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İnvaryant Ortalamalı Vektör Değerli Çarpanlar ve Kompakt Toplam Operatörleri

Mahmut KARAKUŞ

Öne Çıkanlar:
- Çarpan yakınsaklık
- Tam normlu uzaylar (Banach Uzaylar)
- Kompakt operatörler
- Sürekli operatörler

Anahtar Kelimeler:
- σ -yakınsaklık
- Sınırlı çarpan yakınsak seriler
- $c_0(X)$-çarpan yakınsak seriler
- Toplam operatörler
- Toplanabilme

ÖZET:

Çarpan yakınsaklık gösterimiyle, bir dizi uzayının genelleştirilmiş Köthe-Toeplitz duali kavramı yeniden tanımlanabilir. Bir dizi uzayı N nin (e^n) ile verilen bazı $(v_n) \in N^\beta$ dizisini domine ettiğinden, N nin β-(genelleştirilmiş Köthe-Toeplitz) duali $N^\beta = \{(v_n) | (e^n) \gtrsim (v_n)\}$ şeklinde temsil edilebilir. Alışılmış terminoloji ve kavramları kullanarak, bu makalede, sınırlı (sürekli) lineer operatörler dizisinin yanı sıra σ -toplanabilirlik yöntemi aracılığıyla yeni vektör değerli çarpan uzaylarını tanıtıyoruz. Bu alt uzaylar sup norm topolojisi ile donatılmışlardır. Normlu uzayların tamlığı esasına dayanarak, çarpan uzayları ve genel normlu uzaylar arasında verilen $\mathcal{S}$ toplam operatörünün bazı özelliklerini ayrıntılı bir şekilde inceliyoruz. Bu araştırma, operatörün çeşitli özelliklerinin detaylı bir karakterizasyonunu gerektirir. Bu özellikleri bazı tip çarpan serileri çerçevesinde inceleyerek, operatörün davranışının kapsamlı ve rafine bir analizini sunarak, işlevsel özelliklerine ilişkin daha geniş ve zenginleştirilmiş bir bakış açısı sağlıyoruz.

Vector Valued Multipliers of Invariant Means and Compact Summing Operators

Highlights:
- Multiplier convergence
- Complete normed spaces (Banach Spaces)
- Compact Operators
- Continuous Operators

Keywords:
- σ - convergence
- Bounded multiplier convergent series
- $c_0(X)$-multiplier convergent series
- Summing operators
- Summability

ABSTRACT:

In notation of multiplier convergence, one can redefine the notion generalized Köthe-Toeplitz dual of a sequence space. Since the basis (e^n) of a sequence space N dominates the sequence $(v_n) \in N^\beta$, the β-(generalized Köthe-Toeplitz) dual of N can be represented as $N^\beta = \{(v_n) | (e^n) \gtrsim (v_n)\}$. Employing usual terminology and concepts, in this paper, we introduce novel vector-valued multiplier spaces through the σ-summability method alongside a sequence of bounded linear operators. These spaces are equipped with the sup norm topology. Building on the foundational comprehension of completeness of normed spaces, we examine some properties of the summing operator $\mathcal{S}$ in detail, which acts between multiplier spaces and general normed spaces. This investigation entails a meticulous characterization of the operator's various properties. By examining these properties through the frameworks of some types of multiplier series, we deliver a thorough and refined analysis of the operator's behavior, providing a more expansive and enriched perspective on its functional characteristics.

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INTRODUCTION

In this context, N and B denote a real normed and Banach spaces, respectively. The continuous dual of N, denoted N^*, refers to the space of all bounded (continuous) linear functionals defined on N. The series $\sum_k v_k$ in N is termed unconditionally convergent (denoted as uc) or unconditionally Cauchy (denoted as uC) if the series $\sum_k v_{\pi(k)}$ converges or if the sequence of partial sums is Cauchy for any permutation π of $\mathbb{N}$, the set of positive integers. Furthermore, a series $\sum_k v_k$ in N is classified as weakly unconditionally Cauchy, denoted wuC, if for every permutation π of $\mathbb{N}$, the sequence $(\sum_{k=1}^{n} v_{\pi(k)})$ forms a weakly Cauchy sequence. It is well-established that a series is wuC if $v^*(v_k) \in \ell_1$ for every $v^* \in N^*$. Let us recall that any wuC series in B is uc if and only if B does not contain any copies of c_0, space of all null sequences. For more detailed exploration of Banach spaces, readers are encouraged to consult Diestel's comprehensive book on the theory of sequences and series (Diestel, 1984), as well as Albiac and Kalton's work (Albiac & Kalton, 2006).

The β-dual or known as the generalised Köthe-Toeplitz dual of sequence spaces have strongly connected to the theory of multiplier convergent series (briefly mc-series). The duality theory also has significant implications in the fields of topological sequence space theory and the theory of summability. Let cs denote the space of sequences has convergent sum, then the β-dual is defined as

$$S^\beta := \{t = (t_k) \in \mathbb{R}^{\mathbb{N}} : ts = (t_k s_k) \in cs, \text{for all } s = (s_k) \in S\}.$$

