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Original Article

On Generalized Digital Topology and Root Images of Median Filters

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Abstract – In this paper, we extend the concepts of semi-open sets and λ-open sets in the digital topology. In addition, we introduce the concepts of regular semi-open and regular λ-open sets. A relationship between digital topology and image processing is established.

Keywords – *Digital topology, median filter, root image, regular open,* λ-open *and semi-open.*

1 Introduction

Over the last decades, digital topology has proved to be an important concept in image analysis and image processing. Rosenfeld [15] introduced the fundamentals of digital topology, which provides a sound mathematical basis for image processing operations such as image thinning, border following, contour filling, object counting, and signal processing. Whenever spatial relations are modeled on a computer, digital topology is needed.

Digital topology aims to transfer concepts from classical topology to digital spaces such as: connectivity, boundary, neighborhood, and continuity which are used to model computer images. The classes of semi-open and λ-open sets are finer than the class of the open sets in the 8-semi-topology which is studied in Alpers [3] proved that if $B \subseteq \mathbb{Z}^2$ is a regular open set in 8-semi-topology, then $(\mathbb{Z}^2, B)$is a root image of median filter Med_4. We found that the converse of this implication holds for the regular semi-open sets in *Marcus-Wyse* topology on $\mathbb{Z}^2$ and the regular λ-open sets in *Marcus-Wyse* or *Khalimsky* topologies on $\mathbb{Z}^2$.

In this paper, we extend the concepts of semi-open, λ-open, regular semi-open, and regular λ-open sets in the digital topology. We study the connections between these concepts and the root images of the median filters in the digital picture.

This paper is organized as following: In section 3 we study the notions of digital picture, median filter, root image, and some of its properties. In section 4 and 5 we study these notions in digital topology. Furthermore, we obtained a relationship between regular semi-open sets, regular λ-open sets, and root images of median filters.

2 Preliminaries

Definition 2. 1. [4, 11] Let (X,τ) be a topological space. A subset A of X is called:
(1) Semi-open if $A \subseteq \overline{A^{\circ}}$.
(2) Semi-closed if its complement is a semi-open.
(3) Λ-set if $A := \cap \{G \mid A \subseteq G, G \in \tau\}$.
(4) V-set if $A := \cup \{F \mid F \subseteq A, F^C \in \tau\}$.
(5) λ-closed if $A = G \cap H$; G is a closed set and H is a Λ-set.
(6) λ-open if A^c is a λ-open.

Definition 2. 2. [5] Let (X,τ) be a topological space and $A \subseteq X$. The semi-interior (λ-interior) of A, denoted by $int_s(A)$ ($int_\lambda(A)$), is the union of all semi-open (λ-open) subsets of A.

Definition 2. 3 [5] Let (X,τ) be a topological space and $A \subseteq X$. The semi-closure (λ-closure) of A, denoted by $cl_s(A)$ ($cl_\lambda(A)$), is the intersection of all semi-closed (λ-closed) supersets of A.

In this paper, for any topological space (X,τ), let $N_{min}(p)$ (respectively, $N^{\lambda}_{min}(p)N^{\lambda}_{min}(p)$, $N^{S}_{min}(p)$) denotes the smallest open (respectively, λ-open and semi-open) set containing p. Let O^{λ}_{p} (respectively, O^{S}_{p}) be a λ- open (respectively, semi-open) set containing p. In addition, the collection of all open singletons of a subset A of X, i.e., one point open subset of A, is denoted by A_{op}.

Definition 2. 4. [14] Let (X,τ) be a topological space. A point $x \in X$ is called an open singleton if $\{x\}$ is an open set. The set of all open points of a subset A of X is denoted by A_{os}.

Definition 2. 4. [7] The digital n-space $\mathbb{Z}^n$ is the set of all n-tuples $p = (p_1, \dots, p_n); p_i \in \mathbb{Z}, i \in \{1, \dots, n\}$. A point $p = (p_1, \dots, p_n)$ of the digital plane $\mathbb{Z}^n$ is called a pure vertex if its coordinates p_i are all even or odd, otherwise p is called mixed vertex. Every point p of the digital space $\mathbb{Z}^n$ has $2n$- and $(3^n - 1)$- neighbors. The $2n$-neighbors is the set $\mathcal{N}_{2n}(x) := (q_1, q_2, \dots, q_n) \in \mathbb{Z}^n \mid \sum_{i=1}^{n} |q_i - p_i| = 1\}$ and the $(3^n - 1)$- neighbors is the set:

$$\mathcal{N}_{(3^n-1)}(x) := (q_1, q_2 \dots, q_n) \in \mathbb{Z}^n \mid \max\{|q_1 - p_1|, |q_2 - p_2|\} = 1\}.$$

