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Meromorphic Function of Fuzzy Complex Variables

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Abstaract − The fuzzy complex set is a fuzzy set whose values lies in the unit circle $|z| \leq 1$ in the complex plane. The Nevanlinna characteristic function playes an important role in the theory of entire and meromorphic function. In this paper we introduce the notion of fuzzy to the Nevanlinna theory and investigate some important properties of Nevanlinna characteristic function of fuzzy complex variables.

Keywords − *Fuzzy complex variavles, fuzzy complex numbers, Nevanlinna characteristic of fuzzy variables.*

1 Introduction.

In the year 1965 Zadeh [8] proposed the fuzzy sets. Many thousand of papers, articles have published in different journals. The idea of fuzzy sets based on real number system. Buckley [1] and [2] introduced the idea of fuzzy complex sets in the year 1987. In Buckley's definition, the representation of fuzzy complex number in the polar form is quite unstable. In the year 2003, Ramot et al. [4] and [5] proposed a new concept of defining a fuzzy complex set.

In this newly defined fuzzy complex number, as the phase and membership grade are present, the Fuzzy complex number(FCN) takes the well-known (wavelike) property of complex numbers. This wavelike property distinguishes the fuzzy complex sets with the traditional fuzzy sets. Many continuations of work of Ramot et al. [4] has been studied by various authors ([5], [9], etc.).

Now we are trying to establish the Nevanlinna Characteristic function in FCN and also investigate some useful properties.

2 Basic Concepts on Fuzzy Complex Numbers

Although this paper is in the line different as that of Buckley [1], we recall the definition of FCN introduced by Buckley.

Definition 2.1. [8] Let X be an universal set. Then the fuzzy subset A of X is defined by its membership function $\mu_A(x) : X \rightarrow [0,1]$ which will assign a real number $\mu_A(x)$ in the interval $[0,1]$ to each element $x \in X$, where the value of $\mu_A(x)$ shows the grade of membership of x in A.

We are not providing the basic definitions and notations such as $\alpha - cut$ or weak $\alpha - cut$ of fuzzy sets as they are available in [6] and [7].

We are now giving two basic definitions introduced by Buckley in [1].

Definition 2.2. Fuzzy complex set: Let $\mathbb{C}$ be a complex field. Then the fuzzy subset $\tilde{Z}$ of $\mathbb{C}$ is defined by the membership function $\mu_{\tilde{z}}(z) : \mathbb{C} \rightarrow [0,1]$.

Definition 2.3. $\tilde{z}$ is a fuzzy complex number if and only if

(i) $\alpha_{\tilde{z}}(z)$ is continuous ,

(ii) ${}^{\alpha-}\mu_{\tilde{z}}(z)$ is open, bounded, connected and simply connected for $0 \leq \alpha < 1$,

(iii) ${}^{1+}\mu_{\tilde{z}}(z)$ is non empty.

We now present the definition introduced by Ramot et al ([4]) .

Definition 2.4. A fuzzy complex set $\mathbb{C}_\mu$, defined on a universe or discourse U is characterized by a membership function $\mu_{\mathbb{C}}(z)$ that assigns any $z \in U$ a complex valued grade of membership in $\mathbb{C}_\mu$, i.e ,

$\mu_{\mathbb{C}}(z) = r_\mu(z) . \exp\left(i \arg_\mu(z)\right)$, where $r_\mu(z)$ and $\arg_\mu(z)$ are both real valued functions and $i = \sqrt{(-1)}$.

Here $\arg_\mu(z)$ is the principal argument and $0 \leq r_\mu(z) \leq 1$.

Also, $Arg_\mu(z) = \arg_\mu(z) + 2k\pi,\ k = 0,\ \pm 1,\ \pm 2, \cdots$

This $\arg_\mu(z)$ gives $\mu_{\mathbb{C}}(z)$ a wavelike property. Clearly $\mu_{\mathbb{C}}(z)$ lies on the unit circle centered at origin in the complex plane.

In this paper we use the notation $\phi_\mu(z)$ for the argument of the fuzzy complex numbers.

