Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (XXXX), 1 – 9
DOI : 10.15672/hujms.xx

Research Article

Submersions of hemi-slant submanifolds

Bayram Şahin*[1], Gülistan Polat[1], Akın Levent[2]

[1] *Ege University, Faculty of Science, Department of Mathematics, 35100 Izmir, Türkiye*
[2] *İnönü University, Faculty of Science and Arts, Department of Mathematics, Malatya, Türkiye*

Abstract

In this paper, we introduce Riemannian submersions of a hemi-slant submanifold of a Kähler manifold by observing the integrability of the anti-invariant distribution of a hemi-slant submanifold and the integrability of the vertical distribution of a Riemannian submersion. Using this notion, we show that the base manifold is a Kähler manifold in the submersion of a hemi-Kaehlerian slant submanifold of an almost Hermitian manifold. We obtain an inequality between the sectional curvature of the hemi-Kaehlerian slant submanifold and the holomorphic sectional curvature of the base manifold. If this inequality becomes equality, a geometric result is given. In addition, the Ricci tensor field on the horizontal distribution along this submersion is also found.

Mathematics Subject Classification (2020). 53B20, 53C15

Keywords. Kähler manifold, Riemannian submersion, hemi-slant submanifold, CR-submanifold, holomorphic sectional curvature

1. Introduction

CR-submanifolds were defined by Bejancu [1] as a generalization of holomorphic submanifolds and anti-invariant submanifolds. It is known that the anti-invariant distribution of a CR-submanifold of a Kähler manifold is always integrable [3].

On the other hand, Riemannian submersions were defined by O'Neill [11] as the submersion counterpart of Riemannian submanifolds. The vertical distribution in a Riemannian submersion is also always integrable.

Observing the integrability of the vertical distribution in Riemannian submersions and the integrability of the anti-invariant distribution of a CR-submanifold of a Kähler manifold [3], Kobayashi defined and studied CR-submersions [9]. This notion later attracted the attention of many authors and a large number of works were published, [5], [6], [7], [12], [13].

On the other hand, generic submanifolds were defined and studied by Chen [4] as a generalization of CR-submanifolds. Unlike the case of CR-submanifolds, the distribution that is orthogonal to the holomorphic distribution in generic submanifolds is not always integrable [4]. Fatima and Ali [8] applied Kobayashi's definition to generic submanifolds,

*Corresponding Author.
Email addresses: bayram.sahin@ege.edu.tr (B. Şahin), glstnyldrmm@gmail.com (G. Polat), akhlev44@gmail.com (A. Levent)
Received: 17.08.2024; Accepted: 25.01.2025

thus assuming the integrability of purely real distribution. Under this assumption, they studied the effects of such submersions on submanifolds and target manifolds.

Another class of submanifolds that is a generalization of CR-submanifolds in the literature is hemi-slant submanifolds [2], [14]. The anti-invariant distribution is always integrable in hemi-slant submanifolds of a Kähler manifold [14].

In this paper, the submersion of a hemi-slant submanifold is defined by analogy with this integrability situation in hemi-slant submanifolds and Riemannian submersions. The paper is organized as follows: In the second section, the concepts and formulas that will be used in the paper are recalled. In the third section, we first define a Riemannian submersion from a hemi-slant submanifold of an almost Hermitian manifold to an almost Hermitian manifold (Definition 3.1). For a Riemannian submersion from a hemi-slant submanifold of a Hermitian manifold to an almost Hermitian manifold, it is then shown that the target manifold is also a Hermitian manifold (Theorem 3.2). It is obtained that the target manifold is a Kaehler manifold if the hemi-slant submanifold is a hemi-Kaehlerian slant submanifold (Theorem 3.3). On the other hand, an inequality is obtained between the ambient manifold of the hemi-slant submanifold and the holomorphic sectional curvatures of the target manifold. The case of the inequality being an equality was also investigated (Theorem 3.4). We also find necessary and sufficient conditions for the horizontal distribution of such a Riemannian submersion to be integrable (Corollary 3.6). Finally, the Ricci curvature tensor field on the horizontal distribution is obtained (Proposition 3.7).

2. Preliminaries

If there is a $(1,1)$ tensor field $\mathfrak{J}$, Riemannian metric g on a differentiable manifold $\bar{\mathfrak{M}}$ and the following conditions are satisfied, $(\bar{\mathfrak{M}}, \mathfrak{J}, g)$ is called an almost Hermitian manifold [10];

$$\mathfrak{J}^2 = -\mathcal{I}, \quad g(\mathfrak{J}\zeta_1, \mathfrak{J}\zeta_2) = g(\zeta_1, \zeta_2),$$

where $\mathcal{I}$ is the identity map. Let $(\bar{\mathfrak{M}}, g)$ be an almost Hermitian manifold. Then the Nijenhuis tensor field is given by

$$N(\zeta_1, \zeta_2) = [\mathfrak{J}\zeta_1, \mathfrak{J}\zeta_2] - [\zeta_1, \zeta_2] - \mathfrak{J}[\mathfrak{J}\zeta_1, \zeta_2] - \mathfrak{J}[\zeta_1, \mathfrak{J}\zeta_2] \tag{2.1}$$

