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APPLYING THE MODIFIED F-EXPANSION METHOD TO FIND THE EXACT SOLUTIONS OF THE BOGOYAVLENSKII EQUATION

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Abstract

The aim of this study is to obtain the new exact solutions of the Bogoyavlenskii equation (BE) using the modified F-expansion method. With the aid of symbolic computation, this method has been successfully implemented in the BE and the exact solutions obtained have been expressed by the hyperbolic functions, trigonometric functions, and rational functions. To the best of our knowledge, the BE has not been previously investigated by the modified F-expansion method. The findings of this study demonstrate that the suggested method is highly effective, powerful, and practical for obtaining the exact solutions of one dimensional and higher-dimensional nonlinear partial differential equations arising in mathematical physics and engineering.

Keywords: Bogoyavlenskii equation, exact solution, modified F-expansion method

1. Introduction

In science, many important complex phenomena in various fields can be described by nonlinear partial differential equations (NPDEs) and these equations are widely used to describe many phenomena and processes in various scientific fields.

In recent years, the study of exact solutions to highly nonlinear partial differential equations has been of great importance. In particular, the investigation of the traveling wave solutions, exact or numerical solutions to NPDEs play an important role in nonlinear science. These equations are mathematical models derived from complex physical phenomena that arise in engineering and applied mathematics, ranging from physics to biology, chemistry, optics, fiber optics, mechanics, and numerous other fields. Furthermore, investigating exact solutions of NPDEs will enhance our understanding of these phenomena.

In the recent years, many powerful and reliable methods have been developed to obtain exact solutions of NPDEs, such as the the modified simple equation method [1-4], different types of F-expansion methods [5-10], several forms of (G'/G)-expansion methods [11-18], exp-function method [19-23] and many more [24-29].

In this paper, we study the following Bogoyavlenskii equation (BE) [41]

$$4u_t + u_{xxy} - 4u^2u_y - 4u_xv = 0\ , \qquad (1)$$
$$uu_y = v_x$$

where $u(x, y, t)$ is the physical field and $v(x, y, t)$ is some potential. The BE is used to describe some kinds of waves on the sea surface. The BE was first proposed by Bogoyavlenskii [41], describing the (2+1)-dimensional interaction of a Riemann wave propagating along the y-axis with long waves propagating along the x-axis.

Several different methods have been applied to find exact solutions to the Bogoyavlenskii equation. In [30], exact solutions of Eq. (1) were obtained by the singular manifold method and the traveling wave method, respectively. The (G'/G)-method was utilized to obtain exact traveling wave solutions of the BE [31]. In [32], the exact traveling wave solutions of the Bogoyavlenskii equation using a modified extended tanh-function method is presented. In [33], the exp(-$\Phi(\xi)$)-expansion method was used to find exact solutions of the BE. The exact traveling wave solutions of the Bogoyavlenskii equation were constructed using modified method of simplest equation in [34]. In [35], Lie symmetry method to find the exact solutions of the BE equation was used. $(G'/G,1/G)$-expansion and $(1/G')$-expansion techniques were used for obtaining traveling wave solutions of the Bogoyavlenskii equation in [36]. In paper [37], the dynamical behavio,r and exact traveling wave solutions for a (2+1)-dimensional Bogoyavlenskii coupled system by using the modified extended tanh method via a Riccati equation were investigated. In paper [38] three various techniques, [39] nonlocal symmetry method and [40] improved G'/G^2 and simplified $\tan(\phi(\xi)/2)$ methods were used for solutions of the BE.

The motivation of this paper is to investigate the new exact solutions of the BE. Using the proposed modified F-expansion method [42], we found some new rational functions, trigonometric functions, and hyperbolic functions that can be the exact solutions to this equation.

The structure of this article is organized as follows: Section 1 provides an introduction to the topic. In section 2, we present a detailed description of the modified F-expansion method. Section 3 applies this method to find exact solutions of the Bogoyavlenskii equation. Finally, section 4 summarizes the results obtained and presents the conclusion of the study.

