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Research Article

General Hardy-type operators on local generalized Morrey spaces

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ABSTRACT. This paper extends the mapping properties of the general Hardy-type operators to local generalized Morrey spaces built on ball quasi-Banach function spaces. As applications of the main result, we establish the two weight norm inequalities of the Hardy operators to the local generalized Morrey spaces, the mapping properties of the Riemann-Liouville integrals on local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces, the Hardy inequalities on the local generalized Morrey spaces with variable exponents.

Keywords: General Hardy-type operator, Hardy inequality, Riemann-Liouville integrals, local generalized Morrey spaces, ball Banach function spaces, rearrangement-invariant, variable exponents.

2020 Mathematics Subject Classification: 42B25, 42B35, 46E40.

This paper extends the mapping properties of the general Hardy-type operators to the local generalized Morrey spaces built on ball quasi-Banach function spaces.

The general Hardy-type operators [25, Definition 2.5] include a number of important operators in analysis. The most important example is the Hardy operator. It also includes the Riemann-Liouville integrals. The mapping properties of the general Hardy-type operators on Lebesgue spaces and extensions of Lebesgue spaces were investigated in [1, 2, 4, 11, 12, 15, 16, 25, 26, 29, 31, 34, 35, 36, 37].

The local generalized Morrey spaces are extensions of the Lebesgue spaces and Morrey spaces [28, 33]. The local generalized Morrey spaces are members of the ball quasi-Banach function spaces introduced in [32]. A number of results from the harmonic analysis, such as the mapping properties of the singular integral operators, the fractional integral operators, the maximal Carleson operators, the geometric maximal functions, the minimal functions and the spherical maximal functions had been extended to the local generalized Morrey spaces [5, 6, 7, 8, 9, 13, 14, 19, 20, 30, 38, 40].

It motivates us to investigate the mapping properties of the general Hardy-type operators on the local generalized Morrey spaces. We find that whenever a given general Hardy-type operator is bounded on a ball quasi-Banach function space, it can be extended to be a bounded operator on the local generalized Morrey space built on this ball quasi-Banach function space. As applications of this main result, we extend the mapping properties of the general Hardy-type operators with Oinarov kernel on the weighted local generalized Morrey spaces, the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces. We also obtain the Hardy-type inequalities on the local generalized Morrey spaces with variable exponents.

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This paper is organized as follows. The definition of the general Hardy-type operator is given in Section 1. This section also contains the mapping properties of the general Hardy-type operators on weighted Lebesgue spaces. The main result is given in Section 2. The definitions of the ball quasi-Banach function spaces and its corresponding local generalized Morrey spaces are also presented in Section 2. The applications of the main result on the weighted local generalized Morrey spaces, the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces and the local generalized Morrey spaces with variable exponents are given in Section 3.

1. Preliminaries and Definitions

Let $\mathcal{M}$ denote the class of Lebesgue measurable functions on $(0,\infty)$. For any Lebesgue measurable set E on $(0,\infty)$, the Lebesgue measure of E is denoted by $|E|$. Define $I_0 = \{(0,r) : r > 0\}$ and $I = \{(s,r) : r > s \geq 0\}$.

Let $p \in (0,\infty)$ and $v : (0,\infty) \to [0,\infty)$, the weighted Lebesgue space $L^p(v)$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L^p(v)} = \left(\int_0^\infty |f(x)|^p v(x)dx\right)^{\frac{1}{p}} < \infty.$$

Let $k : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function satisfying $k(x,y) \geq 0$ when $0 < y < x$. The general Hardy-type operator with kernel k is defined as

$$Kf(x) = \int_0^x k(x,y)f(y)dy, \quad x \in (0,\infty),$$

see [25, Definition 2.5].

When $k(x,y) \equiv 1$, K is the Hardy operator $Hf(x) = \int_0^x f(t)dt$. When $\alpha \in [0,\infty)$ and $k(x,y) = \frac{1}{\Gamma(\alpha)}(x-y)^{\alpha-1}$, K is the Riemann-Liouville operator $R_\alpha f(x) = \frac{1}{\Gamma(\alpha)}\int_0^x (x-y)^{\alpha-1}f(y)dy$, see [2, 37].

Definition 1.1. *Let $k : (0,\infty) \times (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. We say that k is an Oinarov kernel if it satisfies*

(1) $k(x,y) \geq 0$ when $0 < y < x$,
(2) k is non-decreasing in x or non-increasing in y,
(3) there is a constant $D > 0$ such that for any $0 < z < y < x$,

$$D^{-1}(k(x,y) + k(y,z)) \leq k(x,z) \leq D(k(x,y) + k(y,z)).$$

The reader is referred to [25, Example 2.7] for the examples of the Oinarov kernels.

We now recall some well known boundedness results for the general Hardy-type operators with Oinarov kernels in the following. For any $s \in [0,\infty)$, we write

$$K_s f(x) = \int_0^x k(x,y)^s f(y)dy, \quad \tilde{K}_s f(y) = \int_y^\infty k(x,y)^s f(x)dx.$$

We have the following result for the boundedness of general Hardy-type operators on the weighted Lebesgue space.

