\widehat{U} = \frac{1}{i\omega(e^{i\omega}-2)(e^{3i\omega} - e^{2i\omega} + 4e^{i\omega} - 4)} + \frac{i\omega(e^{i\omega}-1)}{i\omega(e^{3i\omega} - e^{2i\omega} + 4e^{i\omega} - 4)}.$$

$\widehat{U}$ can be represented as a partial fraction expansion as follows

$$\widehat{U} = \frac{1/8}{i\omega(e^{i\omega}-2)} - \frac{1/5}{i\omega(e^{i\omega}-1)} + \frac{(43/40)e^{i\omega}}{i\omega(e^{2i\omega}+4)} - \frac{1/20}{i\omega(e^{i\omega}+4)}.$$

By applying inverse Fourier transform of (19), gives

$$U(t) = \frac{1}{25} - \frac{t}{5} + 2^t\left(\frac{1}{8} + \frac{91}{200}sin\left(\frac{\pi t}{2}\right) - \frac{33}{200}cos\left(\frac{\pi t}{2}\right)\right).$$

Example 3.4. Consider the following linear of 4^{th} order difference equation

$$\Delta^4 U_n - 5\Delta^3 U_n + 8\Delta^2 U_n - 4\Delta U_n = n^4, \quad (20)$$

with the conditions $U(0) = 1$, $U(1) = -1$, $U(2) = 2$, $U(3) = 0$. We can rewrite equation (20) as follows

$$U(t+4) - 9U(t+3) + 29U(t+2) - 39U(t+1) + 18U(t) = t^4. \quad (21)$$

By the formula (9), we can write equation (21) as follows

$$U(t+4) - 9U(t+3) + 29U(t+2) - 39U(t+1) + 18U(t) = t + 7t(t-1) + 6t(t-1)(t-2) + t(t-1)(t-2)(t-3) \quad (22)$$

By applying Fourier transform of (22), gives

$$e^{4i\omega}\widehat{U} - \hat{S}_0(e^{4i\omega} - e^{3i\omega} + 2e^{2i\omega}) - 9e^{3i\omega}\widehat{U} + 9\hat{S}_0(e^{3i\omega} - e^{2i\omega} + 2e^{i\omega}) +29e^{2i\omega}\widehat{U} - 29\hat{S}_0(e^{2i\omega} - e^{i\omega}) - 39e^{i\omega}\widehat{U} + 39\hat{S}_0(e^{i\omega}) + 18\widehat{U} = \frac{1}{i\omega(e^{i\omega}-1)} + 7\frac{2!}{i\omega(e^{i\omega}-1)^2} + 6\frac{3!}{i\omega(e^{i\omega}-1)^3} + \frac{4!}{i\omega(e^{i\omega}-1)^4},$$

So

$$\widehat{U} = \frac{1}{i\omega(e^{i\omega}-1)(e^{i\omega}-2)(e^{i\omega}-3)^2}\left[\frac{1}{i\omega(e^{i\omega}-1)} + \frac{14}{i\omega(e^{i\omega}-1)^2} + \frac{36}{i\omega(e^{i\omega}-1)^3} + \frac{24}{i\omega(e^{i\omega}-1)^4} + \frac{1-e^{-i\omega}}{i\omega}(e^{4i\omega} - 10e^{3i\omega} + 40e^{2i\omega} - 86e^{i\omega})\right],$$

and this is just rational function in terms of $e^{i\omega}$, except $\frac{1}{i\omega}$ factor. so by partial fractions we get:

$$\widehat{U} = \frac{1}{i\omega}\left(\frac{75}{e^{i\omega}-2} - \frac{243}{4(e^{i\omega}-1)} - \frac{103}{2(e^{i\omega}-1)^2} - \frac{38}{(e^{i\omega}-1)^3} - \frac{21}{(e^{i\omega}-1)^4} - \frac{6}{(e^{i\omega}-1)^5} - \frac{57}{4(e^{i\omega}-3)} + \frac{5}{(e^{i\omega}-3)^2} +(e^{i\omega}-1)\left[\frac{-38}{e^{i\omega}-2} + \frac{55}{4(e^{i\omega}-1)} + \frac{101}{4(e^{i\omega}-3)} - \frac{29}{2(e^{i\omega}-3)^2}\right]\right). \quad (23)$$

By applying inverse Fourier transform of (23), gives

$$U(t) = \frac{-1269}{4!} - \frac{243}{4}t - \frac{103}{4}t^{2} - \frac{38}{6}t^{3} - \frac{21}{24}t^{4} - \frac{1}{20}t^{5} + 37(2^t) + \frac{405}{4!}(3^t) - \frac{41}{3}t(3^t).$$

Example 3.5. Consider the Fibonacci sequence defined by difference equation

$$\Delta^2 U_n + \Delta U_n - U_n = 0, U(0) = U(1) = 1. \quad (24)$$

We can rewrite equation (24) as follows

$$U(t+2) - U(t+1) - U(t) = 0 \quad (25)$$

By applying Fourier transform of (25), gives

$$e^{2i\omega}\widehat{U} - \hat{S}_0(e^{2i\omega} - e^{i\omega}) - e^{i\omega}\widehat{U} + \hat{S}_0 e^{i\omega} - \widehat{U} = 0,$$

so that

$$\widehat{U}(e^{2i\omega} - e^{i\omega} - 1) = \frac{e^{2i\omega} - e^{i\omega}}{i\omega}.$$

This can be reduced to

$$\widehat{U} = \frac{e^{i\omega}(e^{i\omega}-1)}{i\omega(e^{2i\omega} - e^{i\omega} - 1)}.$$

$\widehat{U}$ can be represented as a partial fraction expansion as follows

$$\widehat{U} = \frac{(e^{i\omega} - 1)}{i\omega}\left(\frac{\frac{\alpha}{\alpha - \beta}}{e^{i\omega} - \alpha} - \frac{\frac{\beta}{\alpha - \beta}}{e^{i\omega} - \beta}\right), \quad (26)$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\beta = \frac{1-\sqrt{5}}{2}$ is the silver ratio.

By applying inverse Fourier transform of (26), gives

$$U(t) = \frac{1}{\alpha - \beta}(\alpha \cdot \alpha^t - \beta \cdot \beta^t) = \frac{1}{\sqrt{5}}\frac{(1+\sqrt{5})^{t+1} - (1-\sqrt{5})^{t+1}}{2^{t+1}}.$$

CONCLUSION

In this study, a new accurate technique (Fourier Transform Method) was applied to solve the linear of m^{th}-order difference equation with constant coefficients, also successfully overcoming the finding of the Fourier transform of a polynomial function. In addition, we gave five examples. One of them is the difference equations from the Fibonacci sequence.

AUTHORSHIP CONTRIBUTIONS

Authors equally contributed to this work.

DATA AVAILABILITY STATEMENT

The authors confirm that the data that supports the findings of this study are available within the article. Raw data that support the finding of this study are available from the corresponding author, upon reasonable request.

CONFLICT OF INTEREST

The author declared no potential conflicts of interest with respect to the research, authorship, and/or publication of this article.

ETHICS

There are no ethical issues with the publication of this manuscript.

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