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A Method for n-Dimensional Reconstruction of Scalar Field Proposals

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Abstract: In the present study, we mainly implement a correspondence between two different concepts of dark energy via the reverse engineering technique. Expressing the self-potential of scalar fields in analytical form is a very important and mathematically challenging problem. In conclusion, after performing the required mathematical calculations, we show that one can find exact expressions for a scalar field function and its self-interacting potential by making use of such correspondence.

Skaler Alan Tasarılarının n-Boyutlu İnşaası İçin Bir Yöntem

Anahtar Kelimeler
Skaler alan, Karanlık enerji, Einstein denklemleri.

Öz: Bu çalışmada, esas olarak iki farklı karanlık enerji kavramı arasında tersine mühendislik tekniği ile bir ilişki kurulmuştur. Skaler alanların öz potansiyelini analitik formda ifade etmek oldukça önemli ve matematiksel zorluklar içeren bir problemdir. Sonuç olarak, gerekli matematiksel hesaplamaları tamamladıktan sonra, bu tür eşleştirmelerden yararlanarak bir skaler alan fonksiyonu ve onun öz-etkileşim potansiyeli için kesin ifadelerin bulunabileceği gösterilmiştir.

1. INTRODUCTION

It is generally known that, in a free fall motion, a particle follows a geodesic in spacetime. The concepts of geodesic and spacetime are linked by the metric:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu, \qquad (1.1)$$

where ds^2, $g_{\mu\nu}$ and x^μ respectively indicate the proper interval, metric tensor and a four-vector. It is significant to emphasize here that we have three possible kinds of proper interval:

1) $\mathrm{ds}^2 < 0$ ***: spacelike,***
2) $\mathrm{ds}^2 = 0$ ***: lightlike (or null),***
3) $\mathrm{ds}^2 > 0$ ***: timelike.***

In the present work, we focus on an n-dimensional Friedmann-Robertson-Walker (nFRW henceforth) type Kaluza-Klein metric [1]:

$$ds^2 = dt^2 - b^2(t)\left[\frac{dr^2}{1-kr^2} + r^2 dx_m^2\right], \qquad (1.2)$$

where $n = m + 2$ is the number of spacetime dimension and

$$dx_m^2 = dy_1^2 + \sin^2 y_1 dy_2^2 + \ldots + \sin^2 y_{m-1} dy_m^2. \qquad (1.3)$$

It is understood that the above interval represents an isotropic and homogenous universe [1]. Moreover, $b(t)$ shows the cosmic scale factor and k is known as the spatial curvature parameter, which can be normalized to three different values -1, 0, $+1$. Here, the case $k = 0$ indicates that $(m + 2)$-dimensional space-time has a flat fabric. On the other hand, $k = -1$ and $k = +1$ cases denote open and closed manifolds, respectively.

In 1915, making use of the diverse frameworks of classical mechanics, Einstein introduced a field equation describing the geometry of spacetime [2]:

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = \kappa\{(\rho + p)u_\mu u_\nu - p g_{\mu\nu}\}, \qquad (1.4)$$

where $R_{\mu\nu}$ is the Ricci tensor, R denotes the curvature scalar, ρ represents the energy density and p indicates the pressure. Also, $\kappa = 8\pi G$, where G is known as the universal gravitational constant. As a matter of fact, Einstein field equations was described by considering $(3+1)$-dimensional Riemannian geometry. Besides, the most recent cosmological observations strongly indicates an accelerated expansion epoch of the universe [3-8] and a large number of scientists believe that dark energy and dark matter are responsible for this mysterious nature of cosmos. In this context, extra dimensional point of view is one of the most studied theoretical candidates of the dark energy [9]. In addition to this idea, there are some other theoretical candidates: scalar fields [10-12], modified gravity [13-16], unified energy densities [17-19] etc. In this work, we mainly discuss our attention on investigating the dynamical evolution of a scalar field dark energy model from the nFRW perspective. As we mentioned before, the scalar field idea is one of the most significant dark energy candidates, which means these proposals may be taken into account as an effective description of the accelerated expansion phase of the cosmos. In physics, there are various scalar field definitions, but, it is so hard to get an exact relation for the self-interacting potential of a scalar field by making use of fundamental theories. On this purpose, we perform an investigation in order to obtain exact expressions for the self-interacting potential of a scalar field model via n-dimensional framework.