Theorem 1.1. *Let $1 < p \le q < \infty$, $u, v : (0,\infty) \to [0,\infty)$ and $k : (0,\infty) \times (0,\infty) \to \mathbb{R}$ be an Oinarov kernel. If K, u and v satisfy*

$$\sup_{t>0} (\tilde{K}_q u)^{1/q} (K_0 v^{1-p'})^{1/p'}(t) < \infty \tag{1.1}$$

$$\sup_{t>0} (\tilde{K}_0 u)^{1/q}(t) (K_{p'} v^{1-p'})^{1/p'}(t) < \infty, \tag{1.2}$$

then there is a constant $C > 0$ such that for any $f \in L^p(v)$

$$\|Kf\|_{L^q(u)} \le C\|f\|_{L^p(v)}.$$

For the proof of the above result, the reader is referred to [25, Theorem 2.10]. We have the following results from [25, Theorem 2.15].

Theorem 1.2. *Let $1 < q < p < \infty$ and $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$. If k is an Oinarov kernel and*

$$\left(\int_0^\infty \left((\tilde{K}_q u)^{1/q}(t) (K_0 v^{1-p'})^{1/q'}(t) \right)^r v^{1-p'}(t) dt \right)^{\frac{1}{r}} < \infty, \tag{1.3}$$

$$\left(\int_0^\infty \left((\tilde{K}_0 u)^{1/p}(t) (K_{p'} v^{1-p'})^{1/p'}(t) \right)^r u(t) dt \right)^{\frac{1}{r}} < \infty, \tag{1.4}$$

then there is a constant $C > 0$ such that for any $f \in L^p(v)$

$$\|Kf\|_{L^q(u)} \le C\|f\|_{L^p(v)}.$$

The above theorems also give the results in [26] where $K(x,y) = \phi(y/x)$ and $\phi : (0,1) \to (0,\infty)$ is a Lebesgue measurable function. When $k(x,y) = g(x-y)$ for some Lebesgue measurable function $g : (0,\infty) \to (0,\infty)$, the above theorems extend the results in [36]. For the mapping properties of the general Hardy-type operators on weighted Herz spaces, the reader is referred to [24].

2. Main results

The main result of this paper is established in this section. We obtain the mapping properties of the general Hardy-type operators on the local generalized Morrey spaces built on ball quasi-Banach function spaces. Notice that the main result given in this section applies to a general kernel k, not necessary restricted to the Oinarov kernel.

We begin with the definition of the ball quasi-Banach function spaces introduced in [32].

Definition 2.2. *A quasi-Banach space $X \subset \mathcal{M}$ is a ball quasi-Banach function space if it satisfies*

1. *there is a constant $C > 0$ such that for any $f, g \in X$, $\|f+g\|_X \le C(\|f\|_X + \|g\|_X)$,*
2. *$\|f\|_X = 0$ if and only if $f = 0$ a.e. on $(0,\infty)$,*
3. *$0 \le g \le f$ and $f \in X$ implies $g \in X$ and $\|g\|_X \le \|f\|_X$,*
4. *$f_n \uparrow f$ and $f \in X$ implies $\|f_n\|_X \uparrow \|f\|_X$,*
5. *for any $E \in I$, we have $\chi_E \in X$.*

Whenever $\|\cdot\|_X$ satisfies (1)-(3) and

$$\chi_E \in \mathcal{M},\ |E| < \infty \Rightarrow \chi_E \in X, \tag{2.5}$$

X is called as a quasi-Banach function space.

Whenever $\|\cdot\|_X$ is a norm and for any $E \in I$, we have a constant $C > 0$ such that for any $f \in X$, we have $\int_E |f(x)|dx < \infty$, X is a ball Banach function space.

The family of the ball quasi-Banach function spaces includes a number of well known function spaces. The weighted Lebesgue spaces, the rearrangement-invariant quasi-Banach function spaces and the Lebesgue spaces with variable exponents are members of the ball quasi-Banach function spaces.

We now give the definition of the local generalized Morrey spaces built on ball quasi-Banach function spaces.

Definition 2.3. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. The local generalized Morrey space LM^X_ω consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM^X_\omega} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} f\|_X < \infty.$$

Whenever X is the Lebesgue space L^p, $p \in (1,\infty)$, LM^X_ω becomes the classical local generalized Morrey space.

The following results identify the conditions that ensure that LM^X_ω is a ball quasi-Banach function space.

Proposition 2.1. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. If ω and X satisfy*

$$1 \le C\omega(r), \quad r \in (1,\infty), \tag{2.6}$$

$$\|\chi_{(0,r)}\|_X \le C\omega(r), \quad r \in (0,1) \tag{2.7}$$

for some $C > 0$, then LM^X_ω is a ball quasi-Banach function space.

Proof. It is easy to see that LM^X_ω satisfies Items (1)-(3) in Definition 2.2. To obtain Item (4) of Definition 2.2, it suffices to show that for any $s > 0$, we have $\chi_{(0,s)} \in LM^X_\omega$.