for $\zeta_1, \zeta_2 \in \Gamma(T\bar{\mathfrak{M}})$, here $\Gamma(T\bar{\mathfrak{M}})$ denotes the set of all vector fields on $\bar{\mathfrak{M}}$. The same notation will be used for the set of vector fields of any other manifold or submanifold under consideration. If $N = 0$, then $\bar{\mathfrak{M}}$ is called Hermitian manifold. For an almost Hermitian manifold $(\bar{\mathfrak{M}}, g)$, if $\mathfrak{J}$ is parallel, $\nabla\mathfrak{J} = 0$, then $(\bar{\mathfrak{M}}, g)$ is called a Kähler manifold. The holomorphic sectional curvature K of $\bar{\mathfrak{M}}$ with respect to ς_1 is given by

$$K = \frac{g(\tilde{R}(\varsigma_1, \mathfrak{J}\varsigma_1)\mathfrak{J}\varsigma_1, \varsigma_1)}{(g(\varsigma_1, \varsigma_1))^2}.$$

where $\tilde{R}$ is the curvature tensor field of $\bar{\mathfrak{M}}$. A complex space form, denoted by $\bar{\mathfrak{M}}(c)$, is a Kähler manifold of constant holomorphic sectional curvature c. In this direction, the curvature tensor $\tilde{R}$ of $\bar{\mathfrak{M}}(c)$ is computed as

$$\begin{aligned}\tilde{R}(\zeta_1, \zeta_2,)\zeta_3 &= \frac{c}{4}\{g(\zeta_2, \zeta_3)\zeta_1 - g(\zeta_1, \zeta_3)\zeta_2 + g(\mathfrak{J}\zeta_2, \zeta_3)\mathfrak{J}\zeta_1 \\ &\quad -g(\mathfrak{J}\zeta_1, \zeta_3)\mathfrak{J}\zeta_2 + 2g(\zeta_1, \mathfrak{J}\zeta_2)\mathfrak{J}\zeta_3\},\end{aligned} \tag{2.2}$$

for every $\zeta_1, \zeta_2, \zeta_3 \in \Gamma(T\bar{\mathfrak{M}})$.

Let M be an m-dimensional submanifold of $\bar{\mathfrak{M}}$. The Riemannian connection $\bar{\nabla}$ on $\bar{\mathfrak{M}}$ induces the Riemannian connections ∇ and $\nabla^\perp$ on M and in the normal bundle of M in $\bar{\mathfrak{M}}$, respectively. These connections are related by Gauss and Weingarten formulas

$$\bar{\nabla}_{\zeta_1}\zeta_2 = \nabla_{\zeta_1}\zeta_2 + h(\zeta_1, \zeta_2), \tag{2.3}$$

$$\bar{\nabla}_{\zeta_1}N' = -\bar{A}_{N'}\zeta_1 + \nabla^\perp_{\zeta_1}N' \tag{2.4}$$

for any $\zeta_1, \zeta_2 \in \Gamma(TM)$ and $N' \in \Gamma(TM^{\perp})$. h and $\bar{A}$ are the second fundamental form and Weingarten map and it is easy to see that $g\left(h(\zeta_1, \zeta_2), N'\right) = g\left(\bar{A}_{N'}\zeta_1, \zeta_2\right)$. Here, TM and $TM^{\perp}$ denote the tangent bundle and normal bundle of the submanifold, respectively.

Let $\tilde{R}$ and R be the curvature tensors of $\bar{\mathfrak{M}}$ and M. The equations of Gauss, Codazzi, and Ricci are given by

$$\begin{aligned} \tilde{R}(\zeta_1, \zeta_2, \zeta_3, \varsigma_2) &= R(\zeta_1, \zeta_2, \zeta_3, \varsigma_2) - g(h(\zeta_1, \varsigma_2), h(\zeta_2, \zeta_3)) \\ &\quad + g(h(\zeta_1, \zeta_3), h(\zeta_2, \varsigma_2)) \end{aligned} \tag{2.5}$$

for any $\zeta_1, \zeta_2, \zeta_3, \varsigma_2 \in TM$ and $N_1, N_2 \in TM^{\perp}$. We now recall two important notions from submanifolds of almost Hermitian manifolds.

Definition 2.1 ([1]). The submanifold M of an almost Hermitian manifold $(\bar{\mathfrak{M}}, \bar{\mathfrak{J}})$ is called a CR-submanifold if there exists a differentiable distribution $D : p \rightarrow D_p \subset T_pM$ such that

(a) TM admits the orthogonal direct decomposition $TM = D^{\perp} \oplus D^{\theta}$.
(b) The distribution D^T is invariant with respect to the complex structure $\bar{\mathfrak{J}}$.
(c) The distribution $D^{\perp}$ is an anti-invariant distribution, i.e., $\bar{\mathfrak{J}}D^{\perp} \subset TM^{\perp}$.

Definition 2.2 ([14]). We say that M is a hemi-slant submanifold an almost Hermitian manifold $(\bar{\mathfrak{M}}, \bar{\mathfrak{J}})$ if there exist two complementary orthogonal distributions $D^{\perp}$ and D^{θ} on M such that

(a) The distribution $D^{\perp}$ is an anti-invariant distribution, i.e., $\bar{\mathfrak{J}}D^{\perp} \subset TM^{\perp}$.
(b) The distribution D^{θ} is slant with slant angle θ.