2. Description of the modified F-expansion method

In this section we describe the modified F-expansion method. Considering a given nonlinear partial differential equation

$$P(u, u_x, u_t, u_{xx}, u_{tt}, \ldots) = 0 \quad (2)$$

with independent variables x, t and dependent variable u.
P is a polynomial in $u = u(x,t)$ and its partial derivatives. In the following, we outline the main steps of the modified F-expansion method [42].

Step1*:* In Eq.(2), wave transformation

$$u(x,t) = u(\xi), \xi = kx + \lambda t \quad (3)$$

where k and λ are constants to determined later is used. Substituting Eq.(3) into Eq.(2), we get a nonlinear ordinary differential equation (NODE) for $u(\xi)$,

$$Q(u, u', u'', \ldots) = 0. \quad (4)$$

Step2: Suppose that the solution u of Eq.(4) can be expressed as a finite series in the form

$$u = a_0 + \sum_{i=1}^{N} (a_i F^i(\xi) + b_i F^{-i}(\xi)) \quad (5)$$

where a_0, a_i, b_i , $(i = 1, 2, \ldots N)$ are constant to be determined later. The right-hand side of Eq.(5) is also a polynomial in $F(\xi)$ and $F(\xi)$ is a solution of the auxiliary NODE

$$F' = A + BF + CF^2 \quad (6)$$

where $F' = \frac{dF(\xi)}{d\xi}$ and A, B, C are constants.

Step3: The positive integer N can be determined by considering the homogeneous balance between the highest order derivative and the highest degree nonlinear term appearing in (4).

Step4: Inserting (5) into (4) together, and using (6), then collecting all terms with the same degree of $F(\xi)$ together, we reach a polynomial in $F(\xi)$. We set each coefficient of the resulting polynomial to zero, yield a set of algebraic equations for a_0, a_i, b_i, k, λ .

Step5: Solving the algebraic equations, we get the constants a_0, a_i, b_i $(i = 1, 2, \ldots N)$, k, λ. The solutions of the first order NODE Eq.(6) have been well known for us (Table 1). Substituting a_0, a_i, b_i $(i = 1, 2, \ldots, N)$, k, λ and solutions of Eq.(6) into Eq.(5) we obtain more exact solutions of Eq.(2).

Table 1. Relations between A, B, C, and corresponding $F(\xi)$ in Eq. (6)

Values of A, B, C			$F(\xi)$
$A = 0$,	$B = 1$,	$C = -1$	$\frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$
$A = 0$,	$B = -1$,	$C = 1$	$\frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$
$A = \frac{1}{2}$,	$B = 0$,	$C = -\frac{1}{2}$	$\coth(\xi) \pm \operatorname{csch}(\xi)$, $\tanh(\xi) \pm i\operatorname{sech}(\xi)$
$A = 1$,	$B = 0$,	$C = -1$	$\tanh(\xi)$, $\coth(\xi)$
$A = \frac{1}{2}$,	$B = 0$,	$C = \frac{1}{2}$	$\sec(\xi) + \tan(\xi)$, $\csc(\xi) - \cot(\xi)$
$A = -\frac{1}{2}$,	$B = 0$,	$C = -\frac{1}{2}$	$\sec(\xi) - \tan(\xi)$, $\csc(\xi) + \cot(\xi)$
$A = 1\ (-1)$,	$B = 0$,	$C = 1\ (-1)$	$\tan(\xi)$, $\cot(\xi)$
$A = 0$,	$B = 0$,	$C \neq 0$	$-\frac{1}{C\xi + d}$, (d is arbitrary constant)
$A \neq 0$,	$B = 0$,	$C = 0$	Aξ
$A \neq 0$,	$B \neq 0$,	$C = 0$	$\frac{-A + \exp(B\xi)}{B}$

3. Exact solutions of Bogoyavlenskii equation

In this section, we will make use of the modified F-expansion method and symbolic computation to find the exact solutions of Bogoyavlenskii equation.