When $r \in (1,\infty)$, (2.6) guarantees that

$$\frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)} \|\chi_{(0,s)}\|_X \le C\|\chi_{(0,s)}\|_X.$$

When $r \in (0,1)$, (2.7) yields

$$\frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)} \|\chi_{(0,r)}\|_X \le C.$$

The above inequalities assure that

$$\sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)}\chi_{(0,s)}\|_X \le C + C\|\chi_{(0,s)}\|_X$$

and, hence, $\chi_{(0,r)} \in LM^X_\omega$. □

We also have the following result with the range for r replaced by $\|\chi_{(0,r)}\|_X$.

Proposition 2.2. *Let X be a ball quasi-Banach function space and $\omega : (0,\infty) \to (0,\infty)$. If ω and X satisfy*

$$1 \le C\omega(r), \quad 1 < \|\chi_{(0,r)}\|_X, \tag{2.8}$$

$$\|\chi_{(0,r)}\|_X \le C\omega(r), \quad 1 \ge \|\chi_{(0,r)}\|_X \tag{2.9}$$

for some $C > 0$, then LM^X_ω is a ball quasi-Banach function space.

Proof. It suffices to show that for any $s > 0$, we have $\chi_{(0,s)} \in LM^X_\omega$.

When r satisfies $1 \le \|\chi_{(0,r)}\|_X$, (2.8) guarantees that

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)}\|\chi_{(0,s)}\|_X \le C\|\chi_{(0,s)}\|_X.$$

When r satisfies $1 \ge \|\chi_{(0,r)}\|_X$, (2.9) yields

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le \frac{1}{\omega(r)}\|\chi_{(0,r)}\|_X \le C.$$

The above inequalities assure that

$$\sup_{r>0}\frac{1}{\omega(r)}\|\chi_{(0,r)}\chi_{(0,s)}\|_X \le C + C\|\chi_{(0,s)}\|_X$$

and, hence, $\chi_{(0,r)} \in LM^X_\omega$. □

We write $(X,\omega) \in \mathcal{N}$ if LM^X_ω is nontrivial. The above propositions assure that $(X,\omega) \in \mathcal{N}$ whenever X and ω satisfy (2.6)-(2.7) or (2.8)-(2.9).

We now present the main result, the mapping properties of the general Hardy-type operators on the local generalized Morrey space LM^X_ω.

Theorem 2.3. *Let X and Y be ball quasi-Banach function spaces and $\omega : (0,\infty) \to (0,\infty)$. Let $k : (0,\infty)\times(0,\infty) \to \mathbb{R}$ be a Lebesgue measurable function satisfying $k(x,y) \ge 0$ when $0 < y < x$. If $(X,\omega) \in \mathcal{N}$ and there is a constant $C > 0$ such that for any $f \in X$*

$$\|Kf\|_Y \le C\|f\|_X,$$

then for any $f \in LM^X_\omega$

$$\|Kf\|_{LM^Y_\omega} \le C\|f\|_{LM^X_\omega}. \tag{2.10}$$

Proof. Let $r > 0$ and $f \in LM^X_\omega$. When $x > r$, we have

$$\chi_{(0,r)}(x)(K|f|)(x) = 0 \le \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy. \tag{2.11}$$

When $x \in (0,r]$, we have

$$\chi_{(0,r)}(x)(K|f|)(x) = \int_0^x k(x,y)|f(y)|dy = \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy \tag{2.12}$$

because for any $y \in (0,x)$, we have $y \in (0,r)$. Hence, $\chi_{(0,r)}(y) = 1$.

Consequently, (2.11) and (2.12) give

$$\chi_{(0,r)}(x)(K|f|)(x) \le \int_0^x \chi_{(0,r)}(y)k(x,y)|f(y)|dy = K(\chi_{(0,r)}|f|)(x). \tag{2.13}$$

By applying the quasi-norm $\|\cdot\|_Y$ on both sides of (2.13), item (2) of Definition 2.2 yields

$$\|\chi_{(0,r)}K|f|\|_Y \le \|K(\chi_{(0,r)}|f|)\|_Y.$$

The boundedness of $K : X \to Y$ and $|Kf| \le K|f|$ guarantee that

$$\|\chi_{(0,r)}Kf\|_Y \le C\|\chi_{(0,r)}f\|_X.$$

By multiplying $\frac{1}{\omega(r)}$ on both sides of the above inequality, we obtain

$$\frac{1}{\omega(r)}\|\chi_{(0,r)}Kf\|_Y \le C\frac{1}{\omega(r)}\|\chi_{(0,r)}f\|_X \le C\|f\|_{LM^X_\omega}.$$

Finally, by taking the supremum over $r > 0$, we have

$$\|Kf\|_{LM^Y_\omega} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} Kf\|_Y \le C\|f\|_{LM^X_\omega}.$$

□

The condition $(X, \omega) \in \mathcal{N}$ ensures that (2.10) is meaningful. In addition, the above result asserts that

$$\|K\|_{LM^X_\omega \to LM^Y_\omega} \le \|K\|_{X\to Y},$$

where $\|K\|_{LM^X_\omega \to LM^Y_\omega}$ and $\|K\|_{X\to Y}$ are the operator norms of $K : LM^X_\omega \to LM^Y_\omega$ and $K : X \to Y$, respectively.