Let M be a hemi-slant submanifold of an almost Hermitian manifold $(\bar{\mathfrak{M}}, \mathfrak{J})$. Then for $\zeta_1 \in \Gamma(TM)$, we write

$$\mathfrak{J}\zeta_1 = P\zeta_1 + \omega\zeta_1, \tag{2.6}$$

where $P\zeta_1$ and $\omega\zeta_1$ are tangential and normal parts of $\mathfrak{J}\zeta_1$, respectively. Similarly, for $\varsigma_2 \in \Gamma(TM^{\perp}$, we write

$$\mathfrak{J}\varsigma_2 = B\varsigma_2 + C\varsigma_2. \tag{2.7}$$

Here $B\varsigma_2$ and $C\varsigma_2$ are tangential and normal parts of $\mathfrak{J}\varsigma_2$, respectively.

Definition 2.3 ([11]). Let (M^m, g_M) and (N^n, g_N) be Riemannian manifolds, where $dim(M) = m$, $dim(N) = n$ and $m > n$. A Riemannian submersion $\mathfrak{S} : M \longrightarrow N$ is a surjective map of M onto N satisfying the following axioms:

(S1) $\mathfrak{S}$ has maximal rank.
(S2) The differential $\mathfrak{S}_*$ preserves the lengths of horizontal vectors.

For each $q \in N$, $\mathfrak{S}^{-1}(q)$ is an $(m-n)-$dimensional submanifold of M. The submanifolds $\mathfrak{S}^{-1}(q)$, $q \in N$, are called fibers. A vector field on M is called vertical if it is always tangent to fibers. A vector field on M is called horizontal if it is always orthogonal to fibers.Thus for every $p \in M$, M has the following decomposition:

$$T_pM = \mathcal{V}_p \oplus \mathcal{H}_p = \mathcal{V}_p \oplus \mathcal{V}_p^{\perp}.$$

A vector field ζ_1 on M is called basic if ζ_1 is horizontal and $\mathfrak{S}-$ related to a vector field $\acute{\zeta}_{1*}$ on N, i.e., $\mathfrak{S}_*\zeta_{1p} = \acute{\zeta}_{1*\mathfrak{S}(p)}$ for all $p \in M$. Note that we denote the projection morphisms on the distributions $ker\mathfrak{S}_*$ and $(ker\mathfrak{S}_*)^{\perp}$ by $\mathcal{V}$ and $\mathcal{H}$, respectively. The geometry of Riemannian submersions is characterized by O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ defined for vector fields ξ_1, ξ_2 on M by

$$\mathcal{A}_{\xi_1}\xi_2 = \mathcal{H}\nabla_{\mathcal{H}\xi_1}\mathcal{V}\xi_2 + \mathcal{V}\nabla_{\mathcal{H}\xi_1}\mathcal{H}\xi_2, \mathcal{T}_{\xi_1}\xi_2 = \mathcal{H}\nabla_{\mathcal{V}\xi_1}\mathcal{V}\xi_2 + \mathcal{V}\nabla_{\mathcal{V}\xi_1}\mathcal{H}\xi_2, \tag{2.8}$$

where ∇ is the Levi-Civita connection of g_M. From (2.8) we have

$$g(R(\zeta_1,\zeta_2)\zeta_3,\zeta_4) = g(R^*(\zeta_1,\zeta_2)\zeta_3,\zeta_4) + 2g(A_{\zeta_3}\zeta_4, A_{\zeta_1}\zeta_2) + g(A_{\zeta_2}\zeta_4, A_{\zeta_1}\zeta_3) - g(A_{\zeta_1}\zeta_4, A_{\zeta_2}\zeta_3) \tag{2.9}$$

for basic vector fields $\zeta_1, \zeta_2, \zeta_3$ and ζ_4, where R^* represents the horizontal lift of the curvature tensor field R' of the manifold N, i.e. $\mathfrak{S}_*(R^*(\zeta_1,\zeta_2)\zeta_3) = R'(\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2))\mathfrak{S}_*(\zeta_3)$.

Observing the similarity between CR-submanifolds and Riemannian submersions, Kobayashi defined the following notion.

Definition 2.4 ([9])**.** Let M be a CR submanifold of an almost Hermitian manifold $(\bar{\mathfrak{M}}, \bar{\mathfrak{J}})$ with distributions D and $D^\perp$ and the normal bundle $T^\perp M$. By a submersion $\mathfrak{S} : M \longrightarrow N$ of M onto an almost Hermitian manifold N we mean a Riemannian submersion $\mathfrak{S} : M \to N$ together with the following conditions:

(1) $D^\perp$ is the kernel of $\mathfrak{S}_*$, that is, $\mathfrak{S}_* D^\perp = \{0\}$,
(2) $\mathfrak{S}_* D_p = T_{\mathfrak{S}(p)}N$ is complex isometry, where $p \in M$ and $T_{\mathfrak{S}(p)}N$ is the tangent space of N at $\mathfrak{S}(p)$,
(3) $\bar{\mathfrak{J}}$ interchanges $D^\perp$ and $T^\perp M$.