Definition 2.5. [9] Let A and B be two fuzzy complex sets on U such that $\mu_A(z) = r_A(z) . \exp(i\phi_A(z))$ and $\mu_B(z) = r_B(z) . \exp(i\phi_B(z))$, then

$$\begin{aligned} (i)\ \ \mu_{A\cup B}(z) &= r_{A\cup B}(z) . \exp i\left(\phi_{A\cup B}(z)\right) \\ &= \max\left(r_A(z), r_B(z)\right) . \exp\left(i \max\left(\phi_A(z), \phi_B(z)\right)\right) \\ (ii)\ \ \mu_{A\cap B}(z) &= r_{A\cap B}(z) . \exp i\left(\phi_{A\cap B}(z)\right) \\ &= \min\left(r_A(z), r_B(z)\right) . \exp\left(i \min\left(\phi_A(z), \phi_B(z)\right)\right) . \end{aligned} \tag{1}$$

Definition 2.6. [9] Let $F_{\mathbb{C}}(U)$ be the set of all fuzzy complex sets on U. Let $C_\alpha \in F_{\mathbb{C}}(U),\ \alpha \in I$ and $\mu_{C_\alpha}(z) = r_{C_\alpha}(z) . \exp(i\phi_{C_\alpha}(z))$ then $\underset{\alpha\in I}{\cup} C_\alpha \in F_{\mathbb{C}}(U)$ and its membership function is

$$\mu_{\underset{\alpha\in I}{\cup} C_\alpha}(z) = \sup_{\alpha\in I}\left(r_{C_\alpha}(z)\right) . \exp\left(i \sup_{\alpha\in I}\left(\phi_{C_\alpha}(z)\right)\right) . \tag{2}$$

Definition 2.7. [9] Let C_μ be a fuzzy complex set on U and

$$\mu_C(z) = r_\mu(z) \,.\, \exp(i\phi_\mu(z))$$

then the fuzzy complex complement of C is denoted by $\bar{C}$ and is specified by the membership function

$$\begin{aligned} \mu_{\bar{C}}(z) &= r_{\bar{C}}(z) \,.\, \exp(i\phi_{\bar{\mu}}(z)) \\ &= (1 - r_C(z)) \,.\, \exp\{i(2\pi - \phi_\mu(z))\} \;. \end{aligned} \tag{3}$$

Definition 2.8. Logarithm of a fuzzy complex number: The logarithm of a fuzzy complex number is defined as $\log z_\mu = \log(r_\mu) + i\phi_\mu(z)\,,\; r_\mu \in (0,1]$ or equivalently $\log(\mu_C(z)) = \log(r_\mu(z)) + i\phi_\mu(z)\,,\; 0 < r_\mu(z) \le 1$.

Definition 2.9. Let A and B be two fuzzy complex sets in the universe U , and $\mu_A(z) = r_\mu(z) \,.\, \exp(i\phi_\mu(z))$ abd $\mu_B(z) = r_\mu(z) \,.\, \exp(i\phi_\mu(z))$ be the membership functions defined on it. Then the fuzzy product of A and B is defined by

$$\mu_{A\circ B}(z) = (r_{\mu_A}(z) \,.\, r_{\mu_B}(z)) \,.\, \exp\left[i\left\{2\pi \frac{\phi_{\mu_A}(z)}{\phi_{\mu_B}(z)}\right\}\right] \;.$$

Definition 2.10. Positive logarithm: The positive logarithm is defined as

$$\begin{aligned} \log^+(x) &= \log x,\; if\; x \ge 1 \\ &= 0,\; if\; 0 \le x < 1 \;. \end{aligned}$$

Remark 2.1. With the above definition it can easily be verified that

$$\log x = \log^+ x - \log^+\left(\frac{1}{x}\right) \;.$$

Now we present some basic definitions of the theory of entire and meromorphic function.

Definition 2.11. A complex valued function which has no singularities other than poles in the finite complex plane is known as meromorphic function.

Definition 2.12. The proximity function: Let $f(z)$ be meromorphic on $|z| \le R$, $(0 < R < \infty)$. Then the proximity function is defined by

$$m(R, f) = \frac{1}{2\pi} \int_0^{2\pi} \log^+ \left| f\left(Re^{i\phi}\right) \right| d\phi \;.$$

Definition 2.13. The counting function: Let $f(z)$ be a non constant meromorphic function on the complex plane. For any complex number $'a'$ we denote by $n(r,a) = n(r,a,f)$, the number of zeros of the equation $f(z) = a$ (counting multiplicities).