Let us consider the Bogoyavlenskii equation (1). We seek the traveling wave solution for Eq.(1) in the form

$$u(x, y, t) = U(\xi)\ , v(x, y, t) = V(\xi), \quad \xi = x + y - \lambda t. \tag{7}$$

By using this transformation into Eq.(1), we have

$$-4\lambda U' + U''' - 4U^2U' - 4U'V = 0\ , \tag{8a}$$
$$UU' = V'\ . \tag{8b}$$

First, we integrate Eq.(8b) w.r.t. ξ, and letting the integration constant equal to zero for simplicity, and substitute in Eq.(8a). Next integrating Eq.(8a) w.r.t ξ, we obtain

$$U'' - 2U^3 - 4\lambda U = 0\ , \tag{9a}$$
$$V = \frac{1}{2}U^2. \tag{9b}$$

Considering the homogeneous balance, the highest order derivative term U'' with the nonlinear term U^3 , we get $N = 1$. Hence for $N = 1$, the form of the solution of the NODE in Eq.(9a) using the Eq.(5) can be expressed as

$$U(\xi) = a_0 + a_1 F(\xi) + \frac{b_1}{F(\xi)} \tag{10}$$

where a_0, a_1, b_1 are constants to be determined later.

Substituting Eq.(10) into Eq.(9a) and using Eq.(6), the left-hand side of Eq.(9a) can be converted into the finite series in powers of $F(\xi)$.

Collecting all the terms with the same powers of $F(\xi)$ and equating all of the obtained coefficients of each power of $F(\xi)$ to zero, we acquire the following system of nonlinear algebraic equations for the unknown constants a_0, a_1, b_1, λ.

$$\begin{aligned}
F^{-3} &: \quad 2b_1A^2 - 2b_1^3 = 0,\\
F^{-2} &: \quad 3b_1AB - 6a_0b_1^2 = 0,\\
F^{-1} &: \quad b_1B^2 + 2b_1AC - 4b_1\lambda - 6a_0^2b_1 - 6a_1b_1^2 = 0,\\
F^{0} &: \quad a_1AB + b_1BC - 12a_0a_1b_1 - 2a_0^3 - 4a_0\lambda = 0,\\
F^{1} &: \quad a_1B^2 + 2a_1AC - 4a_1\lambda - 6a_0^2a_1 - 6a_1^2b_1 = 0,\\
F^{2} &: \quad 3a_1BC - 6a_0a_1^2 = 0,\\
F^{3} &: \quad 2a_1C^2 - 2a_1^3 = 0.
\end{aligned} \tag{11}$$

Solving this algebraic system (11) by Matlab, we get many solution sets as follows:

Case-1: When $B = 0$, we have
Set$_{11}$: $a_0 = 0$, $a_1 = 0$, $b_1 = \pm A$, $\lambda = AC/2$
Set$_{12}$: $a_0 = 0$, $a_1 = \pm C$, $b_1 = \pm A$, $\lambda = -AC$
Set$_{13}$: $a_0 = 0$, $a_1 = \pm C$, $b_1 = \mp A$, $\lambda = 2AC$

Case-2: When $A = 0$, we have
Set$_{21}$: $a_0 = \pm B/2$, $a_1 = \pm C$, $b_1 = 0$, $\lambda = -B^2/8$

Case-3: when $A = 0$, $B = 0$, we have
Set$_{31}$: $a_0 = 0$, $a_1 = \pm C$, $b_1 = 0$, $\lambda = 0$

Many exact solutions of Eq.(1) can be obtained by substituting the various values of A, B, and C and the function F(ξ) in Table 1. The exact solutions of the Bogoyavlenskii equation are in the form of hyperbolic functions, trigonometric functions, and rational functions.