Based on this notion, he showed that if N is a Kähler manifold then $\bar{\mathfrak{M}}$ is a Kähler manifold. He also found relation between holomorphic sectional curvatures.

3. Submersed hemi-slant submanifolds

In this section, we define the submersion of a hemi-slant submanifold to an almost Hermitian manifold, determine the character of the target manifold, and obtain an inequality between the holomorphic sectional curvature of the ambient space of the submanifold and the holomorphic sectional curvature of the target manifold, and discuss the condition of equality.

We first recall from [14] that the anti-invariant distribution of hemi-slant submanifold of a Kähler manifold is always integrable. Thus we are able to present the following definition.

Definition 3.1. Let M be a hemi-slant submanifold of an almost Hermitian manifold $(\bar{\mathfrak{M}}, J_{\bar{\mathfrak{M}}}, g_\circ)$. Let $g = \psi^* g_\circ$ be the induced metric on M, where $\psi : M \longrightarrow \bar{\mathfrak{M}}$ is the given immersion. Let (N, J_N, g_N) be an almost Hermitian manifold. We say that M is submersed over N if there is a Riemannian submersion $\mathfrak{S} : M \longrightarrow N$ of (M, g) onto (N, g_N) such that the following conditions are fulfilled:

(1) The slant distribution D^θ of M and the horizontal distribution $\mathcal{H}$ of the Riemannian submersion $\mathfrak{S}$ coincide , i.e., $D^\theta_p = \mathcal{H}_p$ for any $p \in M$.
(2) The map $\mathfrak{S}_* p : D^\theta_p \longrightarrow T_{\mathfrak{S}(p)}N$ is (P, J_N)- holomorphic i.e.,
$$\sec\theta \mathfrak{S}_{*p}(P_p \zeta_{1p}) = J_{N\,p} \mathfrak{S}_{*p}(\zeta_{1p}) \tag{3.1}$$
for $p \in M$ and $\zeta_1 \in \Gamma(D^\theta)$.
(3) $ker \mathfrak{S}_{*p} = D^\perp_p$.

We say that a slant distribution D^θ is a Kähler slant if $\nabla P = 0$ on D^θ. It is clear that $\widetilde{J} = \sec\theta P$ is a complex structure on D^θ.

First of all from Definition 3.1 and the definition of the Nijenhuis tensor field, we have the following result.

Theorem 3.2. *Let M be a hemi-slant submanifold of a Hermitian manifold $\widetilde{M}$. Suppose that M is submersed over the almost Hermitian manifold (N, J_N, g_N). Then N is a Hermitian manifold.*

Proof. Consider the Nijenhuis tensor field of N as $[J_N, J_N](\zeta_1{}', \zeta_2{}')$, $\zeta_1{}', \zeta_2{}' \in \chi(M)$. Then, we have

$$\begin{aligned}[J_N, J_N](\zeta_1{}', \zeta_2{}') &= [J_N\mathfrak{S}_*(\zeta_1), J_N\mathfrak{S}_*(\zeta_2)] - [\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2)] \\ &- J_N[J_N\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2)] - J_N[\mathfrak{S}_*(\zeta_1), J_N\mathfrak{S}_*(\zeta_2)].\end{aligned}$$

Since $\mathfrak{S}_*(\mathcal{H}[\zeta_1, \zeta_2]) = [\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2)]$ from the second condition Definition 3.1, we get

$$[J_N, J_N](\zeta_1{}', \zeta_2{}') = \mathfrak{S}_*(N_{\widetilde{J}}(\zeta_1, \zeta_2))$$

where $N_{\widetilde{J}}$ denotes the Nijenhuis tensor field of $\widetilde{J}$. On the other hand, since $\mathfrak{S}$ is a isometry between $\mathcal{H}$ and N, we have

$$g_N(\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2)) = g(\zeta_1, \zeta_2)$$

for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$. Then Hermitian manifold $\widetilde{M}$ implies that

$$g(\widetilde{J}\zeta_1, \widetilde{J}\zeta_2) = g(\zeta_1, \zeta_2).$$

Thus from (3.1), we get

$$\begin{aligned}g_N(\mathfrak{S}_*(\zeta_1), \mathfrak{S}_*(\zeta_2)) &= \sec^2\theta g(\mathfrak{S}_*(P\zeta_1), \mathfrak{S}_*(P\zeta_2)) \\ &= g_N(J_N\mathfrak{S}_*(\zeta_1), J_N\mathfrak{S}_*(\zeta_2))\end{aligned}$$

which shows that J_N is compatible with g_N. □

We are now ready to prove the following theorem.