Definition 2. 4. [7] Two points p, q of the digital n-space $\mathbb{Z}^n$ are called k-adjacent if they are k-neighbors, i:e, one of them belongs to the $2n$ -neighbors or $(3^n - 1)$- neighbors of the other, for $k = n$ or $k = (3n - 1)$. Also for two points p, q of the digital n-space $\mathbb{Z}^n$; a k $-$path from p to q is a sequence of points $p = p_1, p_2, \dots, p_j = q$ such that p_i and p_{i+1} are k $-$adjacent, $i = 1,2, \dots, j-1$.

Definition 2. 6 [7] Any subset X of a digital n-space $\mathbb{Z}^n$ is called k $-$connected, $k = 2n$ or $(3^n - 1)$; if for every pair of points p, q of X, there is a k-path contained in X from p to q.

The digital picture is a pair $(\mathbb{Z}^n, B)$, where $B \subseteq \mathbb{Z}^n$. The elements of $\mathbb{Z}^n$ are called the points of the digital picture, the points of B are called the black points of the picture, and the points of $\mathbb{Z}^n \setminus B$ are called the white points of the picture.

Median filters are quite popular tools in image processing. The median filters are firstly introduced by Tuckey [16], they are used to de-noise images. To deal with the median filters we need the following subsets:

Definition 2. 7 [9] For any subset p of a digital space $\mathbb{Z}^n$, consider the subsets

$$U_{2n}(p) = \{(q_1, q_2 \dots, q_n) \in \mathbb{Z}^n \mid \textstyle\sum_{i=1}^{n} |q_i - p_i| \le 1\}$$

and

$$U_{(3^n-1)}(p) = \{\left(q_1, q_2 \dots, q_n\right) \in \mathbb{Z}^n \mid \max_i |q_i - p_i| \le 1\}.$$

The median filter Med_k on a digital picture is a mapping which maps $(\mathbb{Z}^n, B)$ to $(\mathbb{Z}^n, B^*)$ with

$$B^* = \{p \in \mathbb{Z}^n : |U_k(p) \cap B| \ge \frac{|U_k| + 1}{2}\}$$

for $k = 4$ or 8 in $\mathbb{Z}^2$, and $k = 6$ or 26 in $\mathbb{Z}^3$. A root image of Med_k is a digital picture $(\mathbb{Z}^n, B)$ with $Med_k((\mathbb{Z}^n, B)) = (\mathbb{Z}^n, B)$.

It is clear that, if $B \subseteq \mathbb{Z}^n$ is a root image of the median filter Med_k , $x \in B$, then x has at least one of its k $-$neighbors in B.

An important property of the root images of any median filters will be given in the following proposition:

Proposition 2. 1. The root images of any median filter Med_{2n} in the digital n-space $\mathbb{Z}^n$ are $2n$-connected set for $n = 2, 3$.

Proof. Let $(\mathbb{Z}^n, B)$ be a root image of Med_{2n} and suppose that B is not $2n$ $-$connected. Then there exists at least $x \in B$ such that x has no $2n$ $-$neighbors in $B \setminus \{x\}$. Then $|U_{2n}(x) \cap B| = 1$, and so $x \notin Med_{2n}(\mathbb{Z}^n, B)$. Which contradicts that $(\mathbb{Z}^n, B)$ is a root image of Med_{2n}.

The converse of this Proposition is not true. Figure 1 shows a 4-connected set, but it is not root image of Med_4.

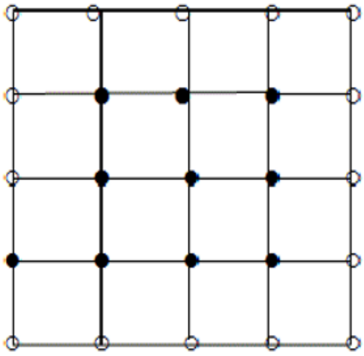

Figure 1. 4-connected set which is neither root image of Med_4 nor root image of Med_8

To study with the concept of connectedness in the digital spaces from the topological point of view, many topological structures are introduced. In this article, we need the following topologies on the digital spaces.

Definition 2. 8 [10] The Kalimsky line is the set of all integers $\mathbb{Z}$ with the topology generated by the following subbase: $\eta = \{\{2n, 2n \pm 1\}, n \in \mathbb{Z}\}$. The Kalimsky n-space is $\mathbb{Z}^n$ with the product space of n-Kalimsky line.

