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Certain Results on Extended Beta and Related Functions Using Matrix Arguments

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Research Article

Abstract — In this study, we present and explore extended beta matrix functions (EBMFs) and their key properties. By utilizing the beta matrix function (BMF), we introduce novel extensions of the Gauss hypergeometric matrix function (GHMF) and Kummer hypergeometric matrix function (KHMF). We delve into their integral representations, recurrence relations, transformation properties, and differential formulas. Additionally, we investigate their statistical applications, mainly focusing on the beta distribution, and derive expressions for the mean, variance, and moment-generating functions. Furthermore, we apply EBMFs to develop the Appell matrix function (AMF) and Lauricella matrix function (LMF) and their integral forms.

Keywords *Beta matrix function, Gauss and Kummer hypergeometric matrix functions, Appell and Lauricella matrix functions*

Mathematics Subject Classification (2020) 33B15, 33E20

1. Introduction

Special matrix functions are a dynamic and intriguing area [1–14] with significant applications in mathematics and physics. When these functions are generalized from scalar to matrix arguments, they offer deeper insights and broaden the scope of their applications. Matrix versions of special functions enhance the utility of their scalar counterparts by extending their relevance to multidimensional and more complex problems. This generalization plays a crucial role in engineering, physics, statistics, and mathematics fields, providing powerful tools for addressing matrix-related challenges and advancing theoretical and practical research. Special matrix functions represent a critical extension of classical special function theory, enabling matrices to be manipulated in ways similar to numbers. This capability proves particularly valuable in applications of fields such as quantum mechanics, statistical mechanics, and signal processing, where matrices are frequently encountered.

The extended beta function is a matrix version of the classical beta function, which arises in various areas of mathematics and physics. Recent studies [1–3, 10, 11, 15] have focused on analyzing the matrix beta function and exploring its convergence regions, integral representations, and differential properties. Similarly, the extended Gauss hypergeometric and Kummer hypergeometric functions are matrix generalizations of their classical counterparts and have been the subject of considerable study in recent years [1–3, 7, 10, 15, 16]. Building on these foundational works, this paper discusses

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the extended beta matrix functions (EBMFs) and their integral representations, recurrence relations, transformation formulas, and differential properties. We also research their applications in statistics. We also define and investigate the integral representations of the extended Appell matrix function (EAMF) and the extended Lauricella matrix function (ELMF).

2. Preliminaries

Throughout this paper, the vector space of r-square matrices with complex entries is designated $\mathbb{C}^{r\times r}$. Spectrum is the set of all the eigenvalues of a matrix $\mathcal{P}\in\mathbb{C}^{r\times r}$ and represented by the symbol $\sigma(\mathcal{P})$. A matrix $\mathcal{P}$ in $\mathbb{C}^{r\times r}$ is called a positive stable matrix (PSM) if $\Re(\lambda)>0$, for all $\lambda\in\sigma(\mathcal{P})$, where $\Re(z)$ represents the real part of a complex number z.

The expression $\Gamma(\mathcal{P})$ for a PSM $\mathcal{P}$ in $\mathbb{C}^{r\times r}$ is as follows [11]:

$$\Gamma(\mathcal{P})=\int_0^\infty e^{-\ell}\ell^{\mathcal{P}-I}d\ell$$

Furthermore, if $\mathcal{P}+\kappa I$ is invertible, for all $\kappa\in\mathbb{Z}^+\cup\{0\}$, then the reciprocal gamma matrix function (GMF) is defined as [11]:

$$\Gamma^{-1}(\mathcal{P})=\mathcal{P}(\mathcal{P}+I)\cdots(\mathcal{P}+(n-1)I)\Gamma^{-1}(\mathcal{P}+nI),\quad n\geq 1$$

If $\mathcal{P}\in\mathbb{C}^{r\times r}$ is a PSM and $n\geq 0$ is an integer, then the GMF can also be defined in the form of a limit as [11]:

$$\Gamma(\mathcal{P})=\lim_{n\to\infty}(n-1)!(\mathcal{P})_n^{-1}n^{\mathcal{P}}$$

The Pochhammer symbol [12] for $\mathcal{P}\in\mathbb{C}^{r\times r}$ is defined as:

$$(\mathcal{P})_n=\begin{cases} I, & n=0\\ \mathcal{P}(\mathcal{P}+I)...(\mathcal{P}+(n-1)I), & n\geq 1\end{cases}$$

Therefore,

$$(\mathcal{P})_n=\Gamma^{-1}(\mathcal{P})\Gamma(\mathcal{P}+nI),\quad n\geq 1$$

If $\mathcal{P}$ and $\mathcal{Q}$ are PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{P}\mathcal{Q}=\mathcal{Q}\mathcal{P}$, then the beta matrix function (BMF) is defined as [11]:

$$\mathscr{B}(\mathcal{P},\mathcal{Q})=\Gamma(\mathcal{P})\Gamma(\mathcal{Q})\Gamma^{-1}(\mathcal{P}+\mathcal{Q})=\int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}d\ell \tag{2.1}$$

Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{H}$ be PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{H}+\kappa I$ be invertible, for all $\kappa\in\mathbb{Z}^+\cup\{0\}$. Then, the Gauss hypergeometric matrix function (GHMF) is [12]:

$${}_2\mathscr{F}_1(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\sum_{n=0}^\infty(\mathcal{P})_n(\mathcal{Q})_n(\mathcal{H})_n^{-1}\frac{z^n}{n!} \tag{2.2}$$

The series in (2.2) converges absolutely for $|z|<1$, and for $z=1$ if $\alpha(\mathcal{P})+\alpha(\mathcal{Q})<\beta(\mathcal{H})$, where $\alpha(\mathcal{P})=\max\{\Re(z)\mid z\in\sigma(\mathcal{P})\}$, $\beta(\mathcal{P})=\min\{\Re(z)\mid z\in\sigma(\mathcal{P})\}$, and $\beta(\mathcal{P})=-\alpha(-\mathcal{P})$.