Theorem 3.3. *Let M be a hemi- Kählerian slant submanifold of a Hermitian manifold and (N, g, J_N) an almost Hermitian manifold. Suppose that M is submersed over the manifold N. Then N is a Kähler manifold.*

Proof. We take the basic vector fields ζ_1, and ζ_2 on D^θ. Then we have

$$(\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N)\mathfrak{S}_*(\zeta_2) = \overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N\mathfrak{S}_*(\zeta_2) - J_N\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)}\mathfrak{S}_*(\zeta_2),$$

where $\overset{2}{\nabla}$ is the Levi-Civita connection of N. From (3.1), we get

$$(\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N)\mathfrak{S}_*(\zeta_2) = \overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)}\sec\theta\mathfrak{S}_*(P\zeta_2) - J_N\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)}\mathfrak{S}_*(\zeta_2)$$

Since $\mathcal{H}\nabla_{\zeta_1}P\zeta_2$ is the basic vector field corresponding to $\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)}\mathfrak{S}_*(P\zeta_2)$, we obtain

$$(\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N)(\mathfrak{S}_*(\zeta_2)) = \sec\theta\mathfrak{S}_*(\nabla_{\zeta_1}P\zeta_2) - J_N\mathfrak{S}_*(\nabla_{\zeta_1}\zeta_2)$$

where ∇ is the Levi-Civita connection on M. Hence we get

$$(\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N)(\mathfrak{S}_*(\zeta_2)) = \sec\theta\mathfrak{S}_*(P\nabla_{\zeta_1}\zeta_2) - J_N\mathfrak{S}_*(\nabla_{\zeta_1}\zeta_2)$$

due to D^θ is a Kählerian slant distribution. Thus using (3.1), we have

$$(\overset{2}{\nabla}_{\mathfrak{S}_*(\zeta_1)} J_N)(\mathfrak{S}_*(\zeta_2)) = \sec\theta\mathfrak{S}_*(P\nabla_{\zeta_1}\zeta_2) - \sec\theta\mathfrak{S}_*(P\nabla_{\zeta_1}\zeta_2) = 0$$

which proves the assertion. □

We now give a relation between the holomorphic sectional curvatures. First of all, we denote the holomorphic sectional curvatures of the manifolds N and $\bar{\mathfrak{M}}$ by $\mathcal{K}^N_{J_N}(\mathfrak{S}_*(\zeta_1))$ and $\bar{\mathcal{K}}_{\widetilde{J}}(\zeta_1)$ with respect to $\zeta_1 \in \Gamma(D^\theta)$, respectively. Accordingly, the following inequality exists between the holomorphic sectional curvatures of N and $\bar{\mathfrak{M}}$.

Theorem 3.4. *Let M be a hemi- Kählerian slant submanifold of a Kähler manifold $(\bar{\mathfrak{M}}, J, g)$. Suppose that M is submersed over an almost Hermitian manifold (N, J_N, g_N). If the morphism ω given by (2.6) is parallel, then we have*

$$\bar{\mathcal{K}}_{\widetilde{J}}(\zeta_1) \geq \mathcal{K}^N_{J_N}(\mathfrak{S}_*(\zeta_1))$$

for $\zeta_1 \in \Gamma(D^\theta)$. The equality is satisfied if and only if M is D^θ- geodesic.

Proof. From (2.9), we have

$$\begin{aligned} g(R(\zeta_1,\zeta_2)\zeta_3,\zeta_4) = g(R^N(\mathfrak{S}_*(\zeta_1),\mathfrak{S}_*(\zeta_2))\mathfrak{S}_*(\zeta_3),\mathfrak{S}_*(\zeta_4)) + 2g(A_{\zeta_3}\zeta_4, A_{\zeta_1}\zeta_2) \\ +g(A_{\zeta_2}\zeta_4, A_{\zeta_1}\zeta_3) - g(A_{\zeta_1}\zeta_4, A_{\zeta_2}\zeta_3) \end{aligned}$$

for $\zeta_1, \zeta_2, \zeta_3, \zeta_4 \in \Gamma(D^\theta)$. Thus we get

$$\begin{aligned} g(R(\zeta_1, \widetilde{J}\zeta_1, \zeta_1, \widetilde{J}\zeta_1) = g(R^N({\zeta_1}', J_N{\zeta_1}'){\zeta_1}', J_N{\zeta_1}') + 3g(A_{\zeta_1}\widetilde{J}\zeta_1, A_{\zeta_1}\widetilde{J}\zeta_1) \\ +g(A_{\widetilde{J}\zeta_1}\widetilde{J}\zeta_1, A_{\zeta_1}\zeta_1), \end{aligned}$$

where ${\zeta_1}' = \mathfrak{S}_*(\zeta_1)$. Since M is a hemi- Kählerian slant submanifold, $(\nabla_{\zeta_1}\widetilde{J})\zeta_2 = 0$ for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$.Using (2.9) we derive

$$h\nabla_{\zeta_1}\widetilde{J}\zeta_2 + A_{\zeta_1}\widetilde{J}\zeta_2 = \widetilde{J}h\nabla_{\zeta_1}\zeta_2 + \widetilde{J}A_{\zeta_1}\zeta_2.$$

Taking the vertical components of this equation, we obtain

$$A_{\zeta_1}\widetilde{J}\zeta_2 = \widetilde{J}A_{\zeta_1}\zeta_2$$

which implies that $A_{\zeta_1}J\zeta_1 = 0$. Therefore we get

$$g(R(\zeta_1, \widetilde{J}\zeta_1)\zeta_1, \widetilde{J}\zeta_1) = g(R^N({\zeta_1}', J_N{\zeta_1}'){\zeta_1}', J_N{\zeta_1}')$$

