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Research Article

# Co-Coatomically Ps-Supplemented Modules

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**Abstract**

In this manuscript, we introduce and study co-coatomically ps-supplemented modules. On condition that each co-coatomic submodule of an $T$-module $A$ has a ps-supplement within $A$, then $A$ is termed co-coatomically ps-supplemented. Each radical module is co-coatomically ps-supplemented module. On condition that each left $T$-module is $PS$-coinjective, then each left $T$-module is co-coatomically ps-supplemented. We show that on condition that $A$ is semilocal and $Rad(A) \ll A$, for $A$, being a co-coatomically ps-supplemented module is equivalent to being ps-supplemented. If $A$ is co-coatomically ps-supplemented $T$-module, at that case the module $\frac{A}{Soc_P(\mathrm{A})}$ has each of its coatomic submodules as direct summands. Assuming $T$ to be a left $SI$-ring with an essential socle, it implies that each left $T$-module is co-coatomically ps-supplemented.

**Keywords:** Ps-supplement submodule, co-coatomically ps-supplemented module, projective semisimple module.

# Eş-Eşatomik Ps-Tümlenmiş Modüller

**Öz**

Bu makalede eş-eşatomik ps-tümlenmiş modülleri tanımlıyor ve çalışıyoruz. Bir $A$ $T$-modülünün her eş-eşatomik alt modülü $A$'da bir ps-tümleyene sahipse bu takdirde $A$ $T$-modülü, eş-eşatomik ps-tümlenmiş modül olarak adlandırılır. Her radikal modül eş-eşatomik ps-tümlenmiş modüldür. Her sol $T$-modül $PS$-koinjektif ise bu takdirde her sol $T$-modül eş-eşatomik ps-tümlenmiştir. Biz gösterdik ki, $A$ yarıyerel ve $Rad(A) \ll A$ ise $A$'nın eş-eşatomik ps-tümlenmiş modül olması icin gerek ve yeter şart $A$'nın ps-tümlenmiş olmasıdır. Eğer $A$ eş-eşatomik ps-tümlenmiş $T$-modül ise bu durumda $\frac{A}{Soc_P(\mathrm{A})}$'nın her eşatom alt modülü direkt toplam terimidir. $T$ halkasının büyük desteğe sahip bir sol $SI$-halka olduğunu varsayarsak her sol $T$-modül eş-eşatomik ps-tümlenmiş $T$-modül olur.

**Anahtar Kelimeler:** Ps-tümlenmiş modül, eş-eşatomik ps-tümlenmiş modül, projektif yarıbasit modül.

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## 1. Introduction

In this paper, $T$ signify an associative ring having an identity element, and assuming that all the modules being considered are unital left $T$-modules. The notation $U \leq A$ ($U \lneqq A$) is employed to signify that $U$ is a (proper) submodule of $A$. Assuming $A$ be an $T$-module. The module $A$ is termed semisimple if it is a direct sum of simple submodules. This is equivalent to the property that every submodule of $A$ is a direct summand (see [4], [9] and [12]). Assume $T$ denote a ring and $A$ denote a left $T$-module. The notation $Soc(A)$ represents the socle of $A$, $Soc_P(A)$ represents the sum of the projective simple submodules of $A$, and $Z(A)$ represents the singular submodule of $A$. If a nonzero submodule $U$ of $A$ has the feature that $U \cap V \neq 0$ for each nonzero submodule $V$ of $A$, it is denoted as essential in $A$ and written as $U \trianglelefteq A$. The singular submodule of a module $A$ is the set $Z(A)$ of these members $a \in A$ for which the annihilator $\{s \in T \mid sa = 0\}$ is an essential left ideal of $T$. Alternatively, $Z(A) = \{a \in A \mid Ia = 0 \; for \; I \trianglelefteq T\}$. The property of being singular (respectively, nonsingular) is attributed to a module A when it meets the condition of $Z(A)$ being equal to $A$ (respectively, $Z(A) = 0$) [6].