Furthermore, the above result does not assume that k is an Oinarov kernel.

3. Applications

We give applications for Theorem 2.3 on some concrete function spaces and general Hardy-type operators in this section. We study the general Hardy-type operators with Oinarov kernel on the weighted local generalized Morrey spaces, the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces. We also establish the Hardy-type inequalities on the local generalized Morrey spaces with variable exponents.

3.1. **Weighted local generalized Morrey spaces.** We extend the mapping properties of the general Hardy-type operators with Oinarov kernel on weighted local generalized Morrey spaces in this section.

Definition 3.4. *Let $p \in (0,\infty)$, v be a locally integrable function and $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. The weighted local generalized Morrey space $LM^p_{v,\omega}$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM^p_{v,\omega}} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} f\|_{L^p(v)} < \infty.$$

Proposition 3.3. *Let $p \in (0,\infty)$, v be a locally integrable function and $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If ω satisfies (2.6) and*

$$\left(\int_0^r v(x)dx\right)^{\frac{1}{p}} \le C\omega(r), \quad r \in (0,1) \tag{3.14}$$

for some $C > 0$, then $LM^p_{v,\omega}$ is a ball quasi-Banach function space.

Proof. As v is a local integrable function, for any $E \in I$, $\int_I v(x)dx < \infty$, we see that $L^p(v)$ is a ball quasi-Banach function space. According to Proposition 2.1, as ω satisfies (2.6) and (3.14), $LM^p_{v,\omega}$ is also a ball quasi-Banach function space. □

Proposition 3.3 guarantees that when v and ω satisfy (2.6) and (3.14), the weighted local generalized Morrey space $LM^p_{v,\omega}$ is nontrivial.

In particular, if $\theta \in (0,1)$ and $\omega_\theta(r) = \left(\int_0^r v(x)dx\right)^{\frac{\theta}{p}}$, then (3.14) is fulfilled and LM^p_{v,ω_θ} is a ball quasi-Banach function space.

Theorems 1.1, 1.2 and 2.3 give the mapping properties of the general Hardy-type operators on the weighted local generalized Morrey spaces.

Theorem 3.4. *Let $p, q \in (1,\infty)$, $u, v : (0,\infty) \to [0,\infty)$ be locally integrable functions, $\omega : (0,\infty) \to (0,\infty)$ satisfy (2.6) and (3.14) and k be a Oinarov kernel.*

(1) If $p \le q$ and K, u and v satisfy (1.1) and (1.2), then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Kf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

(2) If $q \le p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and K, u and v satisfy (1.3) and (1.4), then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Kf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

We now apply the above theorem to establish the mapping properties of the Hardy operator H on the weighted local generalized Morrey spaces. Recall that

$$\tilde{H}f(x) = \int_x^\infty f(y)dy.$$

Theorem 3.5. *Let $p, q \in (1, \infty)$, $u, v : (0, \infty) \to [0, \infty)$ be locally integrable functions and $\omega : (0, \infty) \to (0, \infty)$ satisfy (2.6) and (3.14).*

(1) If $p \le q$ and

$$\sup_{t>0} \left(\int_t^\infty u(y)dy \right)^{\frac{1}{q}} \left(\int_0^t v(y)^{1-p'}dy \right)^{\frac{1}{p'}} < \infty, \tag{3.15}$$

then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Hf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

(2) If $q \le p$, $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$ and

$$\left(\int_0^\infty \left((\tilde{H}u)^{1/q}(t)(Hv^{1-p'})^{1/q'}(t) \right)^r v^{1-p'}(t)dt \right)^{\frac{1}{r}} < \infty, \tag{3.16}$$

$$\left(\int_0^\infty \left((\tilde{H}u)^{1/p}(t)(Hv^{1-p'})^{1/p'}(t) \right)^r u(t)dt \right)^{\frac{1}{r}} < \infty, \tag{3.17}$$

then there is a constant $C > 0$ such that for any $f \in LM^p_{v,\omega}$, we have

$$\|Hf\|_{LM^q_{u,\omega}} \le C\|f\|_{LM^p_{v,\omega}}.$$

The above results are extensions of the two weight norm inequalities of the Hardy operator to the local generalized Morrey spaces.

We consider the case $q \in (0, 1)$ in the following. We first recall the mapping properties of H on the weighted Lebesgue spaces from [35, Theorem 1 (3)].

Theorem 3.6. *Let $0 < q < 1 < p < \infty$ and $u, v : (0, \infty) \to (0, \infty)$ be Lebesgue measurable functions. If u, v satisfy*

$$\int_0^\infty \left(\int_0^t (u(y))^{1-p'}dy \right)^{\frac{r}{p'}} \left(\int_t^\infty v(y)dy \right)^{\frac{r}{p}} dt < \infty, \tag{3.18}$$

where $\frac{1}{r} = \frac{1}{q} - \frac{1}{p}$, then

$$\left(\int_0^\infty |Hf(t)|^q v(t)dt \right)^{\frac{1}{q}} \le C \left(\int_0^\infty |f(t)|^p u(t)dt \right)^{\frac{1}{p}}.$$

We now have the mapping properties of operator H on the weighted local generalized Morrey spaces.