On the other hand from Gauss equation (2.5), we get

$$\begin{aligned} g(R(\zeta_1, \widetilde{J}\zeta_1)\zeta_1, \widetilde{J}\zeta_1) &= g(\bar{R}(\zeta_1, \widetilde{J}\zeta_1)\zeta_1, \widetilde{J}\zeta_1) + g(h(\zeta_1, \widetilde{J}\zeta_1), h(\widetilde{J}\zeta_1, \zeta_1)) \\ &- g(h(\zeta_1,\zeta_1), h(\widetilde{J}\zeta_1, \widetilde{J}\zeta_1)) \end{aligned} \tag{3.2}$$

Since $\bar{\mathfrak{M}}$ is a Kähler manifold we have $\overline{\nabla}J = 0$. Using Gauss and Weingarten formulas we get

$$\begin{aligned} \nabla_{\zeta_1}P\zeta_2 + h(\zeta_1, P\zeta_2) - \bar{A}_{\omega\zeta_2}\zeta_1 + \nabla^\perp_{\zeta_1}\omega\zeta_2 = P\nabla_{\zeta_1}\zeta_2 + \omega\nabla_{\zeta_1}\zeta_2 \\ +Bh(\zeta_1,\zeta_2) + Ch(\zeta_1,\zeta_2) \end{aligned}$$

for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$. Thus we get

$$\sec\theta(\nabla_{\zeta_1}\omega)\zeta_2 = -h(\zeta_1, \widetilde{J}\zeta_2) + \sec\theta Ch(\zeta_1,\zeta_2)$$

and

$$\sec\theta(\nabla_{\zeta_2}\omega)\zeta_1 = -h(\zeta_2, \widetilde{J}\zeta_1) + \sec\theta Ch(\zeta_1,\zeta_2)$$

Hence we have

$$\sec\theta(\nabla_{\zeta_1}\omega)\zeta_2 - \sec\theta(\nabla_{\zeta_2}\omega)\zeta_1 = -h(\zeta_1, \widetilde{J}\zeta_2) + h(\zeta_2, \widetilde{J}\zeta_1)$$

If ω is parallel, we obtain

$$h(\zeta_1, \widetilde{J}\zeta_2) = h(\zeta_2, \widetilde{J}\zeta_1) \tag{3.3}$$

Putting (3.3) in (3.2) we find

$$g(\bar{R}(\zeta_1, \widetilde{J}\zeta_1)\zeta_1, \widetilde{J}\zeta_1) = \left|\left|h(\zeta_1, \widetilde{J}\zeta_1)\right|\right|^2 + ||h(\zeta_1,\zeta_1)||^2 + g(R^*(\zeta_1, \widetilde{J}\zeta_1)\zeta_1, \widetilde{J}\zeta_1)$$

Thus we arrive at

$$\bar{\mathcal{K}}_{\widetilde{J}}(\zeta_1) = \left|\left|h(\zeta_1, \widetilde{J}\zeta_1)\right|\right|^2 + ||h(\zeta_1,\zeta_1)||^2 + \mathcal{K}^N_{J_N}(\mathfrak{S}_*(\zeta_1)).$$

This gives the inequality. The equality is satisfied if and only if $h(\zeta_1, \zeta_2) = 0$ for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$ which completes the proof. □

We now have the following lemma.

Lemma 3.5. *Let M be a hemi-slant submanifold of a Kähler manifold $(\bar{\mathfrak{M}}, J, g)$. Suppose that M is submersed over an almost Hermitian manifold (N, J_N, g_N), then we have*

$$A_{\zeta_1}P\zeta_2 = \bar{A}_{\omega\zeta_2}\zeta_{1_{ker\mathfrak{S}_*}} + PA_{\zeta_1}\zeta_2 + Bh(\zeta_1,\zeta_2)_{ker\mathfrak{S}_*} \tag{3.4}$$

for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$.

Proof. Since $\bar{\mathfrak{M}}$ is a Kähler manifold, we have

$$(\bar{\nabla}_{\zeta_1}J)\zeta_2 = 0.$$

Thus using (2.6), (2.7), O'Neill's tensor fields and Gauss-Weingarten formulas, we get

$$\begin{aligned} A_{\zeta_1}P\zeta_2 + \mathcal{H}\nabla_{\zeta_1}P\zeta_2 + h(\zeta_1, P\zeta_2) - \bar{A}_{\omega\zeta_2}\zeta_1 \quad &+ \quad \nabla^{\perp}_{\zeta_1}\omega\zeta_2 = PA_{\zeta_1}\zeta_2 + P\mathcal{H}\nabla_{\zeta_1}\zeta_2 \\ &+ \quad \omega\nabla_{\zeta_1}\zeta_2 + Bh(\zeta_1,\zeta_2) + Ch(\zeta_1,\zeta_2). \end{aligned}$$

Taking the tangential parts of this equation, we derive

$$A_{\zeta_1}P\zeta_2 + \mathcal{H}\nabla_{\zeta_1}P\zeta_2 - \bar{A}_{\omega\zeta_2}\zeta_1 = PA_{\zeta_1}\zeta_2 + P\mathcal{H}\nabla_{\zeta_1}\zeta_2 + Bh(\zeta_1,\zeta_2).$$

Finally, taking the vertical components of the above equation, we have

$$A_{\zeta_1}P\zeta_2 - \bar{A}_{\omega\zeta_2}\zeta_{1_{ker\mathfrak{S}_*}} = PA_{\zeta_1}\zeta_2 + Bh(\zeta_1,\zeta_2)_{ker\mathfrak{S}_*}.$$

which is (3.4). □

From Lemma 3.5 we have the following corollary.