Assume $A$ be an $T$-module and $B \leq A$. If $C = A$ for each submodule $C$ of $A$ such that $A = B + C$, at that case $B$ is termed a small submodule of $A$ together with denoted by $B \ll A$. Assume $C$ and $B$ be submodules of $A$, $C$ is termed a supplement of $B$ within $A$ if $A = B + C$ and $C$ is minimal with respect to this property, or equivalently, $A = B + C$ and $B \cap C \ll C$. $A$ is termed a supplemented module if for each submodule $B$ of $A$ there exists a submodule $C$ of $A$ such that $A = B + C$ and $B \cap C \ll C$ (see [4] and [12]). It is clear that supplemented modules are a generalization of semisimple modules. It follow from [12, 43.9] a ring $R$ is left perfect if and only if every left $R$-module is supplemented.

In [10], we introduce and study ps-supplement submodules. A submodule $B$ of a module $A$ is termed ps-supplement within $A$ on condition that there exists a submodule $C$ of $A$ such that $A = C + B$ together with $C \cap B$ is projective semisimple. The module $A$ is termed ps-supplemented on the condition that each submodule of $A$ owns a ps-supplement within $A$. For a module $A$, they denote by $Soc_P(A)$ the sum of all projective simple submodules of $A$, that is, $Soc_P(A) = \sum\{S \subseteq A \mid S \; is \; simple \; and \; projective\}$. Then $Soc_P(A) \subseteq Soc(A)$ and $Soc_P(A)$ is the largest projective semisimple submodule of $A$ (see [10]).

Authors, explore particular modules characterized by maximal submodules with supplements and introduce the concept of cofinitely supplemented modules. When the factor module $\frac{A}{L}$ satisfies the property of being finitely generated, then within module $A$, a submodule $L$ is termed as cofinite. $A$ is termed a cofinitely supplemented module on the condition that each cofinite submodule of $A$ possesses a supplement within $A$. It is evident that each module that is supplemented is also cofinitely supplemented, although the converse is not necessarily true in all cases (see [1]).

A module $A$ is identified as coatomic if all its proper submodules are included in maximal submodules (see [14]). In [2] and [7], authors introduced co-coatomically supplemented modules. Assume $L$ becomes a submodule of a module $A$. They state that $L$ is a co-coatomic submodule within $A$ on the condition that $\frac{A}{L}$ is coatomic. It is clear that modules which are semisimple, local, together with finitely generated are coatomic. Due to fact that each factor module of a coatomic module is coatomic, each submodule of semisimple, local together with finitely generated modules is co-coatomic. A module $A$ claimed to becomes co-coatomically supplemented module on the condition that each co-coatomic submodule of $A$ owns a supplement within $A$. They prove that so long as a submodule $L$ of $A$ is co-coatomically supplemented together with $\frac{A}{L}$ lacks of maximal submodule, following that $A$ is co-coatomically supplemented. Also, they said that a co-coatomically supplemented module is cofinitely supplemented together with a coatomic module is co-coatomically supplemented if and only if it is a supplemented module (for detailed information about this modules, see [2]).

In [11], authors studied and introduced cofinitely ps-supplemented modules. According to their definition, a module $A$ is termed cofinitely ps-supplemented when each cofinite submodule of $A$ possesses a ps-supplement within $A$. They obtain some properties of these modules.

If in a ring $T$, each singular left $T$-module possesses the property of being injective, $T$ is referred to as a left $SI$-ring (for detailed information about $SI$-ring, see [5] and [6]). In [13], researchers describe a left $T$-module $A$ as an $SI$-module when each singular left $T$-module is $A$-injective.

Motivated by the above results, it is of interest to investigate a new type of co-coatomically supplemented.