Furthermore, if $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$ and $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ are PSMs, then for $|z|<1$, an integral form of (2.2) is defined as [12]:

$${}_2\mathscr{F}_1(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\int_0^1(1-z\ell)^{-\mathcal{P}}\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})\Gamma(\mathcal{H})$$

Let $\mathcal{P}$, $\mathcal{Q}$, and $\mathcal{A}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$. Then, the EBMF $\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A})$ is defined by Abdalla and Bakhet [2] as follows:

$$\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A})=\int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell(1-\ell)}\right)d\ell$$

They generalized the GHMF and Kummer hypergeometric matrix function (KHMF) using EBMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, and $\mathcal{A}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$, $\mathcal{H}\mathcal{A}=\mathcal{A}\mathcal{H}$, and $\mathcal{Q}\mathcal{A}=\mathcal{A}\mathcal{Q}$. The extended GHMF (EGHMF) and the extended KHMF (EKHMF) are defined as [1]:

$$\mathscr{F}^{(\mathcal{A})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A})\frac{z^m}{m!}\right)\times\Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})$$

and

$$\Phi^{\mathcal{A}}(\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A})\frac{z^m}{m!}\right)\times\Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{Q})\Gamma^{-1}(\mathcal{H}-\mathcal{Q})$$

respectively.

Verma et al. [17] have introduced another extension of BMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$. Then, the EBMF $\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ is defined as [17]:

$$\mathscr{B}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})=\int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell}-\frac{\mathcal{C}}{(1-\ell)}\right)d\ell \tag{2.3}$$

Moreover, they introduced EGHMF and EKHMF by (2.3) as follows [17]:

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}(\mathcal{P})_m\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

and

$$\Phi^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\left(\sum_{m\geq 0}\mathscr{B}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

respectively.

Inspired and motivated by EBMF, GHMF, and KHMF, we introduce their extensions and discuss these extensions' integral representations, differential formulae, recurrence relations, and transformation formulae.

3. An Extension of EBMF

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs and commuting matrices in $\mathbb{C}^{r\times r}$ and $\eta,\mu\in\mathbb{C}$. Then, we introduce an extension of EBMF (EOEBMF) $\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ as follows:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})=\int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}}-\frac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \tag{3.1}$$

By applying Schur decomposition [18] and substituting $\ln\ell<\ell$ and $\ln(1-\ell)<(1-\ell)$, for $0<\ell<1$, respectively, we obtain

$$\mathscr{B}(\alpha(\mathcal{P})+i-\kappa,\alpha(\mathcal{Q})+j-l;\alpha(\mathcal{A}),\alpha(\mathcal{C}))<\infty$$

Thus, an EOEBMF $\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ exists.

Theorem 3.1. The EOEBMF satisfies the following integral representations:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = 2\int_0^{\pi/2} (\cos u)^{2\mathcal{P}-I}(\sin u)^{2\mathcal{Q}-I}\exp\left(-\mathcal{A}sec^{2\eta}u - \mathcal{C}csc^{2\mu}u\right)du \tag{3.2}$$

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^{\infty} u^{\mathcal{P}-I}(1+u)^{-\mathcal{P}-\mathcal{Q}}\exp\left(-\mathcal{A}(1+u^{-1})^{\eta} - \mathcal{C}(1+u)^{\mu}\right)du \tag{3.3}$$

and

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = 2^{I-\mathcal{P}-\mathcal{Q}}\int_{-1}^{1}(1+u)^{\mathcal{P}-I}(1-u)^{\mathcal{Q}-I}\times\exp\left(-2^{\eta}\mathcal{A}(1+u)^{-\eta} - 2^{\mu}\mathcal{C}(1-u)^{-\mu}\right)du \tag{3.4}$$

Proof. Substituting $\ell = \cos^2 u$ into (3.1) yields (3.2) after minor simplifications. Similarly, substituting $\ell = \frac{u}{1+u}$ into (3.1) results in (3.3). Finally, replacing $\ell = \frac{1+u}{2}$ in (3.1) provides (3.4). □

Remark 3.2. If $\eta = \mu = 1$ in (3.2), (3.3), and (3.4), respectively, then the result in [17] is obtained.