By $\omega(N)$, we mean the space of N-sequences where N is normed space. We represent the sets of N-bounded and N-sequences which converges zero, by $\ell_\infty(N)$ and $c_0(N)$, respectively. The spaces of sequences with the convergent and bounded sums in N are denoted by $cs(N)$ and $bs(N)$, respectively. Additionally, $\phi(N)$ is the set of N $-$finitely non-zero sequences. When $\mathcal{N}$ is considered as the collection of N-sequences endowed with l.c. Hausdorff topology, then $\mathcal{N}$ is defined to be a K space if the mappings $v = (v_k) \mapsto v_k$ from $\mathcal{N}$ into N are continuous for all $k \in \mathbb{N}$. If $v \in N$, $e^k \otimes v$ represents the sequence with v as the only non-zero element at the k-th position for each $k \in \mathbb{N}$. Suppose that N_1 and N_2 are two normed spaces. We present the set of bounded linear operators from N_1 to N_2 by $\mathcal{L}(N_1 : N_2)$. When $\mathcal{N}$ is a vector space of N_1-sequences such that $\phi(N_1) \subseteq \mathcal{N}$, $\sum_k L_k$ is called $\mathcal{N}$-mc-series if the series $\sum_k L_k v_k$ converges in N_2 for every $(v_k) \in \mathcal{N}$. Similarly, the series is named $\mathcal{N}$-mC-series if the partial sums of $\sum_k L_k v_k$ forms a norm Cauchy sequence in N_2 for every $(v_k) \in \mathcal{N}$. Details on theory of multiplier spaces can be found in (Swartz, 2009).

Certain helpful characterizations about the convergence of multipliers of a series $\sum_k v_k$ in B can be represented by the following expressions, (Swartz, 2009):

i). $\sum_k v_k$ is wuC if and only if it is a c_0-mc-series.

ii). $\sum_k v_k$ is uc if and only if it is an ℓ_∞-mc-series.

iii). Let $\chi_{\mathfrak{s}}$ be the characteristic function of $\mathfrak{s}$ and consider the set $M_0 = \{\chi_{\mathfrak{s}} | \mathfrak{s} \subset \mathbb{N}\}$. Then $\sum_k v_k$ is subseries convergent if and only if it is an M_0-mc-series.

An important reference without doubt for a detailed researching on the theory of multiplier convergence is (Swartz, 2009). For some of recent investigations on multiplier convergence involving various summability methods and vector valued multiplier spaces see also (McArthur, 1956; Aizpuru & Pérez-Fernández, 1999; Pérez-Fernández et al., 2000; Aizpuru et al., 2006; Aizpuru et al., 2008; Aizpuru et al., 2009; Swartz, 2009; Swartz, 2014; Aizpuru et al., 2014; Kama & Altay, 2017; Altay & Kama, 2018; Kama et al., 2018; Karakuş, 2019; Karakuş & Başar, 2019; Akın, 2020; Karakuş & Başar, 2020a; Karakuş & Başar, 2020b; Karakuş & Başar, 2022a; Karakuş & Başar, 2022b; Karakuş & Başar, 2024).

One of the most significant applications on theorem of Hahn-Banach rises the concept of Banach limits. These are non-negative, normalized, and shift-invariant linear functionals defined on ℓ_∞. Banach limits generalize the ordinary limit and have numerous applications in various mathematical fields, (Lorentz, 1948; Eberlein, 1950; Semenov & Sukochev, 2010; Semenov et al., 2019). In their research paper on functional characteristics and extreme points of the set of Banach limits on ℓ_∞, Semenov et al. provide a thorough introduction to recent results and developments in the theory of Banach limits and almost convergence, (Semenov et al., 2019). Banach limits effectively extend the limit functional on the space of convergent sequences, c, to ℓ_∞. An important result in this area is due to Lorentz (Lorentz, 1948), who, in 1948, presented a beautiful characterization of almost convergence by using Banach limits. Additionally, Eberlein introduced the concept of the Banach-Hausdorff limit, emphasizing the invariance of Banach limits under regular Hausdorff transformations, (Eberlein, 1950). The reader can refer to (Boos, 2000;Başar, 2012) and (Mursaleen, 2014) for the recent results and related topics in summability.

MATERIALS AND METHODS

Raimi described the concept of σ-convergence as a slight generalization of Lorentz almost convergence by means of *motion* which has same role with Banach limit for linear functionals defined on ℓ_∞, (Raimi, 1963). A motion $\sigma: \mathbb{N} \to \mathbb{N}$ is a one-to-one function that does not contain any finite orbits. An invariant mean, often known as a σ-mean, is a continuous linear functional φ defined on ℓ_∞ that satisfies the following conditions:

1. φ is non-negative,
2. $\varphi(x) = \varphi(x_{\sigma(n)})$,
3. $\varphi(e) = 1$, where $e = (1,1,1, \dots)$, (Raimi, 1963).

Let us note that $\sigma^k(j)$ is assumed to be the k^{th} iteration of σ at j and $\sigma^k(j) \neq j$. It is said to be the bounded sequence $x = (x_k)$ σ-converges to the generalized limit $l \in \mathbb{C}$ if $\varphi(x) = l$ for all φ. Invariant mean is a generalization of the well-known "lim" on c which means $\varphi(x) = \lim x, \forall x \in c$ iff σ has no finite orbits and $c \subset V_\sigma \subset \ell_\infty$, (Mursaleen, 1983; Mursaleen & Edely, 2009). Let us recall that, the functional φ is 1-1 such that $\sigma^k(j) \neq j$.

We have the following:

i). $\sigma^j(l) \neq l$ for all $j, l \in \mathbb{N}$.

ii). $\sigma^l(l) = l$, since a motion has no finite orbit and $\sigma^l(l) = (\sigma^l \circ I)(l) = \sigma^l[I(l)] = I(l)$, where I denotes the identity function.

iii). $\sigma^{i+j}(j) = \sigma^i$, since $\sigma^{i+j}(j) = (\sigma^i \circ \sigma^j)(j) = \sigma^i[\sigma^j(j)] = \sigma^i(j)$ for all $i, j \in \mathbb{N}$.

