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HYPERBOLIC HORADAM SPINORS

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Abstract. This study, introduces a new definition of hyperbolic spinors through a transformation from the Horadam split quaternion, which holds significant importance in mathematics and physics. Subsequently, fundamental concepts such as conjugate and norm are elucidated. Leveraging the defined hyperbolic spinor and the recurrence relation of the Horadam sequence, a novel sequence is delineated, and its foundational equations, akin to the generator function and Binet formula, are expressed through theorems.

1. Introduction

Number sequences are of significant interest in mathematics. Among these, the number of sequences attributed to Leonardo Fibonacci (1170–1250) stand out prominently. However, numerous sequences exist akin to the Fibonacci sequence, where each term after the second is the sum of the preceding two terms, albeit with different initial values. Notable examples include the Lucas, Pell, modified Pell, Pell–Lucas, Jacobsthal, and Jacobsthal–Lucas numbers, each defined with distinct starting points [10].

Mathematically, quaternions represent a number system that extends beyond complex numbers, thus enriching the domain of normed division algebra. This algebraic hierarchy comprises the real numbers R, complex numbers C, quaternions H, and octonions O, marking a significant milestone in modern algebra following their discovery in 1843 by Hamilton [8]. Hamilton's seminal work has resonated across diverse disciplines, spanning from quantum physics to computer science [6, 7, 13]. Within algebraic realms, split quaternions or coquaternions emerge as elements within a 4-dimensional associative algebra initially introduced by James Cockle [5]. Diverging from the quaternion algebra established by Hamilton, which delineates a 4-dimensional real vector space equipped with a multiplicative operation, split quaternions exhibit distinctive attributes. They encompass zero divisors, nilpotent elements, and nontrivial idempotent elements, distinguishing themselves from conventional quaternionic structures.

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Spinor is a significant concept in quantum mechanics, particularly in areas such as spacetime geometry and particle physics. Spinors are used to represent a property called spin, which is the rotation of a particle around its axis, determining its magnetic moment and response to magnetic fields. Spinors are mainly employed in defining fermions (particles with spin) in quantum mechanics, particularly in describing the properties of fundamental particles such as electrons, protons, and neutrons. Consequently, spinors are crucial for understanding and predicting the behavior of fundamental particles [3]. However, spinors are not limited to quantum mechanics alone. Mathematically, spinors are also utilized in general relativity and spacetime geometry. Spinors are particularly used to describe the behavior of particles subject to Lorentz transformations (operations rotating and changing the direction of spacetime). This is particularly important for understanding topics such as spacetime curvature and time dilation within the framework of Einstein's general theory of relativity. Spinors represent an essential concept with broad applications in physics and mathematics, utilized in various fields ranging from theoretical physics to practical applications such as magnetic resonance imaging [12].

2. Preliminaries

The recurrence relation defines Horadam number sequence

$$W_{n+2} = pW_{n+1} + qW_n$$

with initial conditions $W_0 = a$, $W_1 = b$, for $n \geq 0$. The characteristic equation of the recurrence relation of this sequence is

$$x^2 - px - q = 0$$

the roots of the equation are

$$\alpha = \frac{1+\sqrt{d}}{2}, \beta = \frac{1-\sqrt{d}}{2}, d = p^2 + 4q.$$

The recurrence relation of the (p,q)-Fibonacci number sequence derived from the Horadam number sequence with initial conditions $a = 0$ and $b = 1$ is

$$U_{n+2} = pU_{n+1} + qU_n.$$

where U_n is nth (p,q)-Fibonacci number, for $n \geq 0$ [4].

Ipek has formulated the recurrence relation for (p,q)-Fibonacci quaternions, represented by the equation

$$QU_{n+2} = pQU_{n+1} + qQU_n, n \geq 0.$$

and has subsequently derived various identities. These include the Binet formula, generating functions, and specific binomial sums incorporating (p,q)-Fibonacci quaternions.