Theorem 3.3. The EOEBMF satisfies the following properties:

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q}+I;\mathcal{A},\mathcal{C}) + \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C}) = \mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) \tag{3.5}$$

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) = \sum_{n=0}^{\infty}\frac{(\mathcal{Q})_n}{n!}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,I;\mathcal{A},\mathcal{C}) \tag{3.6}$$

and

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \sum_{n=0}^{\infty}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,\mathcal{Q}+I;\mathcal{A},\mathcal{C}) \tag{3.7}$$

Proof. From (3.1),

$$\begin{aligned}
\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q}+I;\mathcal{A},\mathcal{C}) + \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C}) &= \int_0^1[\ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}}]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&\quad + \int_0^1[\ell^{\mathcal{P}}(1-\ell)^{\mathcal{Q}-I}]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&= \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}[(1-\ell)+\ell]\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&= \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&= \mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})
\end{aligned}$$

Hence, the proof of (3.5) is done. Moreover,

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^1 \ell^{\mathcal{P}-I}(1-\ell)^{I-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^{\eta}} - \frac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell$$

By using the relation $(1-\ell)^{-\mathcal{Q}} = \sum\limits_{n=0}^{\infty}\frac{(\mathcal{Q})_n}{n!}\ell^n$ in [12],

$$\begin{aligned}
\mathscr{B}_{\eta,\mu}(\mathcal{P},I-\mathcal{Q};\mathcal{A},\mathcal{C}) &= \int_0^1 \ell^{\mathcal{P}-I}\sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\ell^n\exp\left(-\tfrac{\mathcal{A}}{\ell^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&= \sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\int_0^1 \ell^{\mathcal{P}+(n-1)I}\exp\left(-\tfrac{\mathcal{A}}{t^{\eta}} - \tfrac{\mathcal{C}}{(1-\ell)^{\mu}}\right)d\ell \\
&= \sum_{n=0}^{\infty}\tfrac{(\mathcal{Q})_n}{n!}\mathscr{B}_{\eta,\mu}(\mathcal{P}+nI,I;\mathcal{A},\mathcal{C})
\end{aligned}$$

Thus, the proof of (3.6) is done. Similarly, by substituting its series representation for $(1-\ell)^{-I}$ in (3.1),

$$\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^1 (1-\ell)^{\mathcal{Q}} \sum_{n=0}^{\infty} \ell^{\mathcal{P}+(n-1)I} \exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right) d\ell$$

The result (3.7) is obtained by using (3.1) and altering the integration and summation orders. □

4. Application of EOEBMF

Many researchers [2, 11, 17, 19, 20] have investigated different generalizations and extensions of BMFs, showcasing their potential applications in various domains. In this section, we analyze an application of the EOEBMF in (3.1) within the realm of statistics. Specifically, we define the beta distribution and derive its mean, variance, and moment-generating function using the EOEBMF.

For $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be commutative PSMs in $\mathbb{C}^{r\times r}$ and $\Re(\eta), \Re(\mu) > 0$. Define the beta distribution as:

$$u(\ell) = \begin{cases} [\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}\ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right), & 0<\ell<1 \\ 0, & \text{otherwise} \end{cases} \tag{4.1}$$

For any matrix $\mathcal{R} \in \mathbb{C}^{r\times r}$, the moment of a random variable X is as follows:

$$E\left(X^{\mathcal{R}}\right) = \mathscr{B}_{\eta,\mu}(\mathcal{P}+\mathcal{R},\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

If $\mathcal{R} = I$, then the mean of the beta distribution is as follows:

$$\rho = E\left(X^{I}\right) = \mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

Therefore, the variance of the distribution is defined as:

$$\begin{aligned}\sigma^2 &= E(X^{2I}) - \left\{E(X^I)\right\}^2 \\ &= \mathscr{B}_{\eta,\mu}(\mathcal{P}+2I,\mathcal{Q};\mathcal{A},\mathcal{C}[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1} - \{\mathscr{B}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}\}^2\end{aligned}$$

Besides, the moment generating matrix function of the distribution in (4.1) is as follows:

$$M(\ell) = \sum_{\kappa=0}^{\infty} \frac{\ell^\kappa}{\kappa!} E(X^{\kappa I}) = [\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1} \sum_{\kappa=0}^{\infty} \mathscr{B}_{\eta,\mu}(\mathcal{P}+\kappa I,\mathcal{Q};\mathcal{A},\mathcal{C}) \frac{\ell^\kappa}{\kappa!}$$

The cumulative distribution of (4.1) is defined as:

$$\mathscr{F}(x) = \int_0^x u(\ell)d\ell = \mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})]^{-1}$$

where $F(1) = I$ and $\mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C})$ is the incomplete BMF defined as:

$$\mathscr{B}_{x,\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{A},\mathcal{C}) = \int_0^x \ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I} \exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right) d\ell$$

5. Graphical and Numerical Comparison of the Classical and Generalized Matrix-Variate Beta Distributions

The classical beta distribution involving the BMF in (2.1) is defined as:

$$u(\ell) = \begin{cases} [\mathscr{B}(\mathcal{P},\mathcal{Q})]^{-1}\ell^{\mathcal{P}-I}(1-\ell)^{\mathcal{Q}-I}, & 0<\ell<1 \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

Consider $\mathcal{P} = \begin{pmatrix} 2 & 0.5 \\ 0.5 & 3 \end{pmatrix}$, $\mathcal{Q} = \begin{pmatrix} 3 & 0.2 \\ 0.2 & 4 \end{pmatrix}$, $\mathcal{A} = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 2 \end{pmatrix}$, $\mathcal{C} = \begin{pmatrix} 1.5 & 0.3 \\ 0.3 & 2.5 \end{pmatrix}$, and $\eta = \mu = 2$.

In Figure 1, taking $\mathcal{P}$ and $\mathcal{Q}$ matrices, compute the eigenvalues of $\mathcal{P} - I$ and $\mathcal{Q} - I$, and using in (5.1) to compute and plot the classical beta distribution over the range $0 < \ell < 1$ for 2×2 matrices.