Definition 2. 8 [13] The Marcus-Wyse topological structure on $\mathbb{Z}^n$; $n = 2,3$ is the topology generated by the following base: $\beta = \{N_{min}(p); p \in \mathbb{Z}^n \ n = 2,3\}$ where

$$N_{min}(p = (p_1, p_2, \dots, p_n)) = \begin{cases} \{p\} & ; \sum_{i=1}^{n} p_i \ odd\ number \\ U_{2n}(p) & ; otherwise \end{cases}$$

Theorem 2. 1 [7] The two topologies Khalimsky and Marcus-Wyse on the digital n $-$space $\mathbb{Z}^n$ satisfies the following two conditions:

If $S \subseteq \mathbb{Z}^n$ is $2n$-connected set, then S is topologically connected.

If $S \subseteq \mathbb{Z}^n$ is not $(3^n - 1)$-connected set, then S is not topologically connected.

Corollary 2. 1 In any digital n $-$space $\mathbb{Z}^n$ with the Khalimsky topology or Marcus-Wyse topologies, any rot image of any median filter Med_{2n} is topologically connected.

Note that, neither of the following sets is in general a root image, the union of two root images, the intersection of two root images or difference between any two root images.

Figure 2 shows that the intersection and the difference between two root images is not necessarily to be root image. Let $A = \{a, b, f, e\}$ and $B = \{b, c, e, d\}$. Then, $A \cap B = \{b, e\}$ and $A - B = \{\{a, f\}$ which is not root image of Med_4.

Figure 3 shows that the union of two root images is not necessarily to be root image. Let $A = \{a, b, \ c, d, e, f, g, h\}$ and $B = \{1, 2, 3, 4\}$. It is clear that, A and B are root images of Med_4 but $A \cup B$ is not since $(A \cup B)^* \neq A \cup B$.

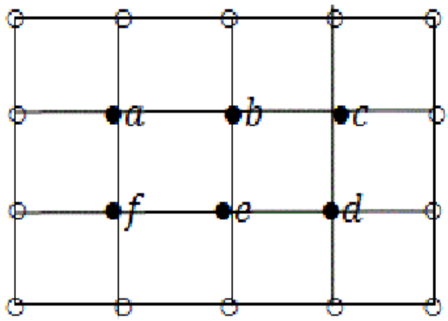

Figure 2. shows that the intersection of two root images is not root image

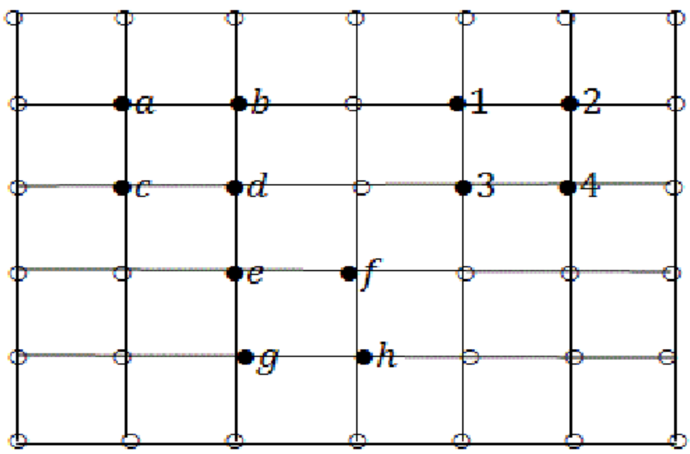

Figure 3. shows that the union of two root images is not root image

3 Semi-openness and Root Images

Theorem 3. 1 [7] A subset A of a digital n-space $\mathbb{Z}^n$ is a semi-open set if and only if $N_{min}(x) \cap A_{op} \neq \emptyset$ for all $x \in A$.

Proposition 3. 1

(1) The collection of the smallest semi-open neighborhoods of a point p in the Marcus-Wyse topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p = (p_1, p_2)\} & ; p_1 + p_2 \text{ is odd number, otherwise} \\ \left.\begin{array}{l} \{(p_1, p_2), (p_1 + i, p_2 + j)\} \\ \{(p_1, p_2), (p_1 - i, p_2 - j)\} \end{array}\right\} & ; p_1 + p_2 \text{ is even number and} \\ & \text{either } i = 0, j = 1 \text{ or } i = 1, j = 0 \end{array} \right\}$$