The recurrence relation defines (p,q)- Lucas sequence

$$V_{n+2} = pV_{n+1} + qV_n$$

with initial conditions $V_0 = 2$, $V_1 = b$, for $n \geq 0$ [9].

Patel and Ray introduced the (p,q)- Lucas quaternion and they define this quaternion as follows [11].

The (p,q)- Lucas quaternion is defined recursively by

$$QV_{n+2} = pQV_{n+1} + qQV_n, n \geq 0.$$

Let's give some information about split quaternions, which play an essential role in our paper. You can find more detailed information in [1].

A split quaternion is defined with $q = q_0 + \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$, where $q_0, q_1, q_2, q_3 \in \mathbb{R}$ and the quaternion basis $\{\mathbf{1}, \mathbf{i}, \mathbf{j}, \mathbf{k}\}$ is given such that

$$\mathbf{i}^2 = -1, \mathbf{j}^2 = \mathbf{k}^2 = 1, \quad \mathbf{ijk} = 1, \quad \mathrm{ij} = -\mathrm{ji} = \mathbf{k}, \quad \mathbf{jk} = -\mathrm{kj} = \mathrm{i}, \quad \mathbf{ki} = -\mathrm{ik} = \mathbf{j}$$

Let $q_0 = S_q$ and $\mathbf{V}_q = \mathbf{i}q_1 + \mathbf{j}q_2 + \mathbf{k}q_3$ be scalar and vectorial parts of the quaternion q. So, we can write the quaternion q as $q = S_q + \mathbf{V}_q$. The set of these quaternions is $\mathbb{K}$. Let $p = S_p + \mathbf{V}_p, q = S_q + \mathbf{V}_q \in \mathbb{K}$ be two real quaternions.

$\overline{q}$ is the conjugate of the quaternion q is equal to $\overline{q} = S_q - \boldsymbol{V}_q$ and it is

$$\overline{q} = \mathbf{q}_0 - \mathbf{i}q_1 - \mathbf{j}q_2 - \mathbf{k}q_3$$

In addition, the norm of a split quaternion

$$N(q) = \sqrt{q_1^2 + q_2^2 + q_3^2 + q_4^2}$$

For $n \geq 0$, define the split Horadam quaternion H_n by

$$H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$$

where, W_n is the nth Horadam number and i,j,k are split quaternionic units [2].

On the other hand, let us consider the vector $(\alpha_1, \alpha_2, \alpha_3)$ with ${\alpha_1}^2 + {\alpha_2}^2 + {\alpha_3}^2 = 0$ in the complex vector space $\mathbb{C}^3$. These vectors form a two-dimensional surface in the two-dimensional $\mathbb{C}^2$ subspace of $\mathbb{C}^3$. If the parameters of this two-dimensional surface are taken as φ_1 and φ_2 , the following equations can be written

$$\begin{aligned} \alpha_1 &= {\varphi_1}^2 - {\varphi_2}^2 \\ \alpha_2 &= i\left({\varphi_1}^2 + {\varphi_2}^2\right) \\ \alpha_3 &= -2\varphi_1\varphi_2 \end{aligned}$$

Thus, each isotropic vector in $\mathbb{C}^3$ corresponds to a vector in $\mathbb{C}^2$ and vice versa. The vector $\varphi = (\varphi_1, \varphi_2) \cong \begin{bmatrix} \varphi_1 \\ \varphi_2 \end{bmatrix}$ obtained in this way is called a spinor [12].

3. Main Theorems and Proofs

Hyperbolic spinors corresponding to the Horadam split quaternion were defined using the transformations provided in the preceding section. Their conjugates, norms and fundamental properties were examined in this section. Additionally, important equalities and theorems, such as the Binet formula and the generating function were proven. Consequently, by determining the initial conditions of the Horadam sequence, special cases of hyperbolic spinous, namely (p, q)- Fibonacci and (p, q)- Lucas hyperbolic spinors, were introduced and fundamental equations for both were derived.