The function $N(r,a) = \int_0^r \frac{n(t,a) - n(0,a)}{t} dt + n(0,a)\log r$ is called the counting function. By $n(r,\infty,f)$ or simply $n(r,f)$ we mean the poles of the function $f(z)$ in $|z| \le r$.

Definition 2.14. Nevanlinna characteristic function: The sum of the proximity function and the counting function is denoted by $T(r,f)$. Rolf Nevanlinna defined the characteristic function as $T(r,f) = N(r,f) + m(r,f) + O(1)$.

We do not explain the basic definition and notation of the Nevanlinna theory as they are available in [3].

In the line of Nevanlinna it can easily be verifed that

$$T(r_\mu, f) = N(r_\mu, f) + m(r_\mu, f) + O(1) \;. \tag{4}$$

3 Known Results

We now present the well-known result of the Nevanlinna theory.

Theorem 3.1. $T(r, f)$ is an increasing function of r and convex function of $\log r$.

We are not providing the proof as it is available in [3].

4 Main Results

We now discuss our main results of this paper.

Theorem 4.1. $T(r_\mu, f)$ is an increasing function of r_μ and convex function of $\log r_\mu$.

Proof. The well known Jensen's formula is given by

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|f\left(R.e^{i\phi_\mu}\right)\right| d\phi_\mu + \sum_{\alpha=1}^{M} \log\frac{|a_\alpha|}{R} - \sum_{\alpha=1}^{N} \log\frac{|a_\beta|}{R},$$

provided $f(0) \neq 0$ *or* ∞ , $a_\alpha\,(\alpha = 1, 2, \cdots, M)$ and $b_\beta\,(\beta = 1, 2, \cdots N)$ are the zeros and poles of $f(z)$ in $|z| < r$.

Now by Jensen's theorem with $R = 1$ and $f(z_\mu) = a_\mu - z_\mu$, we get

$$\frac{1}{2\pi}\int_0^{2\pi} \log^+ \left|R.e^{i\phi_\mu} - a_\mu\right| d\phi_\mu = \log^+|a_\mu|\,;\ for\ all\ a_\mu \in \mathbb{C}_\mu\ . \tag{5}$$

Now applying Jensen's formula to the function $f(z_\mu) - e^{i\theta_\mu}$, we get

$$\begin{aligned}\log\left|f(0) - e^{i\theta_\mu}\right| &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(r_\mu.e^{i\phi_\mu}\right) - e^{i\theta_\mu}\right| d\phi_\mu \\ &\quad -N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) + N\left(r_\mu, f - e^{i\theta_\mu}\right)\ .\end{aligned}$$

On integrating from 0 to 2π, one may get

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta_\mu}\right| d\theta_\mu &= \frac{1}{2\pi}\int_0^{2\pi} \log\left|f\left(r_\mu.e^{i\phi_\mu}\right) - e^{i\theta_\mu}\right| d\phi_\mu \\ &\quad - \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, f - e^{i\theta_\mu}\right) d\theta_\mu\ .\end{aligned}$$

Now replacing a_μ by $f\left(r.e^{i\phi_\mu}\right)$, we get from 5 that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} \log\left|f(0) - e^{i\theta_\mu}\right| d\theta_\mu &= \frac{1}{2\pi}\int_0^{2\pi} \log^+\left|f\left(r_\mu.e^{i\phi_\mu}\right)\right| d\phi_\mu \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + N(r_\mu, f) \\ &= m(r_\mu, f) + N(r_\mu, f) \\ &\quad -\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu \\ &= T(r_\mu, f) - \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu\ .\end{aligned}$$

Thus we get

$$T\left(r_\mu, f\right) = \frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \log^+\left|f\left(0\right)\right| \tag{6}$$

This is the fuzzy version of Cartan's identity. Now differentiating 6 with respect to r_μ we get

$$\begin{aligned}\frac{d\left(T\left(r_\mu, f\right)\right)}{dr_\mu} &= \frac{d}{dr_\mu}\left\{\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu + \log^+\left|f\left(0\right)\right|\right\} \\ &= \frac{d}{dr_\mu}\left\{\frac{1}{2\pi}\int_0^{2\pi} N\left(r_\mu, \frac{1}{f - e^{i\theta_\mu}}\right) d\theta_\mu\right\}\end{aligned}$$