Theorem 3.7. *Let $0 < q < 1 < p < \infty$ and $u, v, \omega : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. If u, v satisfy (3.18) and ω satisfies (2.6) and (3.14), then there is a constant $C > 0$ such that for any $f \in LM^p_{u,\omega}$*

$$\|Hf\|_{L^q_{v,\omega}} \leq C\|f\|_{LM^p_{u,\omega}}. \tag{3.19}$$

Proof. As $Hf(x) = \int_0^x f(y)dy$, H is a general Hardy operator with kernel $k(x, y) \equiv 1$. The preceding theorem asserts that $H : L^p(u) \to L^q(v)$ is bounded. Moreover, $(L^p, \omega) \in \mathcal{N}$ because Proposition 3.3 assures that $LM^p_{u,\omega}$ is a ball quasi-Banach function space. Thus, Theorem 2.3 yields (3.19). □

The above result shows that our main result also applies to local generalized Morrey spaces built on quasi-Banach function space X.

3.2. **Local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces.** In this section, we apply Theorem 2.3 to establish the mapping properties of the Riemann-Liouville integral on the local generalized Morrey spaces built on rearrangement-invariant quasi-Banach function spaces.

We first recall some notations for defining of the rearrangement-invariant quasi-Banach function spaces. For any $f \in \mathcal{M}$ and $s > 0$, write

$$d_f(s) = |\{x \in (0,\infty) : |f(x)| > s\}|$$

and

$$f^*(t) = \inf\{s > 0 : d_f(s) \leq t\}, \quad t > 0.$$

We recall the definition of rearrangement-invariant quasi-Banach function space (r.i.q.B.f.s.) from [17, Definition 2.1].

Definition 3.5. *A quasi-Banach function space X is said to be a r.i.q.B.f.s. if there exists a quasi-norm ρ_X satisfying Items (1)-(3) of Definition 2.2 and (2.5) such that for any $f \in X$, we have*

$$\|f\|_X = \rho_X(f^*).$$

Next, we recall the definition of the Boyd indices. For any $s \geq 0$ and $f \in \mathcal{M}(0,\infty)$, define $(D_s f)(t) = f(st)$, $t \in (0,\infty)$. Let $\|D_s\|_{X\to X}$ be the operator norm of D_s on X. We recall the definition of Boyd indices for r.i.q.B.f.s. from [27].

Definition 3.6. *Let X be a r.i.q.B.f.s. Define the lower Boyd index of X, p_X, and the upper Boyd index of X, q_X, by*

$$p_X = \sup\{p > 0 : \exists C > 0 \text{ such that } \forall\, 0 \leq s < 1,\ \|D_s\|_{X\to X} \leq Cs^{-1/p}\},$$
$$q_X = \inf\{q > 0 : \exists C > 0 \text{ such that } \forall\, 1 \leq s,\ \|D_s\|_{X\to X} \leq Cs^{-1/q}\},$$

respectively.

It is easy to see that the Boyd indices for the Lebesgue space L^p, $0 < p < \infty$ is $\frac{1}{p}$.

Proposition 3.4. *Let X be a r.i.q.B.f.s. and $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function. If ω satisfies (2.6) and there exists a $q > q_X$ and $C > 0$ such that*

$$r^{\frac{1}{q}} < C\omega(r), \quad r \in (0,1), \tag{3.20}$$

then LM^X_ω is a ball quasi-Banach function space.

Proof. As $D_{1/r}\chi_{(0,1)}(t) = \chi_{(0,1)}(t/r) = \chi_{(0,r)}(t)$, for any $r \in (0,1)$, we find that for any $q > q_X$, we have a constant $C > 0$ such that

$$\|\chi_{(0,r)}\|_X = \|D_{1/r}\chi_{(0,1)}\|_X \le Cr^{1/q}.$$

The above inequality and (3.20) guarantee that (2.7) is satisfied. Therefore, Proposition 2.1 asserts that LM^X_ω is a ball quasi-Banach function space. □

We need the following function space for the studies of the Riemann-Liouville integral.

Definition 3.7. *Let $\alpha \ge 0$ and X be a r.i.q.B.f.s. X_α consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{X_\alpha} = \rho_X(t^{-\alpha} f^*(t)) < \infty.$$

For instance, when $X = L^p$, X_α is the Lorentz spaces $L^{\frac{p}{1-p\alpha},\alpha}$, see [17, p.901]. Moreover, X_α has been used in [17, 18] for the studies of the mapping properties of the convolution operators, the Fourier integral operators and the k-plane transform on r.i.q.B.f.s.

We have the following result from [17, Proposition 3.1].