Definition 3.1. *Let $H_n = W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}$ be nth split Horadam quaternion where W_n is nth Horadam number for $n \geq 0$. H_w is the set of split Horadam quaternions. Therefore, we give the linear transformation between the hyperbolic spinors and split quaternions as follows:*

$$\varphi_w : H_w \longrightarrow S$$

$$H_w \longrightarrow \varphi_w\left(W_n + iW_{n+1} + jW_{n+2} + kW_{n+3}\right) = \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$$

Furthermore, a new hyperbolic Horadam spinor sequence can be introduced using the spinor defined. The recurrence relation of this sequence is as follows.

$$SH_{n+1} = pSH_n + qSH_{n-1}$$

where p and q are real numbers,

$$SH_0 = \begin{bmatrix} a + j\left(p^2 b + qpa + qb\right) \\ -b + j(pb + qa) \end{bmatrix}, SH_1 = \begin{bmatrix} b + j\left(p^3 b + p^2 qa + 2pqb + qbpa\right) \\ -pb - qa + j\left(p^2 b + qpa + qb\right. \end{bmatrix}$$

are initial conditions.

The set of hyperbolic Horadam spinor sequences is

$$\{SH_n\}_{n\in \mathrm{N}}^{\infty} = \left\{ \begin{bmatrix} a + j(p^2 b + qpa + qb) \\ -b + j(pb + qa) \end{bmatrix}, \begin{bmatrix} b + j(p^3 b + p^2 qa + 2qbp + q^2 bpa) \\ -pb - qa + j(p^2 b + qpa + qb) \end{bmatrix}, \ldots, \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}, \ldots \right\}$$

where $SH_n = \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$ is nth hyperbolic Horadam spinor and W_n is nth Horadam number.

Definition 3.2. *Let $n \geq 0, n \in \mathbb{Z}$ and the nth hyperbolic (p, q)- Lucas spinor is SV_n. Then, the recurrence relation of hyperbolic (p, q)- Lucas spinors is as follows:*

$$SV_{n+2} = pSV_{n+1} + qSV_n$$

with initial conditions

$$SV_0 = \begin{bmatrix} 2 + j\left(p^2 b + 2pq + pqb\right. \\ -b + j(pb + 2q) \end{bmatrix}$$

$$SV_1 = \begin{bmatrix} b + j\left(p^3 b + 2qp^2 + p^2 qb + pbq + 2q^2\right) \\ -(pb + 2q) + j\left(p^2 b + 2qb + pbq\right) \end{bmatrix}.$$

Similar to number sequences, here, by keeping the coefficients constant in the recurrence relation of the hyperbolic Horadam spinor sequence and changing the initial conditions to $a = 0, b = 1$, the hyperbolic (p, q)- Fibonacci spinor sequence can be obtained. Similarly, when $a = 2, b = b$ is taken, the hyperbolic (p, q)-Lucas spinor sequence can be obtained as follows:

Definition 3.3. *Hyperbolic (p, q)- Fibonacci spinor sequence is defined with*

$$SU_{n+2} = pSU_{n+1} + qSU_n$$

recurrence relation for $n \geq 0$. The initial conditions of this sequence are

$$SU_0 = \begin{bmatrix} j\left(p^2 + q\right) \\ -1 + jp \end{bmatrix}, SU_1 = \begin{bmatrix} 1 + j\left(p^3 + 2pq\right) \\ -p + j\left(p^2 + q\right) \end{bmatrix}.$$

The terms for this two hyperbolic spinor, defined with $a = 0, b = 1$, have been obtained. In hyperbolic (p, q)- Fibonacci spinors, taking $p = 1, q = 1$ results in the recurrence relation of the Fibonacci sequence. Therefore, similar properties provided for hyperbolic Horadam spinors can readily be derived for hyperbolic Fibonacci spinors. A parallel situation also holds for hyperbolic (p, q)-Lucas spinors. By classifying hyperbolic Horadam spinors, such as the Fibonacci, Pell, Pell-Lucas, Jacobsthal, Jacobsthal Lucas sequences obtained through the classification of the coefficients and initial conditions of the Horadam sequence, the following table can be derived.