Moreover, in Figure 2, taking $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ matrices and $\eta = \mu = 2$, compute the eigenvalues of $\mathcal{P} - I$, $\mathcal{Q} - I$, $\mathcal{A}$, and $\mathcal{C}$ and using in (4.1) to compute and plot the generalized beta distribution with parameters $\mathcal{A}$, $\mathcal{C}$, η, and μ. In Figure 3, we compare our generalized beta distribution with the classical beta distribution in matrices.

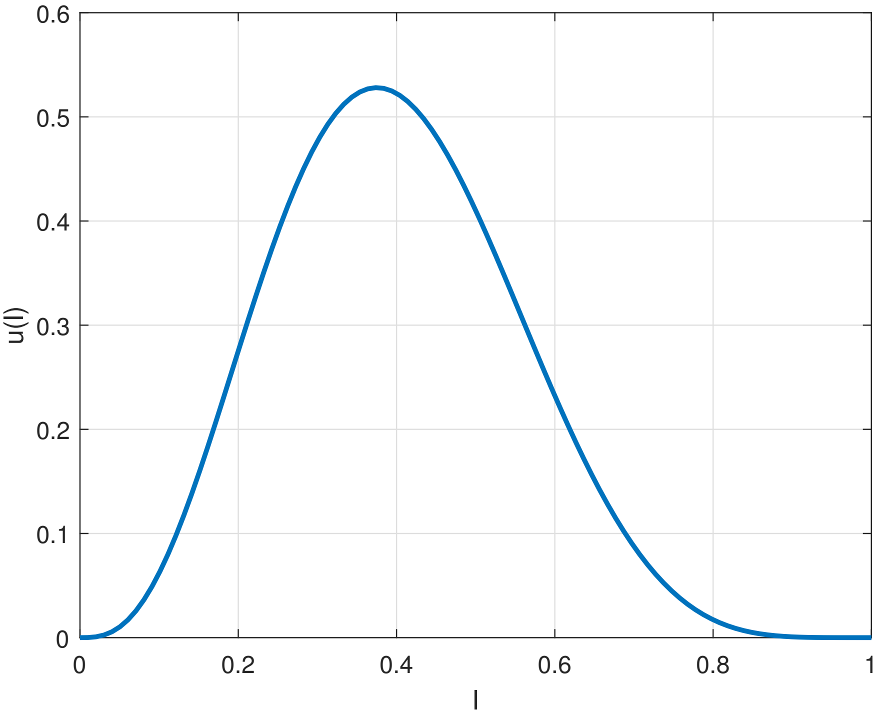

Figure 1. Classical beta distribution for 2×2 matrices $\mathcal{P}$ and $\mathcal{Q}$

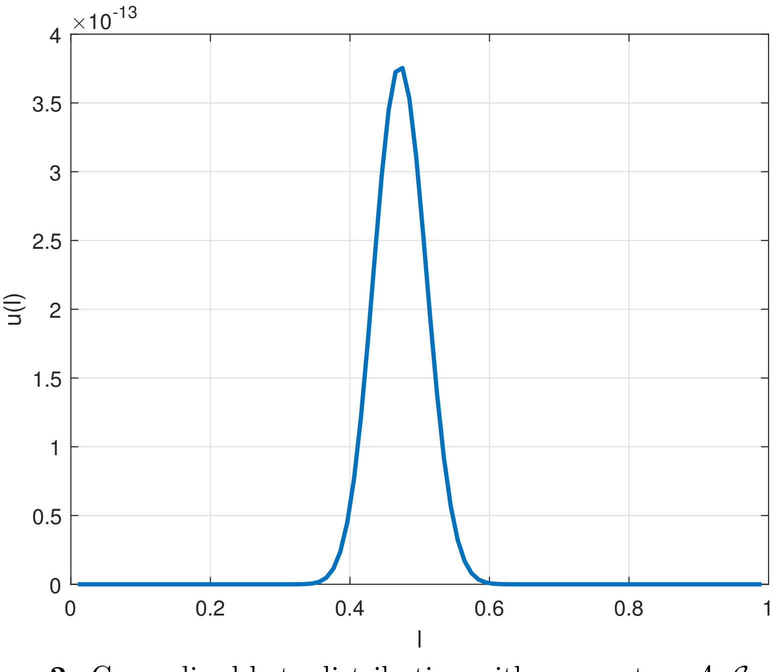

Figure 2. Generalized beta distribution with parameters $\mathcal{A}$, $\mathcal{C}$, η, and μ