Here and after, we take $s_j = \sum_{k=1}^{j} v_k$, $s_{\sigma^{m+n}(n)} = s_n + \sum_{k=1}^{m} v_{\sigma^k(n)}$ and the summation without limits runs from 1 to ∞.

Vector Valued Multiplier Spaces of σ-Summable Sequences

Definition 1. Let $v = (v_k) \subseteq N$. Then, it is said that $v = (v_k)$ is σ-convergent to $v_0 \in N$, i.e., $V_\sigma - lim_k v_k = v_0$ (weakly σ-convergent to $v'_0 \in N$, i.e., $wV_\sigma - lim_k v_k = v_0$) if $\sum_{k=0}^{l} \frac{v_{\sigma^k(j)}}{l+1} \to v_0$ as $l \to \infty$ uniformly in $j \in \mathbb{N}$ ($\sum_{k=0}^{l} \frac{v^*(v_{\sigma^k(j)})}{l+1} \to v^*(v'_0)$ as $l \to \infty$, $\forall v^* \in N^*$ uniformly in $j \in \mathbb{N}$).

We denote the space of all σ-convergent and weakly σ-convergent sequences in N by $V_\sigma(N)$ and by $wV_\sigma(N)$, respectively. So, we have,

$$V_\sigma(N) := \left\{ (v_k) \in \mathbb{R}^{\mathbb{N}}(N) : V_\sigma - \lim_k v_k \text{ exists} \right\}$$

and

$$wV_\sigma(N):=\left\{(v_k)\in\mathbb{R}^{\mathbb{N}}(N): wV_\sigma-\lim_k v_k \text{ exists}\right\}.$$

Definition 2. If $v=(v_i)\subseteq N$, then $\sum_i v_i$ is σ-convergent (weakly σ-convergent) to the point $v_0\in N$ $(v'_0\in N)$ and is denoted by $V_\sigma-\sum_i v_i=v_0$ $(wV_\sigma-\sum_i v_i=v'_0)$ if $V_\sigma-lim_k s_k=v_0$ $(wV_\sigma-lim_k s_k=v'_0)$ holds, where $s_k=\sum_{i=1}^k v_i$, for all $k\in\mathbb{N}$. By a simple calculation, $V_\sigma-\sum_i v_i=v_0$ and $wV_\sigma-\sum_i v_i=v'_0$ if

$$\left(\sum_{i=1}^{j} v_i+\sum_{i=1}^{l}\frac{(l-i+1)v_{\sigma^i(j)}}{l+1}\right)\to v_0$$

as $l\to\infty$ uniformly in $j\in\mathbb{N}$ and for all $v^*\in N^*$

$$\left(\sum_{i=1}^{j} v^*(v_i)+\sum_{i=1}^{l}\frac{(l-i+1)v^*(v_{\sigma^i(j)})}{l+1}\right)\to v^*(v'_0)$$

as $l\to\infty$ uniformly in $j\in\mathbb{N}$ holds, respectively, (Akın, 2020).

We denote the space of all σ-summable and weakly σ-summable sequences in N by $V_\sigma^S(N)$ and by $wV_\sigma^S(N)$, respectively. So, we have,

$$V_\sigma^S(N):=\{(v_i)\in\mathbb{R}^{\mathbb{N}}(N): V_\sigma-\sum_i v_i \text{ is convergent}\}$$

and

$$wV_\sigma^S(N):=\{(v_i)\in\mathbb{R}^{\mathbb{N}}(N): wV_\sigma-\sum_i v_i \text{ is convergent}\}.$$

In (Karakuş & Başar, 2024), the authors demonstrated that $V_\sigma^S(N)$ and $wV_\sigma^S(N)$ are closed in $\ell_\infty(N)$, and $V_\sigma^S(N)$ and $wV_\sigma^S(N)$ are also closed in $bs(N)$ with their usual norm. Moreover, if N is complete, then all of them are complete.

In this study, we present and examine specific classes of vector valued spaces linked to an operator series $\sum_k L_k$ in $\mathcal{L}(N_1:N_2)$. These spaces are considered via σ-summability. We also offer characterizations of $c_0(N_1)$- and $\ell_\infty(N_1)$-mc- (mC-) series in terms of these newly defined spaces. Furthermore, we obtain some results on summing operator.

Lemma 3. The formal series $\sum_k v_k$ in N is wuC if and only if

$$H=\sup_{n\in\mathbb{N}}\{\|\sum_{k=1}^{n}\alpha_k v_k\|:\alpha_k\in[-1,1], k=1,2,\ldots,n.\} \tag{1}$$

holds for some $H>0$, (Diestel, 1984).

RESULTS AND DISCUSSION

We now present the definition of multiplier space associated with σ-convergence, which is essential for the results related to the characterizations of $c_0(N)$- and $\ell_\infty(N)$-mc-series.