(2) The collection of the smallest semi-open neighborhoods of a point p in the Marcus-Wyse topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (p_1, p_2, p_3) \in \mathbb{Z}^3$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p\} & ;p_1 + p_2 + p_3 \text{ is odd number, otherwise} \\ \left. \begin{array}{l} \{p,(p_1+i,p_2+j,p_3+k)\} \\ \{p,(p_1-i,p_2-j,p_3-k)\} \end{array} \right\} & ;p_1+p_2+p_3 \text{ is even number and} \\ & \text{either } i = 1, j = k = 0 \text{ or } j = 1, \\ & i = k = 0 \text{ or } k = 1, i = j = 0 \end{array} \right\}$$

(3) The collection of the smallest semi-open neighborhoods of a point p in the Khalimsky topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1,p_2) \in \mathbb{Z}^2$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p\} & ;p_1 \text{ and } p_2 \text{ are both odd numbers} \\ \{p,(p_1{}^*,p_2{}^*)\} & ;p_i{}^* = p_i \text{ if } p_i \text{ odd number either} \\ & p_i{}^* = p_i + 1 \text{ or } p_i - 1 \text{ if } pi \text{ even for } i = 1,2 \end{array} \right\}$$

(4) The collection of the smallest semi-open neighborhoods of a point p in the Khalimsky topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (p_1,p_2,p_3) \in \mathbb{Z}^3$,

$$N^S_{min}(p) = \left\{ \begin{array}{ll} \{p\} & ;p_i \text{ are odd numbers, otherwise} \\ \{p,(p_1{}^*,p_2{}^*,p_3{}^*)\} \text{ where } p_i{}^* = \left\{ \begin{array}{ll} p_i & \text{if } p_i \text{ is an odd number for } i = 1,2,3 \\ p_i+1 \text{ or } p_i - 1 & \text{if } p_i \text{ is an odd number for } i = 1,2,3 \end{array} \right\} & \end{array} \right\}$$

Proof. (1) Let τ be Marcus-Wyse topology on $\mathbb{Z}^2$. Since $\{p = (p_1,p_2)\}$ is an open set if $p_1 + p_2$ is an odd number, then $\{p\}$ is a semi-open set. Let $A = \{(p_1,p_2),(p_1+1,p_2)\} \subseteq \mathbb{Z}^2$ and $p_1 + p_2$ is an even number. Since $N_{min}(p) \cap A_{op} \neq \emptyset$ for all $p \in A$, then A is a semi-open set by Theorem 3.1. The same results will be given if $A = \{(p_1,p_2),(p_1-1,p_2)\}$ or $A = \{(p_1,p_2),(p_1,p_2+1)\}$ or $A = \{(p_1,p_2),(p_1,p_2-1)\}$. The rest of the proof is of the same argument.

Theorem 3. 2 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_4, then B is a regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.

Proof. Let A be a root image of the median filter Med_4, and $x \in A$. Then at least two of the 4-neighbors of this x are in A. Then A is a semi-open subset by Proposition 3.1 case (1). Let $x \in Cl_s(A)$. Then, $N^S_{min}(x) \cap A \neq \emptyset$. Suppose that $x \notin A$. Then, $x \in A^*$. Which has a contradiction with A is a root image of the cross median filter Med_4.Then A is a semi-closed set, hence A is a regular semi-open set.

The converse of the previous theorem is not true. The following example shows a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$ which is not a root image of the cross median filter Med_4.

Example 3. 1 Let $B \subseteq \mathbb{Z}^2$as shown in Figure 4. Then, B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is not a root image of the cross median filter Med_4.

Solution. Since $|U_4(x) \cap B| = 3$, then $x \in B^* \setminus B$ and so B is not a root image of the cross median filter Med_4. Since $|U_8(x) \cap B| = 5$, then $x \in B^* \setminus B$ and so B is not a root image of the median filter Med_8. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

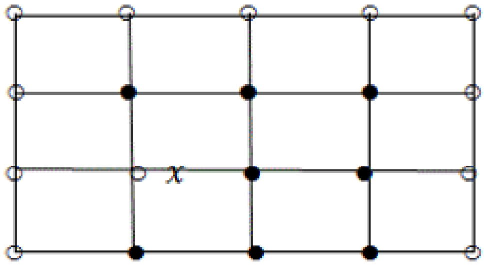

Figure 4. Regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$ which is not root image of Med_4.

Lemma 3. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_8, τ be *Marcus-Wyse* topology on $\mathbb{Z}^2$ and $x \in Cl_s(B)$, then $x \in B$.

Proof. Let $B \subseteq \mathbb{Z}^2$ be a root image of the median filter Med_8, τ be Marcus-Wyse topology on $\mathbb{Z}^2$, and $x \in Cl_s(B)$. Then, $N^S_{min}(x) \cap B \neq \emptyset$, and all the 4-neighbors of x are in B. Suppose that $x \notin B$ as shown in Figure 5. Then the points $a, c \notin B^*$ which is a contradiction with B is a root image of the median filter Med_8.