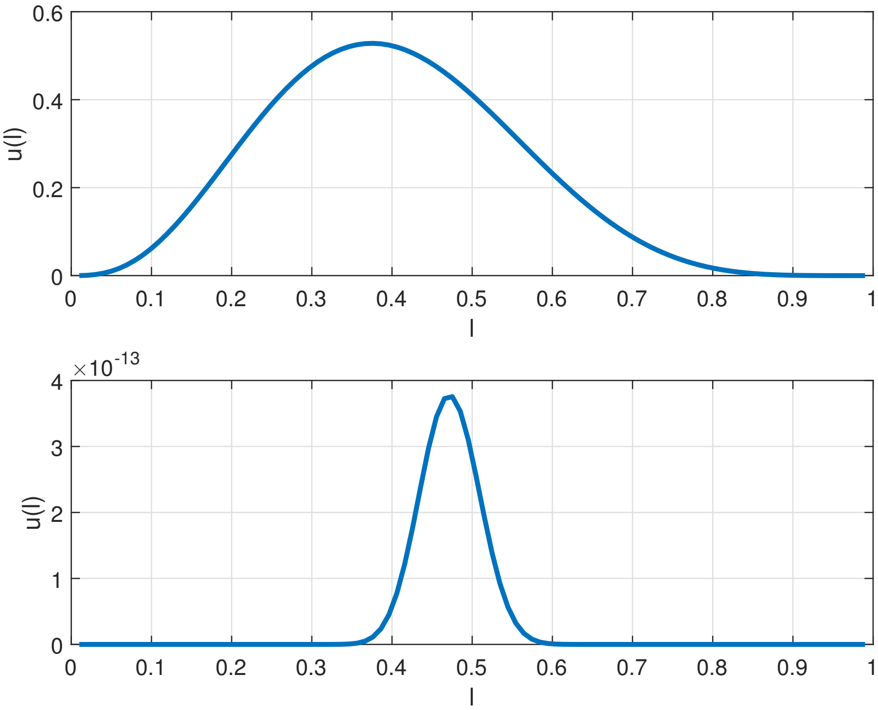

Figure 3. (a) Classical beta distribution and (b) Generalized beta distribution with exponential terms

Both distributions are normalized using a simplified approach based on the scalar beta function. In Figure 1, the distribution is closely related to the scalar classical beta distribution, generalized to matrix arguments $\mathcal{P}$ and $\mathcal{Q}$.

The simpler matrix beta distribution directly relates to random matrix theory, which has applications in signal processing, wireless communications, and finance. The simpler form is also used for matrix-variate generalizations of Bayesian analysis or weighting in optimization problems, particularly in multivariate or matrix-based Bayesian methods. However, the flexibility to model more complex real-world phenomena is restricted because it lacks additional factors like essential terms.

However, in our result, we provided the additional terms $\exp\left(-\frac{\mathcal{A}}{\ell^\eta} - \frac{\mathcal{C}}{(1-\ell)^\mu}\right)$ introduce exponential decay, which can allow for greater flexibility in fitting data or modeling more complex systems. This distribution could be used in more advanced Bayesian frameworks where the priors need to account for additional penalization or constraints, often seen in hierarchical models or models with specific tail behavior. The exponential terms can capture the behavior that decays rapidly, which is helpful in stochastic modeling, particularly in systems with non-linear dynamics or time-varying processes. In areas like financial modeling or signal processing, where matrix-valued variables may represent volatility or correlation, the exponential decay allows better control over tail risks or sensitivity. The exponential terms provide much more flexibility in controlling the shape and behavior of the distribution. This is particularly useful in real-world applications where tail behavior, constraints, or penalizations are needed. Parameters like $\mathcal{A}$, $\mathcal{C}$, η, and μ offer additional degrees of freedom for fine-tuning the distribution, making it more adaptable to complex data or phenomena.

6. EGHMF and EKHMF

The main aim of this section is to introduce extensions of GHMF and KHMF. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ be positive stable and commuting matrices in $\mathbb{C}^{r\times r}$. Extensions of GHMF and KHMF, i.e., EGHMF and EKHMF, are defined as follows:

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z) = \left(\sum_{m\geq 0}(\mathcal{P})_m \mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times \mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.1}$$

and

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z) = \left(\sum_{m\geq 0}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times \mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.2}$$

respectively.

Theorem 6.1. For PSMs $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{A}$, and $\mathcal{C}$ in $\mathbb{C}^{r\times r}$, the EGHMF and EKHMF have following integral representation, respectively.

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z) = \int_0^1 (1-z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.3}$$

and

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z) = \left(\int_0^1 \ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)d\ell\right)\times\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1} \tag{6.4}$$

PROOF. Using (6.1),

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z) = \left(\sum_{m\geq 0}(\mathcal{P})_m \mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\frac{z^m}{m!}\right)\times \mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})^{-1}$$

Using (3.1),

$$\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P},\mathcal{Q};\mathcal{H},z) = \left(\sum_{m\geq 0}(\mathcal{P}_m)\left(\int_0^1 \ell^{\mathcal{Q}+(m-1)I}(1-\ell)^{(\mathcal{H}-\mathcal{Q})-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)d\ell\right)\frac{z^m}{m!}\right)\times\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))^{-1}$$

Moreover, the following matrix identity is valid:

$$(1-z\ell)^{-\mathcal{P}} = \sum_{m=0}^{\infty}(\mathcal{P})_m\frac{(z\ell)^m}{m!}$$

Thus,

$$\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H},z) = \int_0^1 (1-z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{(\mathcal{H}-\mathcal{Q})-I}d\ell\times[\mathscr{B}(\mathcal{Q},(\mathcal{H}-\mathcal{Q}))]^{-1}$$

Similarly, by (6.2), (6.4) is obtained. □

Theorem 6.2. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}\mathcal{H}=\mathcal{H}\mathcal{Q}$. Then, the following differential equations are satisfied by EGHMF and EKHMF, respectively:

$$\frac{d^n}{dz^n}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = (\mathcal{P})_n\mathscr{F}_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{P}+nI,\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1}$$

and

$$\frac{d^n}{dz^n}\Phi_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{Q};\mathcal{H};z) = \Phi_{\eta,\mu}^{\mathcal{A},\mathcal{C}}(\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1} \tag{6.5}$$