p	q	a	b	Hyperbolic Horadam spinor
p	q	0	1	Hyperbolic (p,q)-Fibonacci spinor
p	q	2	p	Hyperbolic (p,q)-Lucas spinor
2	1	0	1	Hyperbolic Pell spinor
1	2	0	1	Hyperbolic Jacobsthal spinor
1	1	2	1	Hyperbolic Lucas spinor
2	1	2	2	Hyperbolic Pell Lucas spinor
2	1	2	1	Hyperbolic Jacobsthal Lucas spinor

TABLE 1. Various hyperbolic spinor types.

For example, let's construct the hyperbolic Lucas spinor sequence using the numerical values specified in the table. Let the general term of the sequence be denoted as SHL_n. Then, the initial conditions SHL_0 and SHL_1 are as follows.

$$SHL_0 = \begin{bmatrix} 2+4j \\ -1+3j \end{bmatrix},$$

$$SHL_1 = \begin{bmatrix} 1+7j \\ -3+4j \end{bmatrix}.$$

Since the recurrence relation of the Lucas sequence is satisfied, the other terms of the sequence are obtained using the relation

$$SHL_{n+1} = SHL_n + SHL_{n-1}.$$

Definition 3.4. *Let the conjugate of the split Horadam quaternion* $\overline{H_n} = W_n - iW_{n+1} - jW_{n+2} - kW_{n+3}$. *The hyperbolic Horadam spinor* $\overline{SH_n}$ *corresponding to the conjugate of the split Horadam quaternion is written by*

$$\left(\overline{SH_n}\right) = \begin{bmatrix} W_n - jW_{n+3} \\ -W_{n+1} - jW_{n+2} \end{bmatrix}$$

Furthermore, by utilizing conjugate definitions, we can obtain the following.
The hyperbolic conjugate of hyperbolic Horadam spinor SH_n *is*

$$SH_n^* = \begin{bmatrix} W_n - jW_{n+3} \\ W_{n+1} - jW_{n+2} \end{bmatrix}.$$

Hyperbolic Horadam spinor conjugate $\tilde{SH_n} = jC\overline{SH_n}$ *is*

$$\tilde{SH_n} = \begin{bmatrix} -W_{n+2} - jW_{n+1} \\ W_{n+3} - jW_n \end{bmatrix}$$

The mate of hyperbolic Horadam spinor $\breve{SH_n} = -C\overline{SH_n}$ *is*

$$\breve{SH_n} = \begin{bmatrix} W_{n+1} + jW_{n+2} \\ W_n - jW_{n+3} \end{bmatrix}$$

where

$$C = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Let's give an example for each of the numerical counterparts of the given conjugate definitions.

For $n = 0$ hyperbolic Horadam spinor SH_0, the conjugatesof this spinor are as follows.

$$(\overline{SH_0}) = \begin{bmatrix} a - j(pc+qb) \\ -b - jc \end{bmatrix},$$

$$SH_0^* = \begin{bmatrix} a - j(pc+qb) \\ b - jc \end{bmatrix},$$

$$S\tilde{H}_0 = \begin{bmatrix} -a - j(pc+qb) \\ b - jc \end{bmatrix},$$

$$S\breve{H}_0 = \begin{bmatrix} a + j(pc+qb) \\ b - jc \end{bmatrix}.$$

Corollary 3.5. *For the hyperbolic Horadam spinor SH_n and its conjugates, the following equalities are valid.*

$$\cdot\, CS\hat{H}_n = \overline{SH_n},$$
$$\cdot\, j\tilde{H}_n = -S\hat{H}_n,$$
$$\cdot\, jCS\tilde{H}_n = -\overline{SH_n}.$$