Hence

$$\frac{d\left(T\left(r_\mu, f\right)\right)}{dr_\mu} = \frac{d}{dr_\mu}\{\frac{1}{2\pi}\int_0^{2\pi}(\frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right) - n\left(0, e^{i\theta_\mu}\right)}{t}dt + n\left(0, e^{i\theta_\mu}\right).\log r_\mu)d\theta_\mu\}$$

That is

$$\begin{aligned}\frac{d\left(T\left(r_\mu, f\right)\right)}{dr_\mu} &= \frac{d}{dr_\mu}\left[\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right)}{t}dt\right\}\right] \\ &= \frac{1}{2\pi}\int_0^{r_\mu}\frac{n\left(t, e^{i\theta_\mu}\right)}{t}dt\ .\end{aligned}$$

Similarly differentiating 6 with respect to $\log r_\mu$, we get

$$\frac{d\left(T\left(r_\mu, f\right)\right)}{d\left(\log r_\mu\right)} = \frac{1}{2\pi}\int_0^{r_\mu} n\left(t, e^{i\theta_\mu}\right) dt\ .$$

Now $n\left(t, e^{i\theta_\mu}\right)$ is counting function and hence non negative and non decreasing. That is $\frac{d(T(r_\mu,f))}{d(\log r_\mu)} \geq 0$ and $\frac{d(T(r_\mu,f))}{dr_\mu} \geq 0$ for all $r_\mu \in \mathbb{C}_\mu$.

Therefore $T\left(r_\mu, f\right)$ is a convex function of $\log r_\mu$ and an increasing function of r_μ. □

Theorem 4.2. If A and B be two fuzzy complex sets on the universe U and $\mu_A\left(z\right) = r_A\left(z\right).e^{i\phi_{\mu_A}(z)}$ and $\mu_B\left(z\right) = r_B\left(z\right).e^{i\phi_{\mu_B}(z)}$ then $T\left(r_{\mu_{A\cup B}}, f\right) \geq T\left(r_{\mu_A}, f\right)$ and $T\left(r_{\mu_{A\cup B}}, f\right) \geq T\left(r_{\mu_B}, f\right)$.

Proof. By theorem 4.1, it is obvious that if $r_{\mu_A} \geq r_{\mu_B}$, then $T\left(r_{\mu_A}, f\right) \geq T\left(r_{\mu_B}, f\right)$. Also by definition 2.5 we have

$$\begin{aligned}\mu_{A\cup B}\left(z\right) &= r_{\mu_{A\cup B}}\left(z\right)e^{i\left(\phi_{\mu_{A\cup B}}(z)\right)} \\ &= \max\left(r_{\mu_A}\left(z\right), r_{\mu_B}\left(z\right)\right).e^{\left(i\max\left(\phi_{\mu_A}(z),\phi_{\mu_B}(z)\right)\right)}\ .\end{aligned}$$

Therefore

$$\begin{aligned} T\left(r_{\mu_{A\cup B}}, f\right) &= T\left(\max\left(r_{\mu_A}(z), r_{\mu_B}(z)\right), f\right) \\ &\geq T\left(r_{\mu_A}, f\right) \ . \end{aligned}$$

Similarly

$$T\left(r_{\mu_{A\cup B}}, f\right) \geq T\left(r_{\mu_B}, f\right)$$

□

Example 4.1. To give an example we use the popular convention notation for fuzzy sets. When the universe U is infinite and continuous, the fuzzy set A in the universe U can be expressed as

$$A = \int \frac{\mu_A(z)}{z}, \ z \in U \ .$$

In this notation the integral sign is not the integral used in calculus or algebraic integral, but a set union notation for continuous variable.

When the discourse is finite or discrete, the fuzzy set can be expressed as

$$A = \frac{\mu_A(z_1)}{z_1} + \frac{\mu_A(z_2)}{z_2} + \frac{\mu_A(z_3)}{z_3} + \cdots + \frac{\mu_A(z_n)}{z_n} \ .$$

In both notation the fraction or horizontal bar is not a quotient but a delimiter. In both notation the numerator represents the membership value in the set A associated with the element of the universe indicated in the denominator.