Proof. From (6.1),

$$\frac{d}{dz}\mathscr{F}^{\mathcal{A},\mathcal{C}}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = \frac{d}{dz}\sum_{n=0}^{\infty}(\mathcal{P})_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+nI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!}$$
$$= \sum_{n=1}^{\infty}(\mathcal{P})_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+nI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^{n-1}}{(n-1)!}$$
$$= \sum_{n=0}^{\infty}(\mathcal{P})_{(n+1)}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+(n+1)I,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!}$$
$$= \mathcal{P}\sum_{n=0}^{\infty}(\mathcal{P}+I)_n\mathscr{B}_{\eta,\mu}(\mathcal{Q}+(n+1)I,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q}+I,\mathcal{H}-\mathcal{Q})]^{-1}\frac{z^n}{n!}(\mathcal{Q})(\mathcal{H})^{-1}$$
$$= (\mathcal{P})_1\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)(\mathcal{Q})_1(\mathcal{H})_1^{-1}$$

Repeat this process n times. The differential formula appears as

$$\frac{d^n}{dz^n}\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = (\mathcal{P})_n\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P}+nI,\mathcal{Q}+nI;\mathcal{H}+nI;z)(\mathcal{Q})_n(\mathcal{H})_n^{-1}$$

Similarly, (6.5) is obtained by (6.2). □

7. Transformation Formulae

In this section, we provide the transformation formulae for EGHMF and EKHMF.

Theorem 7.1. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ and $\mathcal{QH}=\mathcal{HQ}$. Then, the following formulae are satisfied by EGHMF:

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = (1-z)^{-\mathcal{P}}\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\mu,\eta}\left(P,H-Q;H;\frac{z}{(z-1)}\right) \tag{7.1}$$

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};1-\frac{1}{z}) = z^{\mathcal{P}}\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\mu,\eta}(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};1-z) \tag{7.2}$$

and

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};\frac{z}{z+1}) = (1+z)^{\mathcal{P}}\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\mu,\eta}(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};-z) \tag{7.3}$$

Proof. In (6.3), if ℓ is changed to $(1-\ell)$, then

$$\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H};z) = \int_0^1(1-z(1-\ell))^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^\eta}-\frac{\mathcal{C}}{\ell^\mu}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}$$
$$= \int_0^1(1-z+z\ell)^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^\eta}-\frac{\mathcal{C}}{\ell^\mu}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}$$
$$= (1-z)^{-\mathcal{P}}\int_0^1(1-\frac{z\ell}{z-1})^{-\mathcal{P}}\exp\left(-\frac{\mathcal{A}}{(1-\ell)^\eta}-\frac{\mathcal{C}}{\ell^\mu}\right)(1-\ell)^{\mathcal{Q}-I}\ell^{\mathcal{H}-\mathcal{Q}-I}d\ell[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}$$
$$= (1-z)^{-\mathcal{P}}\mathscr{F}^{(\mathcal{A},\mathcal{C})}_{\mu,\eta}\left(\mathcal{P},\mathcal{H}-\mathcal{Q};\mathcal{H};\frac{z}{z-1}\right)$$

To determine (7.2) and (7.3), we replace z in (7.1) with $(1-\frac{1}{z})$ and $\frac{z}{1+z}$, respectively. □

Setting $z=1$ and allowing $\mathcal{P}$ to commute with $\mathcal{Q}$ and $\mathcal{H}$ provides the link between the EGHMF and EBMF that is shown in (6.1):

$$\mathscr{F}^{\mathcal{A},\mathcal{C}}_{\eta,\mu}(\mathcal{P},\mathcal{Q};\mathcal{H},1) = \left(\int_0^1\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{P}-\mathcal{Q}-I}\exp\left(-\frac{\mathcal{A}}{\ell^\eta}-\frac{\mathcal{C}}{(1-\ell)^\mu}\right)d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}$$
$$=\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{P}-\mathcal{Q};\mathcal{A},\mathcal{C})[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \tag{7.4}$$

Using (7.4), we can formulate a novel generalization of Kummer's first theorem.

Theorem 7.2. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{QH}=\mathcal{HQ}$. Then, Kummer's first theorem for new extension is provided as:

$$\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\exp(z)\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{H}-\mathcal{Q};\mathcal{H};-z)$$

Theorem 7.3. Let $\mathcal{A}$, $\mathcal{C}$, $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{H}$, and $\mathcal{H}-\mathcal{Q}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{QH}=\mathcal{HQ}$. Then, EGHMF and EKHMF satisfy the following recurrence relations:

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)\mathcal{Q}\mathcal{H}^{-1} \tag{7.5}$$

$$\frac{d}{dz}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=\frac{\mathcal{P}}{z}\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) \tag{7.6}$$

$$\mathcal{Q}\Delta_{\mathcal{Q}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H}+I;z)+\mathcal{H}\Delta_{\mathcal{H}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=0 \tag{7.7}$$

and

$$\frac{d}{dz}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z)=\mathcal{Q}\mathcal{H}^{-1}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H}+I;z)-\Delta_{\mathcal{H}}\Phi_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{Q};\mathcal{H};z) \tag{7.8}$$

where $\Delta_{\mathcal{P}}$ is the shift operator relative to $\mathcal{P}$.