## 2. Prelimneries

The aim of the current essay is to present the concept of co-coatomically ps-supplemented modules. A submodule $B$ of a module $A$ is termed co-coatomically ps-supplemented or briefly ccps-supplemented on the condition that each co-coatomic submodule of $A$ owns a ps-supplement within $A$. Consider $A$ as a module. Clearly, each ps-supplemented module is ccps-supplemented. We show that on condition that each left $T$-module $PS$-coinjective, then each left $T$-module is ccps-supplemented. Also, each ccps-supplemented module is cofinitely ps-supplemented.

## 3. Main Theorem and Proof

In this part of the text, we analyze the essential properties of ccps-supplemented modules. We begin with the definition below.

**Definition 3.1.** If each co-coatomic submodule of an $T$-module $A$ possess a ps-supplement within $A$, at that case $A$ is termed co-coatomically ps-supplemented or briefly ccps-supplemented.

**Lemma 3.2.** Let $A$ remain a module together with $A = Rad(A)$. Then $A$ is ccps-supplemented.

**Proof.** Due to $A$ being radical, the only co-coatomic submodule of $A$ is $A$. Therefore the trivial submodule $0$ of $A$ is a ps-supplement of $A$. Hence $A$ is ccps-supplemented.

While a ps-supplemented module undoubtedly qualifies as ccps-supplemented, the reverse doesn't hold true, as evidenced in the example that follows.

**Example 3.3.** Given the left $\mathbb{Z}$-module $\mathbb{Q}$. According to Lemma 3.2, $\mathbb{Q}$ is identified as ccps-supplemented, while in contrast, $A$ does not possess ps-supplemented due to the fact that $Soc({}_{\mathbb{Z}}\mathbb{Q}) = 0$.

**Proposition 3.4.** Assume $T$ be a ring, and if $\frac{T}{Soc_P(T)}$ qualifies as a semisimple module, then it follows that each left $T$-module is ccps-supplemented.

**Proof.** Due to fact that each left $T$-module is ps-supplemented by [10, Theorem 4.2.6], the proof is clear.

In [10], the author introduced the $PS$ class, which is the class of all short sequences $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$ such that $Im(f)$ is a ps-supplement submodule in $B$. A module $A$ is $PS$-coinjective, if each short sequence of left $T$-modules, starting with $A$, is in the class $PS$, where $0 \longrightarrow A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow 0$.

**Corollary 3.5.** Assuming that each left $T$-module is $PS$-coinjective, then each of them also becomes ccps-supplemented.

**Proof.** Due to fact that each left $T$-module is ps-supplemented, the proof is clear.

**Proposition 3.6.** Let $A$ becomes a semilocal module together with $Rad(A) \ll A$. Then $A$ is ccps-supplemented if and only if $A$ is ps-supplemented.

**Proof.** Assume $C$ becomes a submodule within $A$. Due to fact that $A$ is semilocal, $\frac{A}{Rad(A)}$ is semisimple, that is, $\frac{A}{Rad(A)}$ is coatomic. In this situation, we have the capability to write the following isomorphism: $\frac{A}{C+Rad(A)} \cong \frac{\frac{A}{Rad(A)}}{\frac{C+Rad(A)}{Rad(A)}}$. Since the factor modules of coatomic modules are also coatomic, $\frac{A}{C+Rad(A)}$ is coatomic. On account of this $C + Rad(A)$ owns a ps-supplement in $A$, say $L$. Then $A = C + Rad(A) + L$ together with $(C + Rad(A)) \cap L$ is projective semisimple. Since $Rad(A) \ll A$, $A = C + L$ becomes and $C \cap L \leq (C + Rad(A)) \cap L$ is projective semisimple. Thereby $A$ is ps-supplemented. The converse is obvious.

Suppose $T$ is a ring. It is termed as a left max ring if $Rad(A) \ll A$ for each left $T$-module $A$. Put another way, $T$ is termed a left max ring if each nonzero left $T$-module includes a maximal submodule. Also, a ring $T$ is termed a perfect ring if it satisfies the conditions of being a left max ring and $\frac{T}{Rad(T)}$ is semisimple when considered as a left $T$-module. For each module $A$ over a left perfect ring $\frac{A}{Rad(A)}$ is semisimple (see [3]).