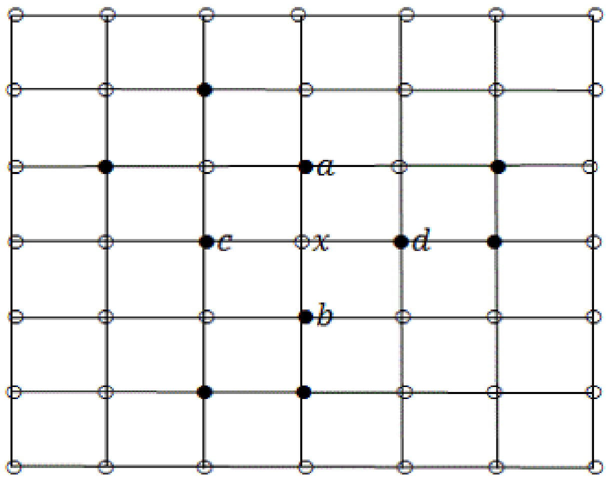

Figure 5. A contradiction with root image of Med_8 if $x \in Cl_s(B)$ and $x \notin B$.

Theorem 3. 3 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_8, then B is a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

Proof. Let B be a root image of the median filter Med_8 and $x \in B$. Then x has at least 4 of its 8-neighbors in B and at least one the 4-neighbors of x is in B. Then, B is a semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$. Let $x \in Cl_s(B)$. Then according to Lemma 3.2, $x \in B$.

The converse of the previous theorem is not true in general. Example 3.1 shows a regular semi-open set in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is not a root image of the median filter Med_8.

Example 3. 2 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 6 and let $x = (x_1, x_2) \in B$ such that x_1, x_2 are even numbers. Then, B is a root image of the median filter Med_4, but it is semi-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since there is no $O^S(x)$ such that $O^S(x) \subseteq B$, then $x \notin int_S(B)$ and so B is not semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$. Since $|U_4(y) \cap B| \geq 3$ for all $y \in B$ and $|U_4(y) \cap B| < 3$ for all $y \notin B$, then B is a root image of the median filter Med_4.

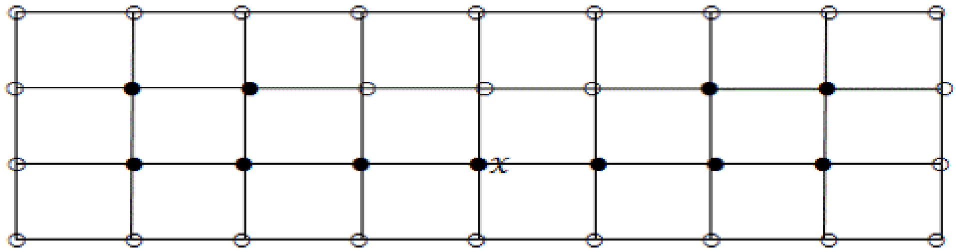

Figure 6. A root image of Med_4 which is not semi-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Example 3. 3 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 7 and $x = (x_1, x_2) \in B$ such that x_1, x_2 are even numbers. Then B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$, but it is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_4(\mathbb{Z}^2, B) = \{y, z, w, t\}$ and $Med_8(\mathbb{Z}^2, B) = \{z, y\}$, then B neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

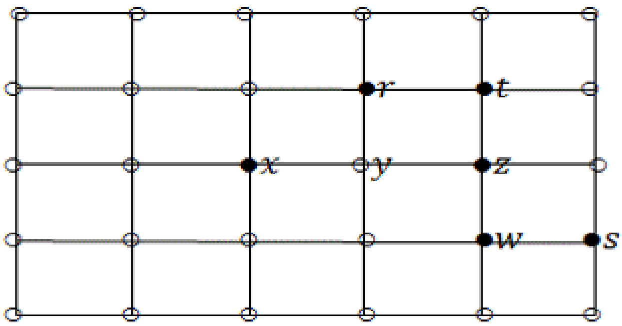

Figure 7. A regular semi-open subset in *Khalimsky* topology on $\mathbb{Z}^2$ which is neither root image of Med_4 nor root image of Med_8.

Example 3. 4 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 8. Let $x = (x_1, x_2) \in B$ such that x_1, x_2 are even numbers. Then, B is a root image of the median filter Med_8, but it is not a semi-open set in the *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since x has no $O^S(x) \subseteq BO^S(x) \subseteq B$, then B is not a semi-open set in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $|U_8(y) \cap B| \geq 5$ for all $y \in B$ and $|U_8(y) \cap B| < 5$ for all $y \notin B$, then B is a root image of the median filter Med_8.