PROOF. By using $\Delta_{\mathcal{P}}$ as the shift operator about $\mathcal{P}$ and the integral representation of the EGHMF (6.1),

$$\begin{aligned}\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z) &= \mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q};\mathcal{H};z)-\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)\\ &= \left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}(1-(1-z\ell))\exp\left(-\tfrac{\mathcal{A}}{\ell^\eta}-\tfrac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1}\end{aligned}$$

Therefore,

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}\exp\left(-\tfrac{\mathcal{A}}{\ell^\eta}-\tfrac{\mathcal{C}}{(1-\ell)^\mu}\right)\ell^{\mathcal{Q}}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\times[\mathscr{B}(\mathcal{Q},\mathcal{H}-\mathcal{Q})]^{-1} \tag{7.9}$$

We can see from (6.1) that

$$\begin{aligned}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z) &= \left(\int_0^1(1-z\ell)^{-\mathcal{P}-I}(1-(1-z\ell))\exp\left(-\tfrac{\mathcal{A}}{\ell}-\tfrac{\mathcal{C}}{(1-\ell)}\right)\ell^{\mathcal{Q}-I}(1-\ell)^{\mathcal{H}-\mathcal{Q}-I}d\ell\right)\\ &\quad\times[\mathscr{B}(\mathcal{Q}+I,\mathcal{H}-\mathcal{Q})]^{-1}\end{aligned} \tag{7.10}$$

From (7.9) and (7.10),

$$\Delta_{\mathcal{P}}\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P},\mathcal{Q};\mathcal{H};z)=z\mathscr{F}_{\eta,\mu}^{(\mathcal{A},\mathcal{C})}(\mathcal{P}+I,\mathcal{Q}+I;\mathcal{H}+I;z)\mathcal{Q}\mathcal{H}^{-1}$$

Another differential recurrence relation can be found using the EGHMF's differentiation formula, as illustrated in (7.6). The results in (7.7) and (7.8) can be obtained by using the same steps as the proof in (7.5) and (7.6). □

8. EAMF and ELMF

This section extends the Appell matrix function (AMF) and Lauricella matrix function (LMF) to three variables. Specifically, we present the extended forms of the AMF, i.e., $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w)$ and $\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w)$, and the LMF with three variables, $\mathscr{F}_{\mathcal{D}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H};z;w;v)$. These extensions are formulated using the new EBMF [7,16,21]. Additionally, we provide integral representations for these extended hypergeometric matrix functions.

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, $\mathcal{A}$, and $\mathcal{C}$ commutes, $\mathcal{HQ}=\mathcal{QH}$, and $\mathcal{HQ}'=\mathcal{Q}'\mathcal{H}$. Then, we define an extension of EAMF as:

$$\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})=\Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n\geq0}\mathscr{B}_{\eta,\mu}(\mathcal{P}+(m+n)I,\mathcal{H}-\mathcal{P};\mathcal{A},\mathcal{C})(\mathcal{Q})_m(\mathcal{Q}')_n\frac{z^m w^n}{m!n!}$$

where

$$\Gamma\begin{pmatrix} \mathcal{H} \\ \mathcal{P}, \mathcal{H}-\mathcal{P} \end{pmatrix} = \Gamma(\mathcal{H})\Gamma^{-1}(\mathcal{P})\Gamma^{-1}(\mathcal{H}-\mathcal{P})$$

Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{H}'-\mathcal{Q}'$, $\mathcal{A}$, and $\mathcal{C}$ in $\mathbb{C}^{r\times r}$ be commutative PSMs such that $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{A}$, and $\mathcal{C}$ commutes. We define the new extended Appell hypergeometric matrix function (EAHMF) $\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C})$ as:

$$\begin{aligned}\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \sum_{m,n\geq 0}(\mathcal{P})_{m+n}\mathscr{B}_{\eta,\mu}(\mathcal{Q}+mI,\mathcal{H}-\mathcal{Q};\mathcal{A},\mathcal{C})\mathscr{B}_{\eta,\mu}(\mathcal{Q}'+nI,\mathcal{H}'-\mathcal{Q}';\mathcal{A},\mathcal{C})\frac{z^m w^n}{m!n!} \\ \times\Gamma\begin{pmatrix} \mathcal{H},\mathcal{H}' \\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}' \end{pmatrix}\end{aligned} \tag{8.1}$$

Suppose $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, and $\mathcal{A}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, and $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$. Then, we define the extension of the new Lauricella hypergeometric matrix functions (LHMF) defined as:

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v) = \Gamma\begin{pmatrix} \mathcal{H} \\ \mathcal{P},\mathcal{H}-\mathcal{P} \end{pmatrix}\sum_{m,n,p\geq 0}\mathscr{B}_{\eta,\mu}(\mathcal{P}+(m+n+p)I,\mathcal{H}-\mathcal{P};\mathcal{A},\mathcal{C})(\mathcal{Q})_m(\mathcal{Q}')_n(\mathcal{Q}'')_p\frac{z^m w^n v^p}{m!n!p!} \tag{8.2}$$

We focus on identifying the integral representations of the three variable extensions of the AMF and the LMF. We start by representing the integral of $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$ determined in the following theorem.