2. MATERIAL AND METHOD

Theorem 2.1. ***n-dimensional description of a scalar field model can be obtained via the Reverse Engineering method. Thus, one can reconstruct any scalar field model with the help of this technique.***

Proof. First of all, we focus on the n-dimensional field equations [1]:

$$\frac{(n-2)(n-1)}{2}\{H^2 + \frac{k}{b^2}\} = \kappa\rho, \tag{2.1}$$

$$(n-2)\{\dot{H} + H^2\} + \frac{(n-2)(n-3)}{2}\{H^2 + \frac{k}{b^2}\} = -\kappa p. \tag{2.2}$$

Here, $H = \frac{\dot{b}}{b}$ is the cosmic Hubble parameter and the dot indicates a derivative with respect to the cosmic time t. In order to write a relation for the equation-of-state (EoS henceforth) parameter, i.e. $\omega = \frac{p}{\rho}$, we can use equations (2.1) and (2.2). It is generally known that this parameter yields very interesting conclusions in modern cosmology. From this point of view, we need this quantity in further calculations. So, making use of equations (2.1) and (2.2) gives

$$\omega = -\frac{(n-2)\{\dot{H} + H^2\} + \frac{(n-2)(n-3)}{2}\{H^2 + \frac{k}{b^2}\}}{\frac{(n-2)(n-1)}{2}\{H^2 + \frac{k}{b^2}\}}. \tag{2.3}$$

It is important to mention at this point that the condition for the late-time speedy expansion epoch is $\omega < -\frac{1}{3}$ [20].

Now, we are in a position to consider scalar field dark energy models and we use the quintessence type scalar field description [12] as an example in this section. For the quintessence field, the corresponding energy density and pressure are written [21-22] respectively as given below;

$$\rho_Q = \frac{\dot{\varphi}^2}{2} + \nu(\varphi), \tag{2.4}$$

$$p_Q = \frac{\dot{\varphi}^2}{2} - \nu(\varphi). \tag{2.5}$$

Therefore, it can be written that

$$\omega_Q = \frac{\dot{\varphi}^2 - 2\nu}{\dot{\varphi}^2 + 2\nu}. \tag{2.6}$$

One can see from here that the universe has a speedy expansion phase for the case $\dot{\varphi}^2 < \nu$ if we have the condition $\omega < -\frac{1}{3}$ [22-23].

At this step, we can now perform the reverse engineering method for the n-dimensional description of the quintessence scalar field model. Subsequently, we should assume $\rho_Q = \rho$ and $\omega_Q = \omega$, because both of these scenarios are proposed to explain nature of the dark energy. Consequently, from equations (2.4) and (2.5) together with $p = \rho\omega$, we get

$$\dot{\varphi}^2 = (1 + \omega_Q)\rho_Q = (1 + \omega)\rho, \tag{2.7}$$

$$\nu(\varphi) = \frac{1}{2}(1 - \omega_Q)\rho_Q = \frac{1}{2}(1 - \omega)\rho. \tag{2.8}$$

Now, we substitute expressions (2.1), (2.2) and (2.3) into the above results which lead to

$$\dot{\varphi}^2 = \frac{(n-2)}{\kappa b^2}\left[k + \dot{b}^2 - b\ddot{b}\right], \tag{2.9}$$

$$\nu = \frac{(n-2)}{2\kappa b^2}\left[(n-2)(k + \dot{b}^2) + b\ddot{b}\right]. \tag{2.10}$$

As it is seen from the above expressions, we can obtain an exact relation for the self-interacting potential via the reverse engineering method.

3. RESULTS

In the previous section, we give a correspondence between the quintessence proposal and the n FRW scenario. It is important to emphasize here that our investigation can be extended by considering other well-known dark energy descriptions such as the tachyon, dilaton and the $k-$essence type scalar field models etc.

(i) $n-$dimensional tachyonic model

$$\dot{\varphi}^2 = (1+\omega_T) = (1+\omega), \qquad (3.1)$$

$$v = \rho_T (1-\dot{\varphi}^2)^{\frac{1}{2}} = \rho\sqrt{-\omega}. \qquad (3.2)$$

(ii) $n-$dimensional dilaton field

$$\frac{h}{2}\dot{\varphi}^2 e^{\delta\varphi} = \frac{\omega_{n-1}}{3\omega_{n-1}} = \frac{\omega-1}{3\omega-1}, \qquad (3.3)$$

where h and δ are positive constants.

(iii) $n-$dimensional $k-$essence definition,

$$f(\varphi) = \frac{\rho_k}{3\chi^2-\chi} = \frac{\rho}{3\chi^2-\chi}, \qquad (3.4)$$

where

$$\chi = \frac{\omega_k - 1}{3\omega_k - 1} = \frac{\omega-1}{3\omega-1}. \qquad (3.5)$$

4. DISCUSSION AND CONCLUSION

It is known that our universe expands with an accelerated velocity and there is no theory explaining the entire nature of dark universe successfully. In this context, the scalar field models and extra dimensional frameworks are possible theoretical candidates [13,14,16-18,24,25]. In the present work, we have shown that one can redefine any scalar field dark energy models in the n FRW type framework and obtain exact expressions for the corresponding scalar field and its self-interacting potential. It can be said that such calculations are important to discuss how various cosmological proposals are related to each other.