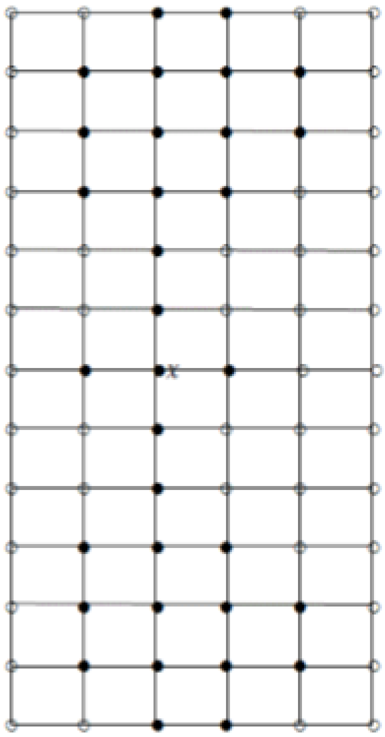

Figure 8. A root image of Med_8 which is not a semi-open subset in the *Khalimsky* topology on $\mathbb{Z}^2$

Theorem 3. 4 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_6, then B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

Proof. Let B be a root image of the median filter Med_6 and $x \in B$. Then x has at least three of its 6-neighbors in B and so B is a semi-open set in Marcus-Wyse topology on $\mathbb{Z}^3$ by Proposition 3. 1 (2). Let $x \in Cl_S(B)$. Then $N^S_{min}(x) \cap B \neq \emptyset$ and $x \in B^* = B$. Then B is a semi-closed, hence B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

The converse of the previous theorem is not true in general as shown in the following example:

Example 3. 5 Let $B \subseteq \mathbb{Z}^3$ such that

$$B = \{(0,1,1), (2,1,1), (1,0,1), \ (1,1,0), \ (0,2,1), (2,2,1), (1,0,2), (1,2,0)\}.$$

Then, B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$, but B is not a root image of the median filter Med_6.

Solution. Since $|U_6(x) \cap B| < 4$ for all $x \in B$, then B is not a root image of the median filter Med_6. Since int_s(B) = B and $Cl_s(B) = B$, then B is a regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$.

The previous example shows also that the regular semi-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^3$ is not necessary to be a root image of the median filter Med_6 or a root image of the median filter Med_{26}.

Example 3. 6 Let $B \subseteq \mathbb{Z}^3$ such that:

$B = \{(0,0,0), (0,1,0), (1,0,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,0,2), (1,2,0), (0,1,2), (0,0,2), (1,1,1), (1,1,2), (0,2,0), (1,2,1), (1,2,2), (0,2,1), (0,2,2)\}$

Then B is a root image of the median filter Med_6, but it is not semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$.

Solution. Since there is no $O^S(x)$ such that $O^S(x) \subseteq B$, then B is not semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$ while B is a root image of the median filter Med_6.

Example 3. 7 Let $B \subseteq \mathbb{Z}^3$ such that:

$$B = \{(1,1,1), (2,1,1), (0,0,0,), (0,1,1), (1,0,1), (1,1,0)\}.$$

Then, B is a regular semi-open set in the *Khalimsky* topology on $\mathbb{Z}^3$, but it is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

Solution. Since int_s(B) = B and $Cl_s(B) = B$, then, B is a regular semi-open set in *Khalimsky* topology on $\mathbb{Z}^3$. Since $Med_6(\mathbb{Z}^3, B) = \{(1,1,1)\}$ and $Med_{26}(\mathbb{Z}^3, B) = \emptyset$, then B is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

4. λ-open and Root Image

A digital topology is an *Alexandroff* space [2]. So, a subset A of a digital topology is called λ-open set if it can be written as a union of an open and a closed set. Then, every open set is also a λ-open set, and every closed set is also a λ-open set. Since *Marcus-Wyse* topology is $T_{1/2}$-space, then every singleton in *Marcus-Wyse* topology is a λ-open set. Consequently,

Corollary 4. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filters Med_4 or Med_8, then B is a regular λ-open set in the *Marcus-Wyse* topology on $\mathbb{Z}^2$.

Corollary 4. 2 If $B \subseteq \mathbb{Z}^3$ is a root image of the median filters Med_6 or Med_{26}, then B is a regular λ-open set in *Marcus-Wyse* topology on $\mathbb{Z}^3$.