Now consider

$$A = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z}, \ z \geq 1, \ B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z}, \ z \geq 1,$$

and $f(z) = \exp z$. Now

$$A \cup B = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z} \ .$$

Then

$$\begin{aligned} T\left(r_{\mu_A}, f\right) &= \frac{r_\mu}{\pi} = T\left(r_{\mu_{A\cup B}}, f\right), \\ T\left(r_{\mu_B}, f\right) &= \frac{r_\mu}{2\pi}, \ (0 \leq r_\mu \leq 1) \ . \end{aligned}$$

Thus

$$\begin{aligned} T\left(r_{\mu_{A\cup B}}, f\right) &\geq T\left(r_{\mu_A}, f\right) \ \textit{and} \\ T\left(r_{\mu_{A\cup B}}, f\right) &\geq T\left(r_{\mu_B}, f\right) \ . \end{aligned}$$

and the result follows .

Theorem 4.3. Let A and B be two fuzzy complex sets on the universe U and $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$ and $\mu_B(z) = r_B(z).e^{i\phi_{\mu_B}(z)}$. Then $T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_A}, f\right)$ and $T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right)$.

Proof. By theorem 4.1,it is obvious that if $r_{\mu_A} \geq r_{\mu_B}$ then $T(r_{\mu_A}, f) \geq T(r_{\mu_B}, f)$. Also by definition 2.5 we have

$$\mu_{A\cap B}(z) = r_{\mu_{A\cap B}}(z).e^{i\left(\phi_{\mu_{A\cap B}}(z)\right)}$$
$$= \min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right).e^{\left(i\min\left(\phi_{\mu_A}(z),\phi_{\mu_B}(z)\right)\right)} .$$

Therefore

$$\begin{aligned} T\left(r_{\mu_{A\cap B}}, f\right) &= T\left(\min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right), f\right) \\ &\leq T\left(r_{\mu_A}, f\right) . \end{aligned}$$

Similarly

$$T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right) .$$

□

Example 4.2. We go with the same sets as in example 4.1 .

Let,

$$A = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z},\ z \geq 1,\ B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ z \geq 1,$$

and $f(z) = \exp z$. Now,

$$A \cap B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z} .$$

Then

$$T\left(r_{\mu_B}, f\right) = \frac{r_\mu}{2\pi} = T\left(r_{\mu_{A\cap B}}, f\right),\ T\left(r_{\mu_A}, f\right) = \frac{r_\mu}{\pi},\ \ (0 \leq r_\mu \leq 1) .$$

Thus

$$T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_B}, f\right) \ and\ T\left(r_{\mu_{A\cap B}}, f\right) \leq T\left(r_{\mu_A}, f\right) ,$$

and the result follows .

Theorem 4.4. For any three fuzzy complex sets A, B and C on the universe U , $T(r_\mu, f)$ follows the associativity property with respect to the union of fuzzy sets, *i.e,*

$$T\left(r_{\mu_{(A\cup B)\cup C}}, f\right) = T\left(r_{\mu_{A\cup(B\cup C)}}, f\right) .$$

Proof. As

$$\begin{aligned} \mu_{(A\cup B)\cup C}(z) &= \max\left\{r_{\mu_{(A\cup B)}}(z), r_{\mu_C}(z)\right\}.e^{i\left\{\max\left(\phi_{\mu_{A\cup B}}(z),\phi_{\mu_C}(z)\right)\right\}} \\ &= \max\left\{r_{\mu_A}(z), r_{\mu_B}(z), r_{\mu_C}(z)\right\}.e^{i\left\{\max\left(\phi_{\mu_A}(z),\phi_{\mu_B}(z),\phi_{\mu_C}(z)\right)\right\}} \end{aligned}$$

and $T(r_\mu, f)$ is an increasing function, therefore

$$\begin{aligned} T\left(r_{\mu_{(A\cup B)\cup C}}, f\right) &= T\left(\max\left\{r_{\mu_{(A\cup B)}}(z), r_{\mu_C}(z)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), r_{\mu_B}(z), r_{\mu_C}(z)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), \left(r_{\mu_B}(z), r_{\mu_C}(z)\right)\right\}, f\right) \\ &= T\left(\max\left\{r_{\mu_A}(z), r_{\mu_{B\cup C}}(z)\right\}, f\right) \\ &= T\left(r_{\mu_{A\cup(B\cup C)}}, f\right) . \end{aligned}$$