Main Theorems on the Space $M_\sigma^\infty(\sum_k L_k)$ and Summing Operator

Definition 4. Let N_1 and N_2 be normed spaces (from now on, we assume that N_1 and N_2 are normed spaces) with $L_k\in\mathcal{L}(N_1:N_2)$ for every $k\in\mathbb{N}$. The space $M_\sigma^\infty(\sum_k L_k)$ is defined as follows:

$$M_\sigma^\infty(\textstyle\sum_k L_k)=\{v=(v_k)\in\ell_\infty(N_1): V_\sigma-\sum_k L_k v_k \text{ exists}\}. \tag{2}$$

This space is also a normed space with the sup norm and one can simply check if the following inclusions hold:

$$\phi(N_1)\subseteq M_\sigma^\infty(\textstyle\sum_k L_k)\subseteq\ell_\infty(N_1). \tag{3}$$

We define the summing operator $\mathcal{S}$ as follows:

$$\begin{array}{lllll}\mathcal{S} & : & M_\sigma^\infty(\sum_k L_k) & \longrightarrow & N_2 \\ & & v=(v_k) & \longmapsto & \mathcal{S}(v)=V_\sigma-\sum_k L_k v_k.\end{array} \tag{4}$$

Theorem 5. If B_1 and B_2 are complete (from now on, we assume that B_1 and B_2 are Banach spaces) with $L_k\in\mathcal{L}(B_1:B_2)$ for every $k\in\mathbb{N}$. Then, $\sum_k L_k$ is a $c_0(B_1)$-mc-*series* if and only if $M_\sigma^\infty(\sum_k L_k)$ is complete.

Proof. By using (1), if $\sum_k L_k$ is $c_0(B_1)$-mc-series, then we can find $H > 0$ satisfying

$$H = \sup_{n\in\mathbb{N}}\{\|\sum_{k=1}^{n} L_k v_k\|: \| v_k \| \leq 1, k \in \{1,2, \dots, n\}\}.$$

Now, suppose that $v^m = (v_k^m)$ is a Cauchy sequence in the space $M_\sigma^\infty(\sum_k L_k)$. So, we get $v^0 = (v_k^0) \in \ell_\infty(B_1)$ satisfying $v^m \to v^0$, as $m \to \infty$, since $\ell_\infty(B_1)$ is complete and from the inclusions (3). We shall prove $v^0 \in M_\sigma^\infty(\sum_k L_k)$. For this, suppose that $u_m = V_\sigma - \sum_k L_k v_k^m$ for all $m \in \mathbb{N}$. Since v^m is a Cauchy sequence, $\forall\epsilon > 0$ we can find $m_0 \in \mathbb{N}$ satisfying $\| v^p - v^q \| < \epsilon/(3H), \forall p, q \geq m_0$. So, if $p, q \geq m_0$ are fixed, then there exist $m \in \mathbb{N}$ such that the following inequalities hold, uniformly in $n \in \mathbb{N}$:

$$u_p^C = \left\|u_p - \left[\sum_{k=1}^{n} L_k v_k^p + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} v_{\sigma^k(n)}^p\right]\right\| < \frac{\epsilon}{3}, \tag{5}$$

$$u_q^C = \left\|u_q - \left[\sum_{k=1}^{n} L_k v_k^q + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} v_{\sigma^k(n)}^q\right]\right\| < \frac{\epsilon}{3}, \tag{6}$$

$$u_{pq}^C = \left\|\sum_{k=1}^{n} L_k (v_k^p - v_k^q) + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} (v_{\sigma^k(n)}^p - v_{\sigma^k(n)}^q)\right\| < \frac{\epsilon}{3}. \tag{7}$$

Therefore, by using the inequalities (5), (6), (7), $\forall\epsilon > 0, \exists n_0 \in \mathbb{N}$ such that

$$\| u_p - u_q \| \leq u_p^C + u_q^C + u_{pq}^C < \epsilon, \forall p, q \geq n_0.$$

Since B_2 is also complete, there exists a $u_0 \in B_2$ such that $u_m \to u_0$, as $m \to \infty$. Let us show that $V_\sigma - \sum_k L_k v_k^0 = u_0$. We see for every $\epsilon > 0$ and fix j that $\| v^j - v^0 \| < \epsilon/(3H)$ and

$$\| u_j - u_0 \| < \frac{\epsilon}{3}. \tag{8}$$

Therefore, there exists $m_0 \in \mathbb{N}$ such that

$$\left\|u_j - \left[\sum_{k=1}^{n} L_k v_k^j + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} v_{\sigma^k(n)}^j\right]\right\| < \frac{\epsilon}{3}, \tag{9}$$

uniformly in $n \in \mathbb{N}$, for all $m \geq m_0$. Since $\sum_k L_k$ is a $c_0(B_1)$-mc-series, by supposing

$$u_j = V_\sigma - \sum_k L_k v_k^j$$

for every $j \in \mathbb{N}$, we achieve

$$A_L = \left[\sum_{k=1}^{n} L_k \frac{(v_k^j - v_k^0)}{\|v^j - v^0\|} + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} \frac{(v_{\sigma^k(n)}^j - v_{\sigma^k(n)}^0)}{\|v^j - v^0\|}\right] \leq H,$$

from Lemma 3. So, $\forall\epsilon > 0$ and $\exists m_0 \in \mathbb{N}$,

$$\begin{aligned} &\left\|u_0 - \left[\sum_{k=1}^{n} L_k v_k^0 + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} v_{\sigma^k(n)}^0\right]\right\| \leq (8) + (9) + \\ &+ \left\|\sum_{k=1}^{n} L_k (v_k^j - v_k^0) + \sum_{k=1}^{m} \frac{(m-k+1)}{m+1} L_{\sigma^k(n)} (v_{\sigma^k(n)}^j - v_{\sigma^k(n)}^0)\right\| < \\ &< \frac{2\epsilon}{3} + \| v^j - v^0 \| \cdot A_L \leq \frac{2\epsilon}{3} + \frac{\epsilon}{3H} \cdot H = \epsilon, \end{aligned}$$

uniformly in $n \in \mathbb{N}$, for every $m \geq m_0$. This leads us to the result $v^0 = (v_k^0) \in M_\sigma^\infty(\sum_k L_k)$.