Example 4. 1 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 9. Then B is a regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$, but it is neither a root image of the median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since $x \in B^* \setminus B$, then B is neither root image of the median filter Med_4 nor a root image of the median filter Med_8. It is clear that B is both λ-open set and λ-closed set, hence B is a regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$.

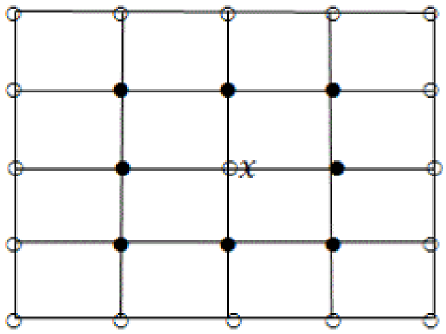

Figure 9. A regular λ-open in *Marcus-Wyse* topology on $\mathbb{Z}^2$ which is neither a root image Med_4 nor a root image of Med_8.

Different results are found with the *Khalimsky* topology:

(1) The collection of the smallest λ-open neighborhoods of a point p in *Khalimsky* topology on $\mathbb{Z}^2$ can be given as follows: for every $p = (p_1, p_2) \in \mathbb{Z}^2$,

$$N^{\lambda}_{\min}(p) = \begin{cases} \{p = (p_1, p_2)\} & \text{if } p \text{ is pure vertex} \\ \begin{Bmatrix} \{(p_1, p_2), (p_1 \pm 1, p_2)\} \\ \{(p_1, p_2), (p_1, p_2 \pm 1)\} \end{Bmatrix} & \text{if } p \text{ is mixed vertex} \end{cases}$$

(2) The collection of the smallest λ-open neighborhoods of a point p in the *Khalimsky* topology on $\mathbb{Z}^3$ can be given as follows: for every $p = (l, m, n) \in \mathbb{Z}^3$,

$$N^{\lambda}_{\min}(p) = \begin{cases} \{p = (l, m, n\} & \text{if } p \text{ is pure vertex} \\ \{(l, m, n), (l \pm 1, m, n)\} & \text{if } \begin{Bmatrix} l \text{ even}, m, n \text{ odd, or} \\ l \text{ odd } m, n, \text{even} \end{Bmatrix} \\ \{(l, m, n), (l, m \pm 1, n)\} & \text{if } \begin{Bmatrix} m \text{ even}, l, n \text{ odd, or} \\ m \text{ odd } l, n, \text{even} \end{Bmatrix} \\ \{(l, m, n), (l, m, n \pm 1)\} & \text{if } \begin{Bmatrix} n \text{ even}, l, m \text{ odd, or} \\ n \text{ odd } l, m \text{ even} \end{Bmatrix} \end{cases}$$

Theorem 4. 1 If $B \subseteq \mathbb{Z}^2$ is a root image of the median filter Med_4 , then B is a λ-closed in the *Khalimsky* topology on $\mathbb{Z}^2$.

Proof. Let B be a root image of the median filter Med_4. Let $x \in Cl_{\lambda}(B)$. Then $N^{\lambda}_{min}(x) \cap B \neq \phi$ and all the 4-neighbors of x are in B. Then, $x \in B^* = B$.

Example 4. 2 Let $B \subseteq \mathbb{Z}^2$ as shown in Figure 10. Let $x \in B$ is pure vertex. Then, B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^2$, but B is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

Solution. Since int_{λ}(B) = B and $Cl_{\lambda}(B) = B$, then B is a regular λ-open in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_4(\mathbb{Z}^2, B) = \{y, z, s\}$ and $Med_8(\mathbb{Z}^2, B) = \{w\}$, then B is neither a root image of the cross median filter Med_4 nor a root image of the median filter Med_8.

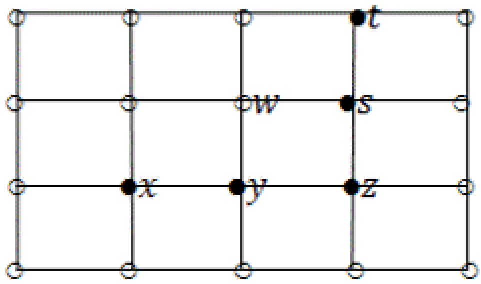

Figure 10. A regular λ-open in *Khalimsky* topology on $\mathbb{Z}^2$ which is neither root image of Med_4 nor a root image of Med_8.

Example 4. 3 Let $B \subseteq \mathbb{Z}^2$ be as shown in Figure 11 and let $x \in B$ be mixed vertex. Then, B is a root image of the median filter Med_8, but it is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^2$.