□

Example 4.3. Let,

$$A = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z},\ B = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ C = \int \frac{\frac{1}{3z}e^{i(3z\pi)}}{z},\ z \geq 1$$

and $f(z) = \exp z$,then

$$\begin{aligned} A \cup B &= \int \frac{\frac{1}{z}e^{i(z\pi)}}{z} \\ B \cup C &= \int \left(\frac{\frac{1}{2z}e^{i(2z\pi)}}{z} + \frac{\frac{1}{3z}e^{i(3z\pi)}}{z} \right) \\ A \cup B \cup C &= \int \frac{\frac{1}{e^z}e^{i(z\pi)}}{z} = (A \cup B) \cup C = A \cup (B \cup C), \end{aligned}$$

and the result follows trivially.

Theorem 4.5. For any three fuzzy complex sets A, B and C on the universe U , $T(r_\mu, f)$ follows the associativity property with respect to the intersection of fuzzy sets, *i.e,*

$$T\left(r_{\mu_{(A\cap B)\cap C}}, f\right) = T\left(r_{\mu_{A\cap(B\cap C)}}, f\right) .$$

Proof. The proof is similar as the proof of previous theorem. So we left the proof. □

Example 4.4. Let,

$$A = \int \frac{\frac{1}{2z}e^{i(2z\pi)}}{z},\ B = \int \frac{\frac{1}{3z}e^{i(3z\pi)}}{z},\ C = \int \frac{\frac{1}{4z}e^{i(4z\pi)}}{z},\ z \geq 1$$

and $f(z) = \exp z$, therefore

$$\begin{aligned} A \cap B &= \int \frac{\frac{1}{6z}e^{i(6z\pi)}}{z},\ B \cap C = \int \frac{\frac{1}{12z}e^{i(12z\pi)}}{z}, \\ A \cap B \cap C &= \int \frac{\frac{1}{12z}e^{i(12z\pi)}}{z} = (A \cap B) \cap C = A \cap (B \cap C), \end{aligned}$$

and the result follows trivially.

Theorem 4.6. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha \in F(U)$, $\alpha \in I$. Also let $\mu_{A_\alpha}(z) = r_{\mu_{A_\alpha}}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$\begin{aligned} (i)\ T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}}, f\right) &= T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\} \\ (ii)\ T\left(r_{\mu_{A_1\cap A_2\cap\cdots\cap A_n}}, f\right) &= T\left(\inf_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \leq \inf_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\} \end{aligned}$$

Proof. (i) Clearly, since

$$\mu_{A_1\cup A_2\cup\cdots\cup A_n}(z) = \left[\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}(z)\right\}\right].\exp\left\{i\left(\sup_{\alpha\in[1,n]}\left\{\phi_{\mu_{A_\alpha}}(z)\right\}\right)\right\}$$

therefore, we have

$$\begin{aligned} T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}}, f\right) &= T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \\ &\geq T\left(r_{\mu_{A_\alpha}}, f\right),\ for\ all\ \alpha\in I\ . \end{aligned}$$

Thus

$$T\left(r_{\mu_{A_1\cup A_2\cup\cdots\cup A_n}}, f\right) = T\left(\sup_{\alpha\in[1,n]}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in[1,n]}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

(ii) Similarly like above. □

Example 4.5. Let

$$A_\alpha = \int \frac{\frac{1}{\alpha z}e^{i(\alpha z\pi)}}{z},\ z\geq 1,\ \alpha\in\mathbb{N}$$

and $f(z) = z$,then

$$\bigcup_{\alpha=1}^{n} A_\alpha = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z}\ and\ \bigcap_{\alpha=1}^{n} = \int \frac{\frac{1}{\beta z}e^{i(z\beta\pi)}}{z},$$

where β is the $\operatorname{lcm}(1, 2, \cdots n)$. Thus the results follows.