On the other hand, let us assume that the space $M_\sigma^\infty(\sum_k L_k)$ is complete and consider $v = (v_k) \in c_0(B_1)$. Then, we have $c_0(B_1) \subseteq M_\sigma^\infty(\sum_k L_k)$ since the space $M_\sigma^\infty(\sum_k L_k)$ is closed and $\phi(B_1) \subset M_\sigma^\infty(\sum_k L_k)$. So, we may assert that the series $\sum_k L_k v_k$ is σ-convergent for all $v = (v_k) \in c_0(B_1)$. Because $c_0(B_1)$ is monotone, we conclude that the series $\sum_k L_k v_k$ is subseries σ-convergent, and consequently weakly subseries σ-convergent. By the Orlicz-Pettis theorem, $\sum_k L_k v_k$ is subseries norm convergent, (Aizpuru et al., 2008).

Remark 6. For each $k \in \mathbb{N}$ let $L_k \in \mathcal{L}(N_1 : N_2)$. The multiplier space $M^\infty(\sum_k L_k)$ is introduced in (Swartz, 2014) and defined as:

$$M^\infty(\textstyle\sum_k L_k) := \{v = (v_k) \in \ell_\infty(N_1) : \sum_k L_k x_k \text{ is convergent}\}. \tag{10}$$

Therefore, we get the following inclusion concerning $M_\sigma^\infty(\sum_k L_k)$ and $M^\infty(\sum_k L_k)$ which are given by (2) and (10), respectively:

$$M^\infty(\sum_k L_k) \subseteq M_\sigma^\infty(\sum_k L_k).$$

Corollary 7. Let $L_k \in \mathcal{L}(B_1: B_2)$ for all $k \in \mathbb{N}$. Then, the following assertions are equivalent:

1. The series $\sum_k L_k$ is a $c_0(B_1)$-mc-series.
2. $M^\infty(\sum_k L_k)$ is complete.
3. $M_\sigma^\infty(\sum_k L_k)$ is complete.
4. $c_0(B_1) \subseteq M^\infty(\sum_k L_k)$.
5. $c_0(B_1) \subseteq M_\sigma^\infty(\sum_k L_k)$.

Proposition 8. Let $L_k \in \mathcal{L}(N_1: N_2)$ for all $k \in \mathbb{N}$ and define the space $CM^\infty(\sum_k L_k)$ as:

$$CM^\infty(\sum_k L_k) := \{v = (v_k) \in \ell_\infty(N_1): \sum_k L_k v_k \text{ is Cauchy series}\}. \tag{11}$$

Then, the equality,

$$CM_\sigma^\infty(\sum_k L_k) = M_\sigma^\infty(\sum_k L_k) \cap CM^\infty(\sum_k L_k) = M^\infty(\sum_k L_k)$$

holds.

Proof. If $v = (v_k) \in M^\infty(\sum_k L_k)$, then it is evident that $v = (v_k) \in M_\sigma^\infty(\sum_k L_k) \cap CM^\infty(\sum_k L_k)$, meaning the inclusion $M^\infty(\sum_k L_k) \subseteq CM_\sigma^\infty(\sum_k L_k)$ holds true.

Consider that $v = (v_k) \in CM_\sigma^\infty(\sum_k L_k)$. Thus, $\sum_k L_k x_k$ is σ-convergent and is also a Cauchy series. Consequently, $\sum_k L_k v_k$ converges according to Theorem 5.1 of (Karakuş & Başar, 2022a). This concludes the proof.

Corollary 9. Let $L_k \in \mathcal{L}(B_1: B_2)$ for all $k \in \mathbb{N}$. Then the following assertions are equivalent:

1. $\sum_k L_k$ is $c_0(B_1)$-mc-series.
2. $CM^\infty(\sum_k L_k)$ is complete.
3. $CM_\sigma^\infty(\sum_k L_k)$ is complete.
4. $c_0(B_1) \subseteq CM^\infty(\sum_k L_k)$.

Theorem 10. Let B be a complete normed space and N be any normed space. If L_k is element of $\mathcal{L}(B: N)$ for all $k \in \mathbb{N}$, then the following i). and ii). are equivalent:

i). N is complete.

ii). $M_\sigma^\infty(\sum_k L_k)$ is complete for every $c_0(B)$-mC-series.

Proof. We omit the details; see (Altay & Kama, 2018), (Swartz, 2014) and (Karakuş & Başar, 2020b).

Next, we characterizes the continuity property of the summing operator $\mathcal{S}$ with $c_0(N_1)$-mC-series.

Theorem 11. Let $L_k \in \mathcal{L}(N_1: N_2)$ for every $k \in \mathbb{N}$. Then, $\mathcal{S}$ given by (4) is continuous if and only if $\sum_k L_k$ is $c_0(N_1)$-*mC-series.*

Proof. Assume that $\mathcal{S}$ is continuous and consider $\mathcal{G}$ given by

$$\mathcal{G} := \{\|\sum_{k=1}^n L_k v_k\| : \| v_k \| \leq 1, k = 1,2, \dots, n\}. \tag{12}$$

Given that the inclusion $\phi(X) \subset M_\sigma^\infty(\sum_k L_k)$ is valid, the series $\sum_k L_k$ is $c_0(N_1)$-mC-series due to the inequality $H = \sup_{n\in\mathbb{N}} \mathcal{G} \leq |\mathcal{S}|$.