Proposition 3.5. *Let $\alpha > 0$ and X be a r.i.q.B.f.s. If $0 < p_X \le q_X < \frac{1}{\alpha}$, then X_α is a r.i.q.B.f.s.*

Theorem 3.8. *Let $\alpha > 0$, $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function and X be a r.i.q.B.f.s. If $0 < p_X \le q_X < \frac{1}{\alpha}$ and ω satisfies (2.6) and (3.20) for some $q > q_X$, then there is a constant $C > 0$ such that for any $f \in LM^X_\omega$*

$$\|R_\alpha f\|_{LM^{X_\alpha}_\omega} \le C\|f\|_{LM^X_\omega}.$$

Proof. It is well known that for any $p \in (1, \frac{1}{\alpha})$, $R_\alpha L^p \to L^q$ is bounded where $\frac{1}{q} = \frac{1}{p} - \alpha$. By applying [18, Theorem 4.1], we find that $R_\alpha : X \to X_\alpha$ is bounded. Consequently, as Riemann-Liouville integral is a member of general Hardy-type operator, Theorem 2.3 yields the boundedness of $R_\alpha : LM^X_\omega \to LM^{X_\alpha}_\omega$. □

The above result is new even for the local generalized Morrey space LM^p_ω. Notice that we have the following inequality

$$|R_\alpha f(x)| \le \int_0^\infty \frac{|f(y)|}{|x-y|^{1-\alpha}} dy = (I_\alpha |f|)(x), \quad x \in (0,\infty),$$

where I_α is the fractional integral operator on $(0,\infty)$. Therefore, by using the idea in [21, Theorem 3.1], we can obtain the mapping properties I_α and hence, the mapping properties of R_α on LM^X_ω. While by using the idea in [21, Theorem 3.1], we need to impose a stronger condition on ω, a condition similar to [21, (2.10)].

We now give another concrete example for Theorem 3.8. A function $\Phi : [0,+\infty] \to [0,+\infty]$ is a Young function if there exists an increasing and left-continuous function ϕ satisfying $\phi(0) = 0$ and that ϕ is neither identically zero nor identically infinite such that

$$\Phi(s) = \int_0^s \phi(u)du, \quad s \ge 0.$$

Let ϕ be a Young function. The Orlicz space L_Φ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi\left(\frac{|f(x)|}{\lambda}\right)dx \le 1\right\} < \infty.$$

Let $\alpha \in \mathbb{R}$ and Φ be a Young function. The Lorentz-Orlicz space $L_{\Phi,\alpha}$ consists of all Lebesgue measurable functions f satisfying

$$\|f\|_{L_{\Phi,\alpha}} = \inf\left\{\lambda > 0 : \int_0^\infty \Phi(t^{-\alpha/n} f^*(t)/\lambda)dt \le 1\right\} < \infty.$$

In view of [3, Chapter 4, Theorem 8.18], the Boyd indices of L_Φ are given by

$$p_{L_\Phi} = \lim_{t\to\infty} \frac{\log t}{\log g(t)}, \quad \text{and} \quad q_{L_\Phi} = \lim_{t\to 0^+} \frac{\log t}{\log g(t)},$$

where

$$g(t) = \limsup_{s\to\infty} \frac{\Phi^{-1}(s)}{\Phi^{-1}(s/t)}.$$

Theorem 3.8 yields the following mapping properties of the Riemann-Liouville integral on the local Orlicz-Morrey space $M_\omega^{L_\Phi}$.

Corollary 3.1. *Let $\alpha > 0$, $\omega : (0,\infty) \to (0,\infty)$ be a Lebesgue measurable function and Φ be a Young function. If $0 < p_{L_\Phi} \le q_{L_\Phi} < \frac{1}{\alpha}$ and ω satisfies (2.6) and (3.20) for some $q > q_{L_\Phi}$, then there is a constant $C > 0$ such that for any $f \in LM_\omega^X$*

$$\|R_\alpha f\|_{LM_\omega^{L_\Phi,\alpha}} \le C\|f\|_{LM_\omega^{L_\Phi}}.$$

For the studies of boundedness of the Calderón-Zygmund operators, the nonlinear commutators of the Calderón-Zygmund operators, the oscillatory singular integral operators, the singular integral operators with rough kernels and the Marcinkiewicz integrals on the local Orlicz-Morrey spaces on the local Orlicz-Morrey spaces, the reader is referred to [39].

3.3. **Local generalized Morrey spaces with variable exponents.** In this section, we extend the Hardy-type inequalities in [12] to the local generalized Morrey spaces with variable exponent. Notice that the kernel for the operators studied in this section is not necessarily an Oinarov kernel. Thus, the results in this section give examples for the use of Theorem 2.3 is not restricted to the general Hardy-type operators with Oinarov kernels.

We begin with the definition of the Lebesgue space with variable exponent.