**Proposition 3.7.** Assume $T$ becomes a left max ring together with $A$ becomes a ccps-supplemented module. At that case $A$ is ps-supplemented.

**Proof.** In accordance with the assumption, it is posited that each non-zero left $T$-module is coatomic. Thus, all submodules of $A$ demonstrate coatomic properties, leading to the conclusion that $A$ is ps-supplemented.

Let $T$ becomes a ring. $T$ is defined as a left $V$-ring on the condition that each simple left $T$-module is injective. As a generalization of left $V$-rings, a ring $T$ is termed a left weakly $V$-ring (for short $WV$-ring) on the condition that each simple left $T$-module is $\frac{T}{I}-$injective for every left ideal such that $\frac{T}{I}$ is proper. A $WV$–ring is also a left max ring. For detailed information about left $WV$-rings, we refer to [8]. The next outcome is of paramount importance.

**Corollary 3.8.** Assume $T$ becomes a $WV$-ring. Then each ccps-supplemented $T$-module is ps-supplemented.

**Proposition 3.9.** Assume $T$ becomes a left perfect ring together with $A$ be an $T$-module. At that case $A$ is ccps-supplemented if and only if $A$ is ps-supplemented.

**Proof.** The proof can becomes executed similarly to Proposition 3.6.

**Proposition 3.10.** On condution that a module is ccps-supplemented, by then its homomorphic images are also ccps-supplemented.

**Proof.** Assume $A$ becomes a ccps-supplemented $T$-module together with $B$ a submodule of $A$. At that case arbitrary co-coatomic submodule of $\frac{A}{L}$ is of the format $\frac{B}{L}$ such that the $B$ is co-coatomic submodule of $A$ together with $L \leq B$. Due to fact that $A$ is ccps-supplemented, $A = B + K$ together with $B \cap K$ is projective semisimple for some $K \leq A$. In that case $\frac{A}{L} = \frac{B+K}{L} = \frac{B}{L} + \frac{K+L}{L}$ . Using the canonical epimorphism $\pi: A \longrightarrow \frac{A}{L}$ , consequently, we ascertain that $\pi(B \cap K) = \frac{(B\cap K)+L}{L} = \frac{B\cap(K+L)}{L} = \frac{B}{L} \cap \frac{K+L}{L}$ is projective semisimple by [10, Lemma 4.1.3]. Thereby $\frac{A}{L}$ is ccps-supplemented.

**Proposition 3.11.** Assume $A$ becomes a ccps-supplemented $T$-module. At that case each co-coatomic submodule of the module $\frac{A}{Soc_P(A)}$ is a direct summand.

**Proof.** Assume $\frac{B}{Soc_P(A)}$ becomes a co-coatomic submodule of $\frac{A}{Soc_P(A)}$. At that case $B$ is also co-coatomic submodule of $A$ such that the $Rad(A) \leq B$. Due to fact that $A$ is ccps-supplemented, there exists a submodule $Y$ of $A$ such that the $A = B + Y$ together with $B \cap Y$ is projective semisimple. That's why $B \cap Y \leq Soc_P(A)$. Hence $\frac{A}{Soc_P(A)} = \frac{B+Y}{Soc_P(A)} = \frac{B}{Soc_P(A)} + \frac{Y+Soc_P(A)}{Soc_P(A)}$ together with $\frac{B}{Soc_P(A)} \cap \frac{Y+Soc_P(A)}{Soc_P(A)} = \frac{B\cap(Y+Soc_P(A))}{Soc_P(A)} = \frac{(B\cap Y)+Soc_P(A)}{Soc_P(A)} = \frac{Soc_P(A)}{Soc_P(A)} = 0$. Thereby $\frac{A}{Soc_P(A)} = \frac{B}{Soc_P(A)} \bigoplus \frac{Y+Soc_P(A)}{Soc_P(A)}$.