Now by substituting the values of a_0, a_1, b_1 , and λ corresponding to the parameter values in Case-1, Case-2, and Case-3 and using Table 1., we give some of the exact solutions of the BE:

Solution Set-1:
When $\boldsymbol{A = 1}$, and $\boldsymbol{C = -1}$, from Table 1., then $F(\xi) = \tanh(\xi)$ or $F(\xi) = \coth(\xi)$.
By Case-1, we get the following hyperbolic function exact solutions:

From Set_{11}:
$u_{1a}(x,y,t) = U_{1a}(\xi) = \pm\frac{1}{\tanh(\xi)}$,
$v_{1a}(x,y,t) = V_{1a}(\xi) = \frac{1}{2}\left(\frac{1}{\tanh(\xi)}\right)^2$
or
$u_{1b}(x,y,t) = U_{1b}(\xi) = \pm\frac{1}{\coth(\xi)}$,
$v_{1b}(x,y,t) = V_{1b}(\xi) = \frac{1}{2}\left(\frac{1}{\coth(\xi)}\right)^2$
where $\xi = x + y + \frac{1}{2}t$.

From Set_{12}:
$u_{2a}(x,y,t) = U_{2a}(\xi) = \mp\tanh(\xi) \pm \frac{1}{\tanh(\xi)}$,

$$v_{2a}(x,y,t) = V_{2a}(\xi) = \frac{1}{2}\left(\mp\tanh(\xi) \pm \frac{1}{\tanh(\xi)}\right)^2$$

or
$u_{2b}(x,y,t) = U_{2a}(\xi) = \mp\coth(\xi) \pm \frac{1}{\coth(\xi)}$,

$$v_{2b}(x,y,t) = V_{2b}(\xi) = \frac{1}{2}\left(\mp\coth(\xi) \pm \frac{1}{\coth(\xi)}\right)^2$$

where $\xi = x + y - t$.

When $\boldsymbol{A} = \mathbf{1/2}$, and $\boldsymbol{C} = -\mathbf{1/2}$, from Table 1., then $F(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi)$ or $F(\xi) = \tanh(\xi) \pm i\operatorname{sech}(\xi)$.

By Case-1, we get the following hyperbolic function exact solutions:
From Set_{11}: by using $F(\xi) = \coth(\xi) \pm \operatorname{csch}(\xi)$,
$u_{3a}(x,y,t) = U_{3a}(\xi) = \pm\frac{1}{2(\coth(\xi)\pm\operatorname{csch}(\xi))}$,
$v_{3a}(x,y,t) = V_{3a}(\xi) = \frac{1}{2}\left(\pm\frac{1}{2(\coth(\xi)\pm\operatorname{csch}(\xi))}\right)^2$
where $\xi = x + y + \frac{1}{8}t$.
From Set_{12}:
$u_{4a}(x,y,t) = U_{4a}(\xi) = \mp\frac{1}{2}(\coth(\xi) \pm \operatorname{csch}(\xi)) \pm \frac{1}{2(\coth(\xi)\pm\operatorname{csch}(\xi))}$,

$$v_{4a}(x,y,t) = V_{4a}(\xi) = \frac{1}{2}\left(U_{4a}(\xi)\right)^2$$

where $\xi = x + y - \frac{1}{4}t$.
In a similar manner, by using $F(\xi) = \tanh(\xi) \pm i\operatorname{sech}(\xi)$, one can easily obtained $u_{3b}(x,y,t)$, $v_{3b}(x,y,t)$, and $u_{4b}(x,y,t)$, $v_{4b}(x,y,t)$ solutions.

When $\boldsymbol{A} = \mathbf{1/2}$, and $\boldsymbol{C} = \mathbf{1/2}$, from Table 1., then $F(\xi) = \sec(\xi) + \tan(\xi)$ or $F(\xi) = \csc(\xi) - \cot(\xi)$.