Proposition 3.6. *Let the nth hyperbolic Horadam spinor* SH_n *be the spinor corresponding to the nth split Horadam quaternion W_n. In this case, the hyperbolic Horadam spin or representation of the split quaternion norm is as follows:*

$$N(H_n) = (SH_n^*)^\top SH_n.$$

Proof. Assume that nth hyperbolic Horadam spinor SH_n corresponds to the nth split Horadam quaternion W_n. Then, the following equation can be obtained as:

$$(SH_n^*)^\top SH_n = \begin{bmatrix} W_n - jW_{n+3} & W_{n+1} - jW_{n+2} \end{bmatrix} \begin{bmatrix} W_n + jW_{n+3} \\ -W_{n+1} + jW_{n+2} \end{bmatrix}$$

We can associate to the product of two Horadam split quaternions with a hyperbolic Horadam spinor matrix product as follows:

$$qw \to \hat{q}w \longrightarrow \hat{Q}SH$$

where $\hat{Q}$ is the hyperbolic, unitary, square matrix defined by

$$\hat{Q} = \begin{bmatrix} W_0 + jW_3 & W_1 + jW_2 \\ -W_1 + jW_2 & W_0 - jW_3 \end{bmatrix}.$$

□

We present fundamental equations, such as the Binet formula, generating function.

Theorem 3.7. *The Binet formula for hyperbolic Horadam spinor is as follows.*

$$SH_n = \frac{1}{2\sqrt{d}}\left(\begin{bmatrix} r + \sqrt{d}(a + j(pc+qb)) \\ s + \sqrt{d}(-b + jc) \end{bmatrix}\alpha^n - \begin{bmatrix} r - \sqrt{d}(a + j(pc+qb)) \\ s - \sqrt{d}(-b + jc) \end{bmatrix}\beta^n\right)$$

where $\quad r = 2b - pa + j\left(p^2c + pqb + 2qc\right), s = pb - 2c + j(pc + 2qb),$

$$c = pb + qa, \alpha = \frac{1+\sqrt{d}}{2}, \beta = \frac{1-\sqrt{d}}{2}, d = p^2 + 4q.$$

Proof. The characteristic equation of the recurrence relation of hyperbolic Horadam spinor sequence is $x^2-px-q=0$. The discriminant of this equation is $d=p^2+49$ and the roots of α and β are $\alpha=\frac{p+\sqrt{d}}{2}, \beta=\frac{p-\sqrt{d}}{2}$. The Bines formula for hyperbolic Horadam spinor sequence is $SH_n=A\alpha^n+B\beta^n$ where $A=\begin{bmatrix} a_0 \\ a_1 \end{bmatrix}, B=\begin{bmatrix} b_0 \\ b_1 \end{bmatrix}$ are 2×1 matrices. When substituted for $n=0$ and $n=1$, after performing the necessary operations, the desired result is obtained. □

As a result, the Binet formulas for hyperbolic (p,q)- Fibonacci and hyperbolic (p,q)- Lucas spinors can be expressed as follows.

Corollary 3.8. *The Binet formulas for hyperbolic (p,q)-Fibonacci and hyperbolic (p,q)- Lucas spinor are as follows, respectively.*

$$SU_n=\frac{1}{2\sqrt{d}}\left(\begin{bmatrix} 2+j\left(p^3+3pq\right)+\sqrt{d}\left(jp^2+jq\right) \\ -p+j\left(p^2+2q\right)+\sqrt{d}(-1+jq) \end{bmatrix}\alpha^n-\begin{bmatrix} 2+j\left(p^3+3pq\right)-\sqrt{d}\left(jp^2+jq\right) \\ -p+j\left(p^2+2q\right)-\sqrt{d}(-1+jq) \end{bmatrix}\beta^n.\right.$$