**Lemma 3.12.** Let $A$ be an $T$-module, $B$ together with $C$ submodules of $A$ such that $B$ is ccps-supplemented, $C$ is co-coatomic together with $B + C$ has a ps-supplement $D$ within $A$. At that case $B \cap (C + D)$ has a ps-supplement $U$ within $B$ together with $D + U$ is a ps-supplement of $C$ within $A$.

**Proof.** Due to fact that $D$ is ps-supplement of $B + C$ in $A$, at that case $A = B + C + D$ together with $(B + C) \cap D$ is projective semisimple. Due to fact that $\frac{A}{C}$ is coatomic, $\frac{B}{B\cap(C+D)} \cong \frac{B+C+D}{C+D} = \frac{A}{C+D} \cong \frac{\frac{A}{C}}{\frac{C+D}{C}}$ is coatomic. Thereby $B \cap (C + D)$ is cocoatomic submodule of $B$. Due to fact that $B$ is ccps-supplemented, $B \cap (C + D)$ has a ps-supplement $U$ in $B$, that is, $B \cap (C + D) + U = B$ and $U \cap (C + D)$ is projective semisimple. Then $A = B + C + D = B \cap (C + D) + U + C + D = C + D + U$. Additionally $C \cap (U + D) \leq (D \cap (C + U)) + (U \cap (C + D)) \leq (D \cap (C + B)) + (U \cap (C + D))$ is projective semisimple, that is, $C \cap (U + D)$ is projective semisimple. Hence $U + D$ is a ps-supplement of $C$ within $A$.

**Corollary 3.13.** Assume $B$ together with $K$ becomes submodules of an $T$-module $A$ such that $B$ is co-coatomic, $K$ is ccps-supplemented together with $B + K$ owns a ps-supplement within $A$. At that case $B$ owns a ps-supplement within $A$.

**Proposition 3.14.** Let $A_1$ besides $A_2$ be arbitrary submodules contained within module $A$, with $A$ being expressible as the sum of $A_1$ together with $A_2$. If $A_1$ and $A_2$ are ccps-supplemented, at that case $A$ is ccps-supplemented.

**Proof.** Assume $B$ denote an arbitrary co-coatomic submodule of $A$. Then $A = A_1 + A_2 + B$. Due to fact that $A_2 + B$ is co-coatomic submodule of $A$, $A_1$ is ccps-supplemented besides the submodule 0, which is self-evident, serves as a ps-supplement of $A = A_1 + A_2 + B$ in A, $A_2 + B$ owns a ps-supplement in $A$ by Corollary 3.13. Due to fact that $A_2$ is ccps-supplemented besides $B$ is co-coatomic, once again in that situation by Corollary 3.13, $B$ owns a ps-supplement within $A$. Thereby $A$ is ccps-supplemented.

**Corollary 3.15.** Finitely sums of modules that are ccps-supplemented are also ccps-supplemented.

Suppose $A$ and $B$ are $T$-modules. If an epimorphism $\psi: A^{(I)} \longrightarrow B$ exists for a finite set $I$, at that case $B$ is known as a finitely $A$-generated module. The subsequent corollary is deduced from Proposition 3.10 and Corollary 3.15.

**Corollary 3.16.** If $A$ is ccps-supplemented module, at that case arbitrary finitely $A$-generated module is a ccps-supplemented module.

**Corollary 3.17.** Finitely direct sums of modules that are ccps-supplemented are also ccps-supplemented.

**Theorem 3.18.** Let $K$ denote a ccps-supplemented submodule contained in the $T$-module $A$ and let $\frac{A}{K}$ lacks of maximal submodule. At that case $A$ is a ccps-supplemented module.