Solution. Since there is no $O^\lambda(x)$ such that $O^\lambda(x) \subseteq B$, then B is not λ-open in the *Khalimsky* topology on $\mathbb{Z}^2$. Since $Med_8(\mathbb{Z}^2, B) = B$, then B is root image of the median filter Med_8.

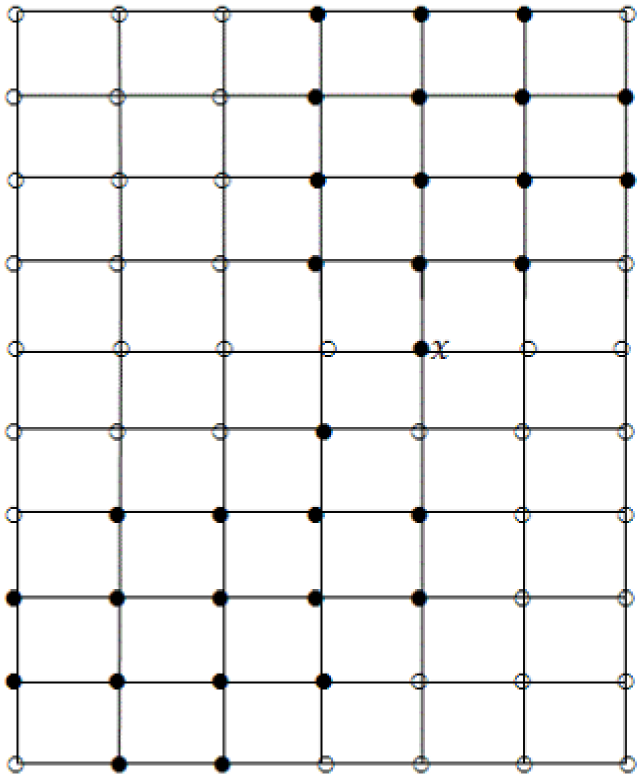

Figure 11. A root image of Med_8 which is not λ-open in *Khalimsky* topology on $\mathbb{Z}^2$.

Example 4. 4 Let $B \subseteq \mathbb{Z}^3$ such that:

$B = \{(0,0,0), (0,1,0),\ (1,0,0), (0,0,1), (1,1,0), (0,1,1), (1,0,1), (1,0,2),\ (1,2,0), (0,1,2),$
$(0,0,2),\ (1,1,1), (1,1,2),\ (0,2,0), (1,2,1), (1,2,2),\ (0,2,1), (0,2,2)\}$

Then, B is a root image of the median filter Med_6, but it is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$.

Solution. Since there is no $O^\lambda((1,0,0))$ such that $O^\lambda((1,0,0)) \subseteq B$, then B is not λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$. But B is a root image of the median filter Med_6 as illustrated in Example 3.6.

Example 4. 5 Let $B \subseteq \mathbb{Z}^3$ such that $B = \{(0,0,0),\ (1,1,1),\ (2,2,2,),\ (3,5,7)\}$. Then, B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$, but it is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

Solution. Since x is a pure vertex for all $x \in B$, then $\{x\}$ is a λ-open set for all $x \in B$ and B is a λ-open set. Since there is $O^\lambda(x)$ such that $O^\lambda(x) \cap B = \phi$ for all $x \notin B$, then B is a λ-closed set, hence B is a regular λ-open set in the *Khalimsky* topology on $\mathbb{Z}^3$. Since $Med_k(\mathbb{Z}^3, B) = \phi$ for k=6 or 26, then B is neither a root image of the median filter Med_6 nor a root image of the median filter Med_{26}.

6. Conclusion

In this paper, we show how the topological concepts such as: λ-open, semi-open, regular λ-open set, regular semi-open set, and topologically connected can be transferred to the digital topology. In addition, we explain how we can apply these concepts in median filter. The results may be summarized as following:

1- Every root image of median filter Med_4 and Med_8 is a regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.
2- Every root image of median filter Med_4 and Med_8 are regular λ-open set in Marcus-Wyse topology on $\mathbb{Z}^2$.
3- Every root image of median filter Med_4 is a every root image of λ-closed set in Khalimsky topology on $\mathbb{Z}^2$ which are the converse of the implication does Alpers have.
4- Every root image of Med_6 is regular semi-open set in Marcus-Wyse topology on $\mathbb{Z}^3$.
5- Every root image of median filter Med_6 and Med_{26} are regular λ-open set in Marcus-Wyse topology on $\mathbb{Z}^3$.
6- Every root image of Med_4 and Med_6 is topologically connected set in the digital topology.

We aim to have a generalization of the digital topology which provide the implication or find another filter that make the root image of this filter is regular semi-open set (regular λ-open) and vice versa.

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