Corollary 4.1. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha \in F(U)$, $\alpha \in I$. Also let $\mu_{A_\alpha}(z) = r_{A_\alpha}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$T\left(r_{\mu_{\cup A_\alpha}}, f\right) = T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

Proof. Clearly, since $\mu_{\cup A_\alpha}(z) = \left[\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}(z)\right\}\right].e^{\ i\ \sup_{\alpha\in I}\left\{\phi_{\mu_{A_\alpha}}(z)\right\}}$, therefore, we have

$$\begin{aligned} T\left(r_{\mu_{\cup A_\alpha}}, f\right) &= T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \\ &\geq T\left(r_{\mu_{A_\alpha}}, f\right),\ for\ all\ \alpha\in I\ . \end{aligned}$$

Thus

$$T\left(r_{\mu_{\cup A_\alpha}}, f\right) = T\left(\sup_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \geq \sup_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\}$$

□

Example 4.6. Let

$$A_\alpha = \int \frac{\frac{1}{\alpha z}e^{i(\alpha z\pi)}}{z},\ z\geq 1,\ \alpha\in\mathbb{N}$$

and $f(z) = \exp z$,then

$$\cup A_\alpha = \int \frac{\frac{1}{z}e^{i(z\pi)}}{z},$$

and the result follows trivially.

The similar expression cannot be obtained for the intersection of fuzzy sets. which can be followed from the following corollary.

Corollary 4.2. Let $F(U)$ be the set of all fuzzy complex sets on the universe U, and $A_\alpha \in F(U),\ \alpha \in I$. Also let $\mu_{A_\alpha}(z) = r_{A_\alpha}(z).e^{i\phi_{\mu_{A_\alpha}}(z)}$ be its membership function. Then

$$T\left(r_{\mu_{\cap A_\alpha}}, f\right) \neq T\left(\inf_{\alpha\in I}\left\{r_{\mu_{A_\alpha}}\right\}, f\right) \leq \inf_{\alpha\in I}\left\{T\left(r_{\mu_{A_\alpha}}, f\right)\right\} ,$$

i.e., no such relationship can be obtained.

Proof. We give a counterexample to establish the result.
for this consider the following fuzzy sets:

$$A_\alpha = \int \frac{\frac{1}{\alpha z}e^{i(\alpha z\pi)}}{z},\ z \geq 1,\ \alpha \in \mathbb{N} .$$

Then $\bigcap_{\alpha=1}^{\infty} A_\alpha = \{0\}$, so no such $T(r, f)$ can be obtained.
Hence the result follows. □

Now we present a theorem with the complement of the fuzzy complex numbers.

Theorem 4.7. If A and B be two fuzzy complex sets in the universe U. Then

$$T\left(r_{\mu_{A\bar{\cap}B}}, f\right) = T\left(r_{\mu_{\bar{A}\cup\bar{B}}}, f\right) .$$

Proof. Clearly $\mu_{A\bar{\cap}B}(z) = r_{\mu_{A\bar{\cap}B}}(z).\exp i\left\{\phi_{\mu_{A\bar{\cap}B}}(z)\right\}$, we have

$$\begin{aligned}
\mu_{A\bar{\cap}B}(z) &= r_{\mu_{A\bar{\cap}B}}(z).\exp i\left\{\phi_{\mu_{A\bar{\cap}B}}\right\} \\
&= \left(1 - r_{\mu_{A\cap B}}(z)\right).\exp i\left\{2\pi - \phi_{\mu_{A\cap B}}(z)\right\} \\
&= \left\{1 - \min\left(r_{\mu_A}(z), r_{\mu_B}(z)\right)\right\}.\exp i\left\{2\pi - \min\left(\phi_{\mu_A}(z), \phi_{\mu_B}(z)\right)\right\} \\
&= \max\left\{\left(1 - r_{\mu_A}(z)\right), \left(1 - r_{\mu_B}(z)\right)\right\} \times \\
&\quad \exp i\left[\max\left\{\left(2\pi - \phi_{\mu_A}(z)\right), \left(2\pi - \phi_{\mu_B}(z)\right)\right\}\right] \\
&= \max\left(r_{\mu_{\bar{A}}}(z), r_{\mu_{\bar{B}}}(z)\right).\exp i\left[\max\left\{\phi_{\mu_{\bar{A}}}(z), \phi_{\mu_{\bar{B}}}(z)\right\}\right] \\
&= \mu_{\bar{A}\cup\bar{B}}(z) .
\end{aligned}$$