In the other hand, let $\sum_k L_k$ be a $c_0(N_1)$-mC-series. Consequently, the set $\mathcal{G}$ defined by (12) is bounded which implies $H = \sup_{n\in\mathbb{N}} \mathcal{G}$, (Swartz, 2014). Now, suppose that $v = (v_k) \in M_\sigma^\infty(\sum_k L_k)$, then we complete the proof because of the following inequality:

$$\| \mathcal{S}(v) \| = \|V_\sigma - \sum_k L_k v_k\| \leq H \| v \|.$$

Corollary 12. Let $L_k \in \mathcal{L}(N_1: N_2)$ for all $k \in \mathbb{N}$. Then, the following assertions are equivalent:

1. The series $\sum_k L_k$ is $c_0(N_1)$-mC-series.

2.
$$\begin{array}{cccc} \mathcal{S} & : & M^{\infty}(\sum_k L_k) & \longrightarrow & N_2 \\ & & v=(v_k) & \longmapsto & \mathcal{S}(v)=\sum_k L_k v_k \end{array}$$
is continuous, (Swartz, 2014).

3. $\mathcal{S}: M^{\infty}_{\sigma}(\sum_k L_k) \rightarrow N_2$ is continuous.

In the next theorem, we provide the description of the compact summing operator $\mathcal{S}$ by using $\ell_\infty(N)$-mc-series.

Theorem 13. Let $L_k \in \mathcal{L}(N:B)$ for every $k \in \mathbb{N}$. Then, the following assertions regarding the formal series $\sum_k L_k$ are equivalent:

(i). $\mathcal{S}: M^{\infty}_{\sigma}(\sum_k L_k) \rightarrow B$ is compact (weakly compact).

(ii). The series $\sum_k L_k$ is $\ell_\infty(N)$-mc-series.

Proof. **(i)⇒(ii):** Assume that $\mathcal{S}$ is compact and $v=(v_k)$ is any N-valued bounded sequence. Then, the following set $\mathcal{H}$ is also bounded:
$$\mathcal{H}:=\{\sum_{l\in\sigma} e^l \otimes v_l : \sigma \text{ finite}, \| v_l \| \leq 1\} \subset M^{\infty}_{\sigma}(\sum_k L_k).$$
According to our assumption,
$$\mathcal{S}(\mathcal{H}):=\{V_\sigma - \sum_{k\in\sigma} L_k v_k : \sigma \text{ finite}, \| v_k \| \leq 1\}$$
is relatively compact. Consequently, the series $\sum_k L_k v_k$ is subseries σ-convergent in norm topology, and hence weakly subseries σ-convergent, as stated in (Swartz, 2009). Moreover, according to the Orlicz-Pettis theorem, the series $\sum_k L_k v_k$ is an $\ell_\infty(N)$-mc-series.

(ii)⇒(i): Let $\sum_k L_k$ be an $\ell_\infty(N)$-mc-series. Consider the operators $\mathcal{S}^{\sigma}_m$ by
$$\begin{array}{cccc} \mathcal{S}^{\sigma}_m & : & M^{\infty}_{\sigma}(\sum_k L_k) & \longrightarrow & B \\ & & v=(v_k) & \longmapsto & \mathcal{S}^{\sigma}_m(v)=V_\sigma - \sum_{k=1}^{m} L_k v_k \end{array}$$
for every $m \in \mathbb{N}$. We need to show that $\| \mathcal{S}^{\sigma}_m - \mathcal{S} \| \rightarrow 0$, as $m \rightarrow \infty$. Since the series $\sum_k L_k$ is $\ell_\infty(N)$-mc-series, then $\sum_k L_k v_k$ is uniformly σ-convergent for $\| v_k \| \leq 1$, (Swartz, 2009). Consequently, if $\| x_k \| \leq 1$, then
$$\lim_{m\to\infty} \|\mathcal{S}^{\sigma}_m - \mathcal{S}\| = \lim_{m\to\infty} \|(V_\sigma - \sum_{k=1}^{m} L_k v_k) - (V_\sigma - \sum_{k=1}^{\infty} L_k v_k)\|$$
$$= \lim_{m\to\infty} \|V_\sigma - \sum_{k=m+1}^{\infty} L_k v_k\| = 0$$
holds.

Corollary 14. The following assertions regarding the formal series $\sum_k L_k$ are equivalent:

1. The series $\sum_k L_k$ is $\ell_\infty(N)$-mc-series.
2. $\mathcal{S}: M^{\infty}(\sum_k L_k) \rightarrow B$ is compact (weakly compact).
3. $\mathcal{S}: M^{\infty}_{\sigma}(\sum_k L_k) \rightarrow B$ is compact (weakly compact).

Main Theorems on the Space $M^{\infty}_{w\sigma}(\sum_k L_k)$ and Weak Summing Operator

We now present the definition of multiplier space associated with weakly σ-convergence and obtain the results related to the characterizations of $c_0(N)$- and $\ell_\infty(N)$-mc-series.

Definition 15. Let $L_k \in \mathcal{L}(N_1:N_2)$ for every $k \in \mathbb{N}$. Vector valued multiplier space $M^{\infty}_{w\sigma}(\sum_k L_k)$ of weakly almost convergence related to the operator series $\sum_k L_k$ is given by
$$M^{\infty}_{w\sigma}(\sum_k L_k):=\{v=(v_k) \in \ell_\infty(N_1): wV_\sigma - \sum_k L_k v_k \text{ exists}\} \tag{13}$$
and the weak summing operator $\mathcal{S}_w$ is also defined as
$$\begin{array}{cccc} \mathcal{S}_w & : & M^{\infty}_{w\sigma}(\sum_k L_k) & \longrightarrow & N_2 \\ & & v=(v_k) & \longmapsto & \mathcal{S}_w(v)=wV_\sigma - \sum_k L_k v_k. \end{array} \tag{14}$$
It is clear that the following inclusions hold:
$$\phi(N_1) \subseteq M^{\infty}_{\sigma}(\sum_k L_k) \subseteq M^{\infty}_{w\sigma}(\sum_k L_k) \subseteq \ell_\infty(N_1). \tag{15}$$
Theorem 16. If B_1 and B_2 are Banach spaces with $L_k \in \mathcal{L}(B_1:B_2)$ for every $k \in \mathbb{N}$. Then, $\sum_k L_k$ is a $c_0(B_1)$-*mc-series* if and only if $M^{\infty}_{w\sigma}(\sum_k L_k)$ is complete.