$$SV_n=\frac{1}{2\sqrt{d}}\left(\begin{matrix} 2b-2p+j\left(p^3b+p^2bq+4qp^2+4q^2\right)+\sqrt{d}\left(4+2p^2bj+4qbj+2pqbj\right) \\ -pb-4q+j\left(p^2b+2pq+2pqb\right)+\sqrt{d}(-2b+2pb+4q) \end{matrix}\right]\alpha^n-$$
$$\begin{bmatrix} 2b-2p+j\left(p^3b+p^2qb+4qp^2+4q^2\right)-\sqrt{d}\left(4+2p^2bj+4qbj+2pbqj\right) \\ -pb-4q+j\left(p^2b+2pq+2pbq\right)-\sqrt{d}(-2b+2pbj+4qj) \end{bmatrix}\beta^n.$$

Theorem 3.9. *The generating function for the nth hyperbolic Horadam spinor is obtained as follows:*

$$G_w(x)=\frac{1}{1-px-qx^2}\left(SH_0(1-px)+SH_1\right),$$

where

$$SH_0=\begin{bmatrix} a+j(pc+qb) \\ -b+jc \end{bmatrix}$$
$$SH_1=\begin{bmatrix} b+j\left(p^2c+qbp+qc\right) \\ -c+j(pc+qb) \end{bmatrix}, \quad c=pb+qa.$$

Proof. Assume that SH_n is the nth hyperbolic Horadam spinor and the generating function of the hyperbolic Horadam spinor is

$$G_w(x)=\sum_{n=0}^{\infty} SH_n x^n.$$

First, the function can be written from the recurrence relation of the hyperbolic Horadam spinor sequence as follows:

$$\sum_{n=0}^{\infty} SH_{n+2}x^n=p\sum_{n=0}^{\infty} SH_{n+1}x^n+q\sum_{n=0}^{\infty} SH_n x^n,$$
$$\sum_{n=2}^{\infty} SH_n x^{n-2}=p\sum_{n=1}^{\infty} SH_n x^{n-1}+q\sum_{n=0}^{\infty} SH_n x^n.$$

Then, the following equation can be obtained

$$\frac{1}{x^2}\left[-SH_0-SH_1+G_w(x)\right]=p\frac{1}{x}\left[-SH_0+G_w(x)\right]+qG_w(x)$$

Consequently, for the hyperbolic Horadam spinors, the generating function is obtained as follows.

$$G_w(x) = \frac{1}{1-px-qx^2}\left(SH_0(1-px)+SH_1\right).$$

□

Corollary 3.10. *The generating functions for hyperbolic (p,q)- Fibonacci spinors and hyperbolic (p,q)-Lucas spinor are as follows, respectively.*

$$G_u(x) = \frac{1}{1-px-qx^2}\begin{bmatrix} 1+j\left(p^3+2pq+p^2+q-p^3x-qpx\right) \\ -1-p+j\left(p-p^2x+p^2+q\right) \end{bmatrix},$$

$$G_v(x) = \frac{1}{1-px-qx^2}\begin{bmatrix} 2+b-2px+j\left((1-x)\left(p^3b+2qp^2+p^2bq\right)+2pq+2pqb+2q^2\right) \\ -b-pb-2q+pbx+j\left((1-x)\left(p^2b+2pq\right)+pbq+2q\right) \end{bmatrix}.$$

4. Conclusion

This study, defined hyperbolic Horadam spinor sequences using the most general form of number sequences, namely Horadam sequences and split Horadam quaternions. Additionally, (p,q)-Fibonacci and (p,q)-Lucas hyperbolic spinor sequences were defined using the general forms of Fibonacci and Lucas sequences with parameters p and q. The relationships between these newly defined sequences, as well as their internal relationships, were demonstrated through provided equalities. As a result, a new hyperbolic spinor sequence was defined based on the properties of number sequences and the definitions of split quaternions and spinors.

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