Therefore the result follows. □

Theorem 4.8. Let A be a fuzzy complex set in the universe U with the membership function $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$. Also let, $f_1, f_2, \cdots, f_n$ be n fuzzy meromorphic functions. Then

$$\begin{aligned}
(i)\ T\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) \\
(ii)\ T(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) .
\end{aligned}$$

Proof. We know by the construction of positive logarithm that (cf. [3])

$$\begin{aligned} \log^+ \left| \prod_{\alpha=1}^{n} r_\alpha \right| &\leq \sum_{\alpha=1}^{n} \log^+ |r_\alpha| \ \textit{and} \\ \log^+ \left| \sum_{\alpha=1}^{n} r_\alpha \right| &\leq \sum_{\alpha=1}^{n} \log^+ |r_\alpha| \end{aligned}$$

therefore on applying it to the counting function and proximity function we get

$$\begin{aligned} N\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} N\left(r_{\mu_A}, f_\alpha\right) \ \textit{and} \\ N\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} N\left(r_{\mu_A}, f_\alpha\right), \end{aligned}$$

also

$$\begin{aligned} m\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} m\left(r_{\mu_A}, f_\alpha\right) \ \textit{and} \\ m\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} m\left(r_{\mu_A}, f_\alpha\right) \ . \end{aligned}$$

Thus $T\left(r_\mu, f\right) = N\left(r_\mu, f\right) + m\left(r_\mu, f\right)$ gives

$$\begin{aligned} (i) \ T\left(r_{\mu_A}, \sum_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) \\ (ii) \ T\left(r_{\mu_A}, \prod_{\alpha=1}^{n} f_\alpha\right) &\leq \sum_{\alpha=1}^{n} T\left(r_{\mu_A}, f_\alpha\right) \ . \end{aligned}$$

□

Theorem 4.9. Let A, B and C be three fuzzy sets in the universe U. Then $T\left(r_{\mu_{(A\circ B)\circ C}}, f\right) = T\left(r_{\mu_{A\circ(B\circ C)}}, f\right)$. That is Nevanlinna Characteristic of the fuzzy complex product is associative.

Proof. Let the membership functions are $\mu_A(z) = r_A(z).e^{i\phi_{\mu_A}(z)}$, $\mu_B(z) = r_B(z).e^{i\phi_{\mu_B}(z)}$ and $\mu_C(z) = r_C(z).e^{i\phi_{\mu_C}(z)}$ for the fuzzy complex sets A, B and C respectively. Then

$$\begin{aligned} \mu_{(A\circ B)\circ C}(z) &= r_{(A\circ B)\circ C}(z).\exp\left\{i.\phi_{\mu_{(A\circ B)\circ C}}(z)\right\} \\ &= \left(r_{A\circ B}(z).r_B(z)\right).\exp\left\{i.2\pi\left(\frac{\phi_{\mu_{A\circ B}}(z)}{2\pi}\cdot\frac{\phi_{\mu_B}(z)}{2\pi}\right)\right\} \\ &= \left(r_A(z).r_B(z).r_C(z)\right).\exp\left\{i.2\pi.\frac{2\pi\left(\frac{\phi_{\mu_A}(z)}{2\pi}\cdot\frac{\phi_{\mu_B}(z)}{2\pi}\right)}{2\pi}\cdot\frac{\phi_{\mu_C}(z)}{2\pi}\right\} \\ &= \mu_{A\circ(B\circ C)}(z), \end{aligned}$$

and the result follows. □

Remark 4.1. From the above theorems it is obvious that, if $f(z) = z$, then

$$T\left(r_{\mu_{(A\circ B)\circ C}}, f\right) \leq T\left(r_{\mu_A}, f\right) + T\left(r_{\mu_B}, f\right) + T\left(r_{\mu_C}, f\right) \ .$$

Remark 4.2. In general, it can be proved that if $f(z) = z$, then

$$T\left(r_{\mu_{\prod_{\alpha=1}^{n} C_\alpha}}, f\right) \leq \sum_{\alpha=1}^{n} T\left(r_{\mu_{C_\alpha}}, f\right),$$

where $\{C_\alpha, \alpha \in I\}$ be the collection of fuzzy complex sets in the universe U and $\prod_{\alpha=1}^{n} C_\alpha = C_1 \circ C_2 \circ \cdots \circ C_n$.

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