Definition 3.8. *Let $p(\cdot) : (0,\infty) \to [1,\infty)$ be a Lebesgue measurable function. The Lebesgue space with variable exponent $L^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{L^{p(\cdot)}} = \inf\left\{\lambda > 0 : \int_0^\infty \left|\frac{f(x)}{\lambda}\right|^{p(x)} dx \le 1\right\} < \infty.$$

For any Lebesgue measurable function $p(\cdot) : (0,\infty) \to (0,\infty)$, define

$$p_- = \inf_{x\in(0,\infty)} p(x) \quad \text{and} \quad p_+ = \sup_{x\in(0,\infty)} p(x).$$

For any Lebesgue measurable function $p(\cdot) : (0,\infty) \to (0,\infty)$, we write $p(\cdot) \in \mathcal{M}_{0,\infty}$ if there exists a constant $C > 0$ such that

(1) $0 \le p_- \le p_+ < \infty$,
(2) the limit $\lim_{x\to 0} p(x)$ exists, $p(0) = \lim_{x\to 0} p(x)$ and

$$|p(x) - p(0)| \le \frac{C}{-\ln x}, \quad \forall x \in (0, 1/2],$$

(3) the limit $\lim_{x\to\infty} p(x) = p_\infty$ exists and

$$|p(x) - p_\infty| \le \frac{C}{\ln x} \quad \forall x \in [2,\infty).$$

We write $p(\cdot) \in \mathcal{P}_{0,\infty}$ if $p(\cdot) \in \mathcal{M}_{0,\infty}$ and $p_- \geq 1$.

We have the following result from [12, Theorems 3.1, 3.3 and Remark 3.2]. To simplify the presentation of the results in the following, for any $a \in \mathbb{R}$ and Lebesgue measurable functions $\alpha(\cdot)$ and $\mu(\cdot)$ on $(0,\infty)$, we write

$$\mathcal{H}^{\alpha(\cdot)} f(x) = x^{\alpha(x)-1} \int_0^x \frac{f(y)}{y^{\alpha(y)}} dy$$

$$\mathcal{H}^{a}_{\mu(\cdot)} f(x) = x^{a+\mu(x)-1} \int_0^x \frac{f(y)}{y^{a}} dy,$$

respectively.

Theorem 3.9. *Let $p(\cdot) \in \mathcal{P}_{0,\infty}$ and $\alpha(\cdot)$ be a bounded function on $(0,\infty)$ such that the limit $\lim_{x\to\infty} \alpha(x) = \alpha_\infty$ exists and satisfies*

$$\alpha(0) < 1 - \frac{1}{p(0)}, \quad \alpha_\infty < 1 - \frac{1}{p_\infty} \tag{3.21}$$

$$|\alpha(x) - \alpha(0)| \leq \frac{C}{|\ln x|}, \quad \forall x \in (0, 1/2], \tag{3.22}$$

$$|\alpha(x) - \alpha_\infty| \leq \frac{C}{\ln x}, \quad \forall x \in (2, \infty) \tag{3.23}$$

for some $C > 0$, then there exists a constant $D > 0$ such that for any $f \in L^{p(\cdot)}$, we have

$$\|\mathcal{H}^{\alpha(\cdot)} f\|_{L^{p(\cdot)}} \leq D \|f\|_{L^{p(\cdot)}}.$$

Theorem 3.10. *Let $a \in \mathbb{R}$, $p(\cdot), \mu(\cdot) : (0,\infty) \to [1,\infty)$ be Lebesgue measurable functions. If*

$$a < \min\left\{1 - \frac{1}{p(0)}, 1 - \frac{1}{p_\infty}\right\}, \tag{3.24}$$

$p(\cdot) \in \mathcal{P}_{0,\infty}$, $\mu(\cdot) \in \mathcal{M}_{0,\infty}$,

$$0 \leq \mu(0) < \frac{1}{p(0)} \quad \text{and} \quad 0 \leq \mu_\infty < \frac{1}{p_\infty}, \tag{3.25}$$

then for any $q(\cdot) \in \mathcal{P}_{0,\infty}$ satisfying

$$\frac{1}{q(0)} = \frac{1}{p(0)} - \mu(0) \quad \text{and} \quad \frac{1}{q_\infty} = \frac{1}{p_\infty} - \mu_\infty, \tag{3.26}$$

we have a constant $D > 0$ such that for any $f \in L^{p(\cdot)}$

$$\|\mathcal{H}^{a}_{\mu(\cdot)} f\|_{L^{q(\cdot)}} \leq D \|f\|_{L^{p(\cdot)}}$$

The above results are generalizations of the Hardy inequalities to the Lebesgue spaces with variable exponents. For the proofs of the above theorems, the reader is referred to [12, Sections 5 and 6].

Notice that the kernels of the operators $\mathcal{H}^{\alpha(\cdot)}$ and $\mathcal{H}^{a}_{\mu(\cdot)}$ are

$$K_1(x,y) = \frac{x^{\alpha(x)-1}}{y^{\alpha(y)}} \quad \text{and} \quad K_2(x,y) = \frac{x^{a+\mu(x)-1}}{y^{a}},$$

respectively. In general, they are not non-decreasing in x nor non-increasing in y, therefore, they do not satisfy Item (1) of Definition 1.1.

Even though the operators $\mathcal{H}^{\alpha(\cdot)}$ and $\mathcal{H}^{a}_{\mu(\cdot)}$ were not covered by the results in Theorems 1.1 and 1.2, our main result, Theorem 2.3 also yields the mapping properties of these operators on the local generalized Morrey spaces with variable exponents.