Corollary 3.6. *Let M be a hemi-slant submanifold of a Kähler manifold $(\bar{\mathfrak{M}}, J, g)$. Suppose that M is submersed over an almost Hermitian manifold (N, J_N, g_N). If*

$$A_{P\zeta_1}P\zeta_2 = \bar{A}_{\omega\zeta_2}P\zeta_{1_{ker\mathfrak{S}_*}} - P\bar{A}_{\omega\zeta_1}\zeta_2 - PBh(\zeta_1,\zeta_2)_{ker\mathfrak{S}_*}$$

for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$, then the distribution D^θ is integrable. Conversely, if the distribution D^θ is integrable, we have

$$\bar{A}_{\omega\zeta_2}P\zeta_{1_{ker\mathfrak{S}_*}} = P\bar{A}_{\omega\zeta_1}\zeta_2 + PBh(\zeta_1,\zeta_2)_{ker\mathfrak{S}_*}$$

Proof. From (3.4), for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$ we have

$$A_{P\zeta_1}P\zeta_2 = \bar{A}_{\omega\zeta_2}P\zeta_{1_{ker\mathfrak{S}_*}} - P\bar{A}_{\omega\zeta_1}\zeta_2 + \cos^2\theta A_{\zeta_2}\zeta_1 - PBh(\zeta_1,\zeta_2)_{ker\mathfrak{S}_*}$$

which completes the proof. □

Finally, we give Ricci tensor of horizontal distribution along a Riemannian submersion.

Proposition 3.7. *Let M be a hemi- Kählerian slant submanifold of a Kähler manifold (M, J, g). Suppose that M is submersed over an almost Hermitian manifold (N, J_N, g_N). If the morphism ω is parallel, then we have*

$$\begin{aligned} Ric_{(ker\mathfrak{S}_*)^\perp}(\zeta_1,\zeta_2) \quad &= \quad \frac{c}{4}rg(\zeta_1,\zeta_2) - trace_{(ker\mathfrak{S}_*)}(\nabla_{.}T)_{\zeta_1}\zeta_2,.) - trace(\nabla_{.}A)_{\zeta_1}\zeta_2,.) \\ &+ \quad g(T_{E_i}\zeta_1, T_{E_i}\zeta_2) + trace_{(ker\mathfrak{S}_*)}A_{\zeta_1}A_{\zeta_2} + 3trace_{(ker\mathfrak{S}_*)^\perp}A_{\zeta_1}A_{\zeta_2} \\ &- \quad \sum_{k=1}^{r} g(\bar{A}_{F_k}\zeta_{2_{(ker\mathfrak{S}*)^\perp}}, \bar{A}_{F_k}\zeta_{1_{(ker\mathfrak{S}*)^\perp}}) \end{aligned} \tag{3.5}$$

for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$ where $\{E_1, ..., E_s\}$ is the orthonormal frame of the vertical distribution and $\{F_1, ..., F_r\}$ is the orthonormal frame of the horizontal distribution.

Proof. Since $\bar{\mathfrak{M}}$ is a complex space form, from Gauss equation, we have

$$\begin{aligned} \frac{c}{4}\{g(\zeta_2,\zeta_3)g(\zeta_1,\varsigma_2) - g(\zeta_1,\zeta_3)g(\zeta_2,\varsigma_2) \quad &+ \quad g(J\zeta_2,\zeta_3)g(J\zeta_1,\varsigma_2) - g(J\zeta_1,\zeta_3)g(J\zeta_2,\varsigma_2) \\ +2g(\zeta_1, J\zeta_2)g(J\zeta_3,\varsigma_2)\} = R(\zeta_1,\zeta_2,\zeta_3,\varsigma_2) \quad &- \quad g(h(\zeta_1,\varsigma_2), h(\zeta_2,\zeta_3)) + g(h(\zeta_1,\zeta_3), h(\zeta_2,\varsigma_2)) \end{aligned}$$

Choose an adapted slant frame for D^θ as $\{e_1, \sec\theta Pe_1, ..., e_n, \sec\theta Pe_n\}$, then we compute the Ricci tensor $Ric(\zeta_1, \zeta_2)$ for $\zeta_1, \zeta_2 \in \Gamma(D^\theta)$. We have

$$\sum_{i=1}^{r} \frac{c}{4}\{g(\zeta_1,\zeta_2)g(e_j,e_j) - g(e_j,\zeta_2)g(\zeta_1,e_j) - g(Je_j,\zeta_2)g(J\zeta_1,e_j) + 2g(e_j,J\zeta_1)g(J\zeta_2,e_j)\}$$
$$= \sum_{i=1}^{r}\{R(e_j,\zeta_1,\zeta_2,e_j) - g(h(e_j,e_j),h(\zeta_1,\zeta_2)) + g(h(e_j,\zeta_2),h(\zeta_1,e_j))\}$$