**Proof.** Assume $B$ becomes a co-coatomic submodule of $A$. In that case $\frac{A}{B}$ is coatomic and also $\frac{A}{K+B}$ is coatomic. Due to fact that $\frac{A}{K}$ has no maximal submodule, $\frac{A}{K+B}$ has no maximal submodule, that's why $A = K + B$. From Corollary 3.13, $B$ owns a ps-supplement in $A$. Thereby $A$ is a ccps-supplemented module.

**Corollary 3.19.** Assume $A$ becomes a module and $\frac{A}{Soc_P(A)}$ lacks of maximal submodule. At that case $A$ is a ccps-supplemented module.

Note that a module $A$ is classified as a co-coatomically supplemented module when each co-coatomic submodule of $A$ owns a supplement within $A$ (see [2]). As illustrated in the example to follow, there is instances where a module being co-coatomically supplemented does not imply it is ccps-supplemented.

**Example 3.20.** Let $T = \mathbb{Z}_{p^n}$, where $p \in P$ together with for values of $n$ equal to or exceeding 2. Therefore $T$ is a local ring and so ${}_TT$ is co-coatomically supplemented. Hovewer, ${}_TT$ isn't ccps-supplemented since all simple $T$-modules are singular.

The proposition to follow delineates how co-coatomically supplemented modules relate to ccps-supplemented modules.

**Proposition 3.21.** Assume $A$ becomes a co-coatomically supplemented $T$-module. On condition that $Rad(A) \leq Soc_P(A)$, at that case $A$ is ccps-supplemented.

**Proof.** Assume $B$ becomes any a co-coatomic submodule of $A$. Due to fact that $A$ is co-coatomically supplemented module, there exists a submodule $D$ within $A$ such that $A = B + D$ together with $B \cap D \ll D$, that is, $B \cap D \leq Rad(D) \leq Rad(A) \leq Soc_P(A)$. Thereby $A$ is ccps-supplemented.

**Proposition 3.22.** Assume $T$ a $V$-ring and $A$ becomes a module over $T$. If $A$ is a co-coatomically supplemented module, at that case $A$ is also a ccps-supplemented module.

**Proof.** Given that $A$ is a co-coatomically supplemented module, it follows from [2, Theorem 2.1] that $\frac{A}{Soc(A)}$ has no maximal submodule. Additionally, since $T$ is a $V$-ring, we have from [12, 23.1] that $\frac{A}{Soc_P(A)} \subseteq \frac{A}{Soc(A)} = Rad\left(\frac{A}{Soc(A)}\right) = 0$. Consequently, $A$ is projective-semisimple, meaning $A$ is a ccps-supplemented module.

**Proposition 3.23.** Every ccps-supplemented module is cofinitely ps-supplemented.

**Proof.** Assume $A$ becomes a ccps-supplemented $T$-module besides $B$ be any cofinite submodule of $A$. At that case $\frac{A}{B}$ is finitely generated, that is, $\frac{A}{B}$ is coatomic. That's why $B$ is co-coatomic submodule of $A$. Due to fact that $A$ is ccps-supplemented module, $B$ possesses a ps-supplement in $A$. Thereby $A$ cofinitely ps-supplemented.

The ring $T$ is called an $SI$-ring if each singular left $T$ −module is injective (see [5] and [6]).

**Corollary 3.24.** If $T$ is a left $SI$-ring with an essential socle, then each left $T$-module is ccps-supplemented.

**Proof.** In accordance with [11, Corollary 2.3].

## 4. Conclusion

In this paper, we discuss the idea of ccps-supplemented modules and analyze their basic characteristics using ps-supplemented submodules and co-coatomically supplemented modules. We have given the relation between the concepts of ccps-supplemented, ps-supplemented, ps-supplement submodule, co-coatomic submodule and projective-semisimple module. In addition, ps-supplemented of ccps-supplemented modules could be achieved with left max rings and also WV-rings. An important result in Corollary 3.24 has been reached with the help of left SI-rings.

### Ethics in Publishing

There are no ethical issues regarding the publication of this study.

### Author Contributions

All authors contributed equally to the study.

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