REFERENCES

[1] Singh GP, Kotambkar S, Srivastava D, Pradhan A, A New Class of Higher Dimensional Cosmological Models of Universe with Variable G and Λ terms, Rom. Journ. Phys., Vol. 53, Nos. 3â€``4, (2008), 607â€``618.

[2] Einstein A, Zur Allgemeinen Relativitätstheorie, Preussische Akademie der Wissenschaften., 47, (1915), 778.

[3] Ade PAR, et al., Planck 2013 results. XII. Diffuse component separation A & A 571, A12 (2014), DOI:10.1051/0004-6361/201321580.

[4] Ade PAR, et al., Planck 2015 results. XIII. Cosmological parameters A&A 594, A13 (2016), DOI: 10.1051/0004-6361/201525830.

[5] Allen SW, et al., Constraints on dark energy from Chandra observations of the largest relaxed galaxy clusters, Mon. Not. R. Astron. Soc., 353 (2) (2004), 457-467.

[6] Bennet CL, et al., First Year Wilkinson Microwave Anisotropy Probe (WMAP) Observations: Preliminary Maps and Basic Results, Astrophys. J. Suppl., (2003), 148, 1.

[7] Perlmutter S, et al., Measurements of Ω and Λ from 42 High-Redshift Supernovae, Astrophys. J., (1998), 517-565.

[8] Tegmark M, et al., Cosmological Parameters from SDSS and WMAP, Phys. Rev. D., (2004), 69, 103501.

[9] Keskin AI, Saltı M, Cosmographic nature of the early universe from extra dimensional perspective, Physics Letters B., 791, (2019),

[10] doi.org/10.1016/j.physletb.2019.02.024, 80-85.

[11] Caldwell RR, A Phantom Menace? Cosmological consequences of a dark energy component with super-negative equation of state, Phys.Lett. B., 545, (2002), 23-29.

[12] Chiba T, Okabe T and Yamaguchi M, Kinetically driven quintessence, Phys. Rev. D., 62, 023511(2000).

[13] Copeland EJ, Sami M and Tsujikawa S, Dynamics of Dark Energy, Int. Journal of Modern Phys. D., 15, 11 (2006), 1753-1935.

[14] Abedi H and M. Saltı M, Multiple field modified gravity and localized energy in teleparallel framework, Gen. Rel. Grav., 47:93, (2015).

[15] Askin M, Abedi H and Saltı M, Thermodynamics in $f(T,\theta)$ Gravity, Rom. J. Phys., 60 (2015), 44-55.

[16] Boisseau B, et al., Reconstruction of a Scalar-Tensor Theory of Gravity in an Accelerating Universe, Phys Rev Lett., 2000, 85(11):2236-9.

[17] Capozziello S, et al., $f(R)$ Cosmology with Torsion,â€,Physica Scripta, 78(6) 2008.

[18] Askin M, Salti M and Aydogdu O, Cosmology via thermodynamics of polytropic gas, Modern Physics Letters A., 32, 1750177, 2017.

[19] Christensen Dalsgaard J, Lecture Notes on Stellar Structure and Evolution, Aarhus Univ. Press, 2004.

[20] Yu A, Kamenshchik A, Moschella U, Pasquier V, An alternative to quintessence, Physics Letters B., 511, 2001, 265-268.

[21] Tripathi A, Sangwan A and Jassal HK, Dark energy equation of state parameter and its evolution at low redshift, Journal of Cosmology and Astroparticle Physics, 06 (2017) 012.

[22] Carrillo JAE, Silva JM and Lima JAS, Astronomy and Relativistic Astrophysics: New Phenomena and New States of Matter in the Universe, in Proceedings of the Third Workshop (Joo Pessoa, Paraba, Brazil, 3-6 October 2007), 183-192.

[23] Jamil M, Karami K and Sheykhi A, Restoring New

Agegraphic Dark Energy in RS II Braneworld, Int. J. Theor. Phys., 50 (2011) 3069-3077.
[24] Pasqua A, Khodam-Mohammadi A, Jamil M, Myrzakulov R, Interacting Ricci dark energy with logarithmic correction, Astrophysics and Space Science, 340 (2012) 199-208.
[25] Zengin B, Askin M, Salti M and Aydogdu O, Thermal Device Beyond the Carnot Limit, Proceedings of the National Academy of Sciences, India Section A: Physical Sciences, 93 (2023) 177-183.
[26] Askin M, Salti M and Aydogdu O, Polytropic Carnot heat engine, Modern Physics Letters A 34 (2018) 1950197.