Proof. Since it is customary, we omit the details to avoid reiterating similar statements.

Remark 17. For each $k \in \mathbb{N}$ let $L_k \in \mathcal{L}(N_1 : N_2)$. The multiplier space $M_w^\infty(\sum_k L_k)$ is introduced in (Swartz, 2014) and defined as:

$$M_w^\infty(\sum_k L_k) := \{v = (v_k) \in \ell_\infty(N_1) : \sum_k L_k x_k \text{ converges weakly}\}. \quad (16)$$

Corollary 18. Let $L_k \in \mathcal{L}(B_1 : B_2)$ for every $k \in \mathbb{N}$. Then, the following are equivalent:

1. The series $\sum_k L_k$ is a $c_0(B_1)$-mc-series.
2. $M_w^\infty(\sum_k L_k)$ is complete.
3. $M_{w\sigma}^\infty(\sum_k L_k)$ is complete.
4. $c_0(B_1) \subseteq M_w^\infty(\sum_k L_k)$.
5. $c_0(B_1) \subseteq M_{w\sigma}^\infty(\sum_k L_k)$.

Remark 19. Let $L_k \in \mathcal{L}(B_1 : B_2)$ for every $k \in \mathbb{N}$ and $\sum_k L_k$ be $c_0(B_1)$-*mc-series*. Then, $\sum_k u^*(L_k v_k)$ converges for each $v = (v_k) \in c_0(B_1)$, $\forall u^* \in B_2^*$, this means that the series converges weakly. By Corollary 7., we have $v = (v_k) \in M_\sigma^\infty(\sum_k L_k)$, so $v = (v_k) \in M_{w\sigma}^\infty(\sum_k L_k)$. That is, there can be find $u_0 \in B_2$ with $wV_\sigma - \sum_k L_k v_k = u_0$ satisfies the following:

$$\sum_k u^*(L_k v_k) = V_\sigma - \sum_k u^*(L_k v_k) = u^*(u_0).$$

Thus, the inclusion $M_\sigma^\infty(\sum_k L_k) \subseteq M_w^\infty(\sum_k L_k)$ is valid. Furthermore, if the series is an $\ell_\infty(B_1)$-mC-series, then the following also holds:

$$M_w^\infty(\sum_k L_k) \subseteq M^\infty(\sum_k L_k) \subseteq M_\sigma^\infty(\sum_k L_k).$$

By the following theorem, one can prove completeness of $\mathcal{L}(B : N)$ due to completeness of N. By the way, since the proof is similar to the case of $M_C^\infty(\sum_k L_k)$ given in (Altay & Kama, 2018), we omit the details.

Theorem 20. Let B be a complete normed space and N be any normed space. If L_k is element of $\mathcal{L}(B : N)$ for all $k \in \mathbb{N}$, then N is complete if and only if $M_{w\sigma}^\infty(\sum_k L_k)$ is complete for every $c_0(B)$-mC-*series.*

Now, we present some theorems and corollaries that are analogous to the previous results. Since they are similar, we omit the details of proofs to avoid reiterating statements.

Theorem 21. Let $L_k \in \mathcal{L}(N_1 : N_2)$ for every $k \in \mathbb{N}$. Then, $\mathcal{S}_w$ given by (14) is continuous if and only if $\sum_k L_k$ is $c_0(N_1)$-mC-*series.*

Corollary 22. Let $L_k \in \mathcal{L}(N_1 : N_2)$ for all $k \in \mathbb{N}$. Then, the following 1. 2. and 3. are equivalent:

1. The series $\sum_k L_k$ is $c_0(N_1)$-mC-series.
2. $$\begin{aligned} \mathcal{S}_w \ : \quad M_w^\infty(\textstyle\sum_k L_k) &\longrightarrow N_2 \\ v = (v_k) &\longmapsto \mathcal{S}_w(v) = \textstyle\sum_k L_k v_k \end{aligned}$$

is continuous.

3. $\mathcal{S}_w : M_{w\sigma}^\infty(\sum_k L_k) \to N_2$ is continuous.

Theorem 23. Let $L_k \in \mathcal{L}(N : B)$ for every $k \in \mathbb{N}$. Then, $\sum_k L_k$ is $\ell_\infty(N)$-*mc-series* iff $\mathcal{S}_w$ is compact (weakly compact).

Corollary 24. For the formal series $\sum_k L_k$ i), ii) and iii) are equivalent:

i). The series $\sum_k L_k$ is $\ell_\infty(N)$-mc-series.

ii). $\mathcal{S}_w : M_w^\infty(\sum_k L_k) \to B$ is compact (weakly compact).

iii). $\mathcal{S}_w : M_{w\sigma}^\infty(\sum_k L_k) \to B$ is compact (weakly compact).