By Case-1, we get the following trigonometric function exact solutions:
From Set_{11}:

$u_{5a}(x,y,t) = U_{5a}(\xi) = \pm\frac{1}{2(\sec(\xi)+\tan(\xi))}$,

$v_{5a}(x,y,t) = V_{5a}(\xi) = \frac{1}{2}\left(\pm\frac{1}{2(\sec(\xi)+\tan(\xi))}\right)^2$

or

$u_{5b}(x,y,t) = U_{5b}(\xi) = \pm\frac{1}{2(csc(\xi)-cot(\xi))}$,

$v_{5b}(x,y,t) = V_{5b}(\xi) = \frac{1}{2}\left(\pm\frac{1}{2(\csc(\xi)-\cot(\xi))}\right)^2$

where $\xi = x + y - \frac{1}{8}t$.

From Set_{12}:

$u_{6a}(x,y,t) = U_{6a}(\xi) = \pm\frac{1}{2}(\sec(\xi)+\tan(\xi)) \pm \frac{1}{2\,(\sec(\xi)+\tan(\xi))}$,

$v_{6a}(x,y,t) = V_{6a}(\xi) = \frac{1}{2}\left(U_{6a}(\xi)\right)^2$

or

$u_{6b}(x,y,t) = U_{6b}(\xi) = \pm\frac{1}{2}\left(\csc(\xi)-\cot(\xi)\right) \pm \frac{1}{2\,(\csc(\xi)+\cot(\xi))}$,

$v_{6b}(x,y,t) = V_{6b}(\xi) = \frac{1}{2}\left(U_{6b}(\xi)\right)^2$

where $\xi = x + y + \frac{1}{4}t$.

Solution Set-2:

When $\boldsymbol{B} = \mathbf{1}$, and $\boldsymbol{C} = -\mathbf{1}$, from Table 1., then $F(\xi) = \frac{1}{2} + \frac{1}{2}\tanh(\frac{1}{2}\xi)$.

By Case-2, we get the following hyperbolic function exact solutions:

From Set_{21}:

$u_{7a}(x,y,t) = U_{7a}(\xi) = \pm\frac{1}{2}\tanh(\xi/2)$,

$v_{7a}(x,y,t) = V_{7a}(\xi) = \frac{1}{8}\tanh^2(\xi/2)$

where $\xi = x + y + \frac{1}{8}t$.

When $\boldsymbol{B} = -\mathbf{1}$, and $\boldsymbol{C} = \mathbf{1}$, from Table 1., then $F(\xi) = \frac{1}{2} - \frac{1}{2}\coth(\frac{1}{2}\xi)$.

By Case-2, we get the following hyperbolic function exact solutions:

From Set_{21}:

$u_{8a}(x,y,t) = U_{8a}(\xi) = \pm\frac{1}{2}\coth(\xi/2)$,

$v_{8a}(x,y,t) = V_{8a}(\xi) = \frac{1}{8}\coth^2(\xi/2)$

where $\xi = x + y + \frac{1}{8}t$.

In a similar manner, one can easily derive some more exact solutions for the Bogoyavlenskii equation.

4. Conclusion

In this study, we successfully applied the modified F-expansion method to derive exact solutions for the Bogoyavlenskii equation. We presented these solutions in terms of both hyperbolic and trigonometric functions, demonstrating the versatility and effectiveness of the method. To the best of our knowledge, the modified F-expansion method is used for the first time in the literature to obtain exact solutions of the Bogoyavlenskii equation. Our results confirm that the modified F-expansion method is a robust and reliable tool for obtaining exact solutions to nonlinear partial differential equations (PDEs) in mathematical physics.

The method has proven to be not only powerful but also the solution procedure is practical and straightforward, making it a valuable tool for generating new solutions. All solutions were computed using MATLAB, and it has been verified that all obtained solutions satisfy the original equation. When comparing our findings with existing literature, we identified that some solutions are novel while others align with previously known results. The method yields a variety of solutions, including hyperbolic functions, rational functions, and trigonometric functions, depending on the coefficients A, B and C in Eq.(6). These solutions are important to help us understand some physical phenomena since they have rich spatial structures.

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