Arranging this equation

$$\frac{rc}{4}g(\zeta_1,\zeta_2) = \sum_{i=1}^{r}\{R(e_j,\zeta_1,\zeta_2,e_j) - g(h(e_j,e_j),h(\zeta_1,\zeta_2)) + g(h(e_j,\zeta_2),h(\zeta_1,e_j))\}$$

From (3.3) if ω is parallel, we get

$$\frac{c}{4}rg(\zeta_1,\zeta_2) = \sum_{i=1}^{r}\{R(e_j,\zeta_1,\zeta_2,e_j) + g(h(e_j,\zeta_2),h(\zeta_1,e_j))\}.$$

If the frame of the vertical distribution is denoted by $\{E_1, ..., E_s\}$,we obtain

$$\begin{aligned}
Ric(\zeta_1,\zeta_2) &= \sum_{i=1}^{s} g(R(E_i,\zeta_1)\zeta_2,E_i) + \sum_{j=1}^{r} g(R(e_j,\zeta_1)\zeta_2,e_j) \\
&+ \sum_{j=1}^{r} g(R(\sec\theta Pe_j,\zeta_1)\zeta_2,\sec\theta Pe_j) \\
&= trace_{(ker\mathfrak{S}_*)}(\nabla_. T)_{\zeta_1}\zeta_2, .) + trace(\nabla_. A)_{\zeta_1}\zeta_2, .) \\
&- \sum_{i=1}^{s} g(T_{E_i}\zeta_1, T_{E_i}\zeta_2) - trace_{(ker\mathfrak{S}_*)}A_{\zeta_1}A_{\zeta_2} \\
&+ Ric_{(ker\mathfrak{S}_*)}(\zeta_1,\zeta_2) + \sum_{i=1}^{r}\{-2g(A_{e_i}\zeta_1, A_{\zeta_2}e_i) \\
&- 2\sec^2\theta g(A_{Pe_i}\zeta_1, A_{\zeta_2}Pe_i) + g(A_{\zeta_1}\zeta_2, A_{e_i}e_i) \\
&+ \sec^2\theta g(A_{\zeta_1}\zeta_2, A_{Pe_i}Pe_i) - g(A_{e_i}\zeta_2, A_{\zeta_1}e_i) \\
&- \sec^2\theta g(A_{Pe_i}\zeta_2, A_{\zeta_1}Pe_i)\}
\end{aligned}$$

Since A and T are skew symmetric with respect to g, and A is symmetric on D^θ, we get (3.5) □

Acknowledgment. The authors are grateful to the referees for their valuable comments and suggestions.

References

[1] A. Bejancu, *Submanifolds of a Kähler manifold I*, Proc. Amer. Math. Soc. **69** (1), 135–142, 1978.
[2] A. Carriazo, *Bi-slant immersions*, in: Proc. ICRAMS 2000, Kharagpur, India, 88–97, 2000.
[3] B. Y. Chen, *CR-submanifolds of a Kähler manifold I*, J. Dif. Geo. **16** (2), 305–322, 1981.
[4] B. Y. Chen, *Differential geometry of real submanifolds in a Kähler manifold*, Monatshefte für Math. **91** (4), 257–274, 1981.
[5] S. Deshmukh, S. Ali and S.I. Husain, *Submersions of CR-submanifolds of a Kaehler manifold*, Indian J. Pure Appl. Math. **19** (12), 1185–1205, 1988.

[6] S. Deshmukh, T. Ghazal and H. Hashem, *Submersions of CR-submanifolds on an almost Hermitian manifold*, Yokohama Math. J. **40** (1), 45–57, 1992.

[7] T. Fatima, M. A. Akyol and A. A. Alzulaibani, *On a submersion of generic submanifold of a nearly Kaehler manifold*, Int. J. Geom. Methods Mod. Phys. **19** (4), 2022.

[8] T. Fatima and S. Ali, *Submersions of generic submanifolds of a Kaehler manifold*, Arab J. Math. Sci. **20** (1), 119–131, 2014.

[9] S. Kobayashi, *Submersions of CR submanifolds*, Tohoku Math. J. **39** (1), 95–100, 1987.

[10] J. Mikes, A. Vanzurova, and I. Hinterleitner, *Geodesic Mappings and Some Generalizations*, Publ. Palacky University Olomouc, 2009.

[11] B. O'Neill, *The fundamental equations of a submersion*, Michigan Math. J. **13** (4), 459–469, 1966.

[12] M. H. Shahid, F. R. Al-Solamy and M. Jamali, *Submersions of CR Submanifolds*, in: Geometry of Cauchy-Riemann Submanifolds, 311–342, Springer-Verlag, Berlin, 2016.

[13] M.D. Siddiqi, *Submersions of contact CR-submanifolds of generalized quasi-Sasakian manifolds*, J. Dyn. Syst. Geom. Theor. **18** (1), 81–95, 2020.

[14] B. Şahin, *Warped product submanifolds of Kaehler manifolds with a slant factor*, Ann. Polon. Math. **95** (3), 207–226, 2009.