CONCLUSION

The study on the sequences, series and summability in Banach spaces have always been interesting and luxuriant research area in the theory of functional analysis. As a generalization of well-known

Banach limits which are considered as an extension of limit functional on c to ℓ_∞, in this study, we intend to generalize the results due to authors (Karakuş & Başar, 2020a) and (Karakuş & Başar, 2020b) by using σ-convergence and σ-summability methods.

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REFERENCES

Aizpuru, A. & Pérez-Fernández, F. J. (1999). Characterizations of series in Banach spaces. *Acta Math. Univ. Comenian. (N.S.)*, 58(2), 337-344.

Aizpuru, A., Gutiérrez-Dávila, A. & Sala. A. (2006). Unconditionally Cauchy series and Cesàro summability. *J. Math. Anal. Appl.*, 324, 39-48.

Aizpuru, A., Armario, R. & Pérez-Fernández, F. J. (2008). Almost summability and unconditionally Cauchy series. *Bull. Belg. Math. Soc. Simon Stevin*, 15, 635-644.

Aizpuru, A., Pérez-Eslava, C. & Seoane-Sepúlveda, J. B. (2009). Matrix summability methods and weakly unconditionally Cauchy series. *Rocky Mountain J. Math.*, 39 (2), 367-380.

Aizpuru, A., Armario, R., García-Pacheco, F. J. & Pérez-Fernández, F. J. (2014). Vector-valued almost convergence and classical properties in normed spaces. *Proc. Indian Acad. Sci. Math.*, 124(1), 93-108.

Albiac, F. & Kalton, N. J. (2006). *Topics in Banach Space Theory*. New York: Springer.

Altay, B. & Kama, R. (2018). On Cesàro summability of vector valued multiplier spaces and operator valued series. *Positivity*, 22 (2), 575-586.

Akın, N. P. (2020). Invariant summability and unconditionally Cauchy series. *IJAA*, 18(4),663-671.

Başar, F. (2012). *Summability Theory and Its Applications*. İstanbul: Bentham Science Publishers.

Boos, J. (2000). *Classical and Modern Methods in Summability*. New York: Oxford University Press.

Diestel, J. (1984). *Sequences and Series in Banach spaces*. New York: Springer-Verlag.

Eberlein, W.F. (1950). Banach-Hausdorff limits. *Proc. Amer. Math. Soc.*, 1(5), 662-665.

Kama, R. & Altay, B. (2017). Weakly unconditionally Cauchy series and Fibonacci sequence spaces. *J. Inequal. Appl.*, 133, 1-9.

Kama, R., Altay, B. & Başar, F. (2018). On the domains of backward difference matrix and the spaces of convergence of a series. *Bull. Allahabad Math. Soc.*, 33 (1), 139-153.

Karakuş, M. (2019). On certain vector valued multiplier spaces and series of operators. *J. Math. Anal.*, 10(2), 1-11.

Karakuş, M. & Başar, F. (2019). A generalization of almost convergence, completeness of some normed spaces with wuC series and a version of Orlicz-Pettis theorem. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 113(4), 3461-3475.

Karakuş, M. & Başar, F. (2020a). Vector valued multiplier spaces of f_λ-summability, completeness through $c_0(X)$-multiplier convergence and continuity and compactness of summing operators. *Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM*, 114:169, 1-17.

Karakuş, M. & Başar, F. (2020b). Operator valued series, almost summability of vector valued multipliers and (weak) compactness of summing operator. *J. Math. Anal. Appl.*, 484 (1), 1-16.

Karakuş, M. & Başar, F. (2022a). On some classical properties of normed spaces via generalized vector valued almost convergence. *Math. Slovaca*, 72 (6), 1551-1566.

Karakuş, M. & Başar, F. (2022b). Characterizations of Unconditionally Convergent and Weakly Unconditionally Cauchy Series via w_p^R-Summability, Orlicz-Pettis Type Theorems and Compact Summing Operator. *Filomat***,** 36 (18), 6347-6358.

Karakuş, M. & Başar, F. (2024). Vector valued closed subspaces and characterizations of normed spaces through σ-summability. *Indian J. Math.*, 66 (1), 85-105.

Lorentz, G. G. (1948). A contribution to the theory of divergent sequences. *Acta Math.***,** 80, 167-190.

McArthur, *On relationships amongst certain spaces of sequences in an arbitrary Banach space*, Canad. J. Math. **8** (1956), 192-197.

Mursaleen, M. (2014). *Applied Summability Methods*. London: Springer.

Mursaleen, M. (1983). On some new invariant matrix methods of summability. *Quart J. Math. Oxford***,** 34, 77-86.

Mursaleen, M. & Edely, O. H. H. (2009). On the invariant mean and statistical convergence. *Appl. Math. Lett.*, 22, 1700-1704.

Pérez-Fernández, F. J., Benítez-Trujillo, F. & Aizpuru, A. (2000). Characterizations of completeness of normed spaces through weakly unconditionally Cauchy series. *Czechoslovak Math. J.*, 50 (125), 889-896.

Raimi, R. A. (1963). Invariant means and invariant matrix methods of summability. *Duke Math. J.*, 30, 81-94.

Semenov, E.M. & Sukochev, F.A. (2010). Invariant Banach limits and applications. *J. Funct. Anal.,* 259 (6), 1517-1541.

Semenov, E.M., Sukochev, F.A. & Usachev, A.S. (2019). The main classes of invariant Banach limits. *Izv. Math.*, 83 (1), 124-150.

Swartz, C. (2009). *Multiplier Convergent Series*. Singapore: World Scientific Publishing.

Swartz, C. (2014). Operator valued series and vector valued multiplier spaces. *Casp. J. Math. Sci.*, 3 (2), 277-288.