Definition 3.9. *Let $p(\cdot) : (0,\infty) \to [1,\infty)$ be a Lebesgue measurable function. The local generalized Morrey space with variable exponent $LM_\omega^{p(\cdot)}$ consists of all $f \in \mathcal{M}$ satisfying*

$$\|f\|_{LM_\omega^{p(\cdot)}} = \sup_{r>0} \frac{1}{\omega(r)} \|\chi_{(0,r)} f\|_{L^{p(\cdot)}} < \infty.$$

When $\omega \equiv 1$, $LM_\omega^{p(\cdot)}$ becomes the Lebesgue space with variable exponent $L^{p(\cdot)}$. Moreover, the local generalized Morrey spaces with variable exponents are extensions of the local generalized Morrey spaces.

For the mapping properties of the fractional integral operators, the maximal Carleson operator, the spherical maximal functions, the geometric maximal functions and the minimal functions on the local generalized Morrey spaces with variable exponent and the Hardy local generalized Morrey spaces with variable exponents, the reader is referred to [19, 22, 23, 38, 40].

Let $p(\cdot) \in \mathcal{P}_{0,\infty}$. Whenever ω satisfies

$$1 \le C\omega(r), \quad \|\chi_{(0,r)}\|_{L^{p(\cdot)}} > 1, \tag{3.27}$$

$$r^{\frac{1}{p_+}} \le C\omega(r), \quad 1 \ge \|\chi_{(0,r)}\|_{L^{p(\cdot)}} \tag{3.28}$$

for some $C > 0$, Proposition 2.2 and [10, Corollary 2.23] guarantee that LM_ω^X is a ball quasi-Banach function space.

We now present the boundedness of $\mathcal{H}^{\alpha(\cdot)}$ on the local generalized Morrey spaces with variable exponents in the following.

Theorem 3.11. *Let $p(\cdot) \in \mathcal{P}_{0,\infty}$, $\omega : (0,\infty) \to (0,\infty)$ be Lebesgue measurable function and $\alpha(\cdot)$ be a bounded function. Suppose that ω satisfies (3.27) and (3.28). If $\alpha(\cdot)$ satisfies (3.21), (3.22) and (3.23), then there is a constant $C > 0$ such that for any $f \in LM_\omega^{p(\cdot)}$*

$$\|\mathcal{H}^{\alpha(\cdot)} f\|_{LM_\omega^{p(\cdot)}} \le C\|f\|_{LM_\omega^{p(\cdot)}}.$$

The above result is a consequence of Theorems 2.3 and 3.9.

Next, we have the mapping properties of $\mathcal{H}_{\mu(\cdot)}^a$ in the local generalized Morrey spaces with variable exponents. The following theorem is guaranteed by Theorems 2.3 and 3.10.

Theorem 3.12. *Let $a \in \mathbb{R}$, $p(\cdot) \in \mathcal{P}_{0,\infty}$, $\mu(\cdot) \in \mathcal{M}_{0,\infty}$ and $\omega(\cdot) : (0,\infty) \to (0,\infty)$ be Lebesgue measurable functions. If a, $p(\cdot)$, $\mu(\cdot)$ and $\omega(\cdot)$ satisfy (3.24), (3.25), (3.27) and (3.28), then for any $q(\cdot) \in \mathcal{P}_{0,\infty}$ satisfying (3.26), there is a constant $C > 0$ such that for any $f \in LM_\omega^{p(\cdot)}$*

$$\|\mathcal{H}_{\mu(\cdot)}^a f\|_{LM_\omega^{q(\cdot)}} \le C\|f\|_{LM_\omega^{p(\cdot)}}.$$

Theorems 3.11 and 3.12 are extensions of [12, Theorems 3.1 and 3.3] from the Lebesgue spaces with variable exponents to the local generalized Morrey spaces with variable exponents.

We now give some concrete examples on ω such that (3.27) and (3.28) are fulfilled. For any $\theta \in (0, \frac{1}{p_+})$, define $\omega_\theta(r) = r^\theta$. In view of [10, Corollary 2.23], we have $\lim_{r\to\infty} \|\chi_{(0,r)}\|_{L^{p(\cdot)}} = \infty$, therefore, ω_θ fulfills (3.27). Moreover, as $p_+ < \infty$, [10, Theorems 2.58 and 2.62] assure that $\|\cdot\|_{L^{p(\cdot)}}$ is an absolutely continuous norm. Thus, $\lim_{r\to 0} \|\chi_{(0,r)}\|_{L^{p(\cdot)}} = 0$. Consequently, (3.28) is fulfilled.

Whenever $p(\cdot)$ satisfies the conditions in Theorem 3.11, $\mathcal{H}^{\alpha(\cdot)}$ is bounded on $LM_{\omega_\theta}^{p(\cdot)}$. Similarly, whenever a, $p(\cdot)$, $q(\cdot)$ and $\mu(\cdot)$ satisfy the conditions in Theorem 3.12, $\mathcal{H}_{\mu(\cdot)}^a : LM_{\omega_\theta}^{p(\cdot)} \to LM_{\omega_\theta}^{q(\cdot)}$ is bounded.

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