Theorem 8.1. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, $\mathcal{A}$, and $\mathcal{C}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, and $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$. Then, the EAMF $\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$ can be presented in the integral form as:

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C}) = \Gamma\begin{pmatrix} \mathcal{H} \\ \mathcal{P},\mathcal{H}-\mathcal{P} \end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}\right. \\ \left.\times\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)du\right)\end{aligned} \tag{8.3}$$

Proof. Using (3.1) in the EAMF $\mathscr{F}_1^{\eta,\mu}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C})$,

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C}) = \Gamma\begin{pmatrix} \mathcal{H} \\ \mathcal{P},\mathcal{H}-\mathcal{P} \end{pmatrix}\sum_{m,n\geq 0}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)\right. \\ \left.\times(\mathcal{Q})_m(\mathcal{Q}')_n\frac{(zu)^m(wu)^n}{m!n!}du\right)\end{aligned} \tag{8.4}$$

By the method discussed by Dwivedi and Sahai [21], the equality

$$(1-z)^{-\mathcal{P}} = \sum_{n=0}^{\infty}(\mathcal{P})_n\frac{z^n}{n!} \tag{8.5}$$

and (8.4),

$$\begin{aligned}\mathscr{F}_1^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H};z,w;\mathcal{A},\mathcal{C}) = \Gamma\begin{pmatrix} \mathcal{H} \\ \mathcal{P},\mathcal{H}-\mathcal{P} \end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{-\mathcal{C}}{(1-u)^\mu}\right)\right. \\ \left.\times(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}du\right)\end{aligned}$$

□

Remark 8.2. After replacing the values $\mu=\eta=1$ in (8.3), the results described in [17] are obtained.

Theorem 8.3. Let $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{H}-\mathcal{Q}$, $\mathcal{H}'-\mathcal{Q}'$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{H}$, $\mathcal{H}'$, $\mathcal{A}$, and $\mathcal{C}$ commutes with each other. Then, the EAMF $\mathscr{F}_2^{\eta,\mu}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C})$ defined in (8.1) has the following integral representation:

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \left(\int_0^1\int_0^1 (1-zu-wv)^{-\mathcal{P}} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I} \times \exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{\mathcal{C}}{(1-u)^\mu}-\frac{\mathcal{A}}{v^\eta}-\frac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.6}$$

Proof. Using (3.1) and (8.1),

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \sum_{m,n\geq 0}\left(\int_0^1\int_0^1 (\mathcal{P})_{m+n}\frac{(zu)^m(wv)^n}{m!n!} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I} \times \exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{\mathcal{C}}{(1-u)^\mu}-\frac{\mathcal{A}}{v^\eta}-\frac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.7}$$

by the interchanging summation and integral in (8.7) via the dominated convergence theorem. Moreover, the following summation formula [22] is valid:

$$\sum_{n\geq 0} f(N)\frac{(z+w)^N}{N!} = \sum_{m,n\geq 0} f(m+n)\frac{z^m w^n}{m!n!}$$

Thus,

$$\mathscr{F}_2^{(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}';\mathcal{H},\mathcal{H}';z,w;\mathcal{A},\mathcal{C}) = \left(\int_0^1\int_0^1 \sum_{N\geq 0}(\mathcal{P})_N\frac{(zu+wv)^N}{N!} u^{\mathcal{Q}-I}(1-u)^{\mathcal{H}-\mathcal{Q}-I} v^{\mathcal{Q}'-I}(1-v)^{\mathcal{H}'-\mathcal{Q}'-I} \times \exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{\mathcal{C}}{(1-u)^\mu}-\frac{\mathcal{A}}{v^\eta}-\frac{\mathcal{C}}{(1-v)^\mu}\right)dudv\right)\Gamma\begin{pmatrix}\mathcal{H},\mathcal{H}'\\ \mathcal{Q},\mathcal{Q}',\mathcal{H}-\mathcal{Q},\mathcal{H}'-\mathcal{Q}'\end{pmatrix} \tag{8.8}$$

Using (8.5) and (8.8), (8.6) is obtained. □

Theorem 8.4. Suppose $\mathcal{P}$, $\mathcal{Q}$, $\mathcal{Q}'$, $\mathcal{Q}''$, $\mathcal{H}$, $\mathcal{H}-\mathcal{P}$, $\mathcal{A}$, and $\mathcal{C}$ be PSMs in $\mathbb{C}^{r\times r}$ such that $\mathcal{P}$, $\mathcal{H}$, and $\mathcal{A}$ commutes with each other, $\mathcal{H}\mathcal{Q}=\mathcal{Q}\mathcal{H}$, $\mathcal{H}\mathcal{Q}'=\mathcal{Q}'\mathcal{H}$, and $\mathcal{H}\mathcal{Q}''=\mathcal{Q}''\mathcal{H}$. Then, the ELMF $\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v)$ in (8.2) provides the following integral representation:

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v) = \Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{\mathcal{C}}{(1-u)^\mu}\right) \times(1-zu)^{-\mathcal{Q}}(1-wu)^{-\mathcal{Q}'}(1-vu)^{-\mathcal{Q}''}du\right) \tag{8.9}$$

Proof. From (3.1) and (8.2),

$$\mathscr{F}_{\mathcal{D},\mathcal{A},\mathcal{C}}^{3(\eta,\mu)}(\mathcal{P},\mathcal{Q},\mathcal{Q}',\mathcal{Q}'';\mathcal{H},;z,w;v) = \Gamma\begin{pmatrix}\mathcal{H}\\ \mathcal{P},\mathcal{H}-\mathcal{P}\end{pmatrix}\sum_{m,n,p\geq 0}\left(\int_0^1 u^{\mathcal{P}-I}(1-u)^{\mathcal{H}-\mathcal{P}-I}\exp\left(-\frac{\mathcal{A}}{u^\eta}-\frac{\mathcal{C}}{(1-u)^\mu}\right) \times(\mathcal{Q})_m(\mathcal{Q}')_n(\mathcal{Q}'')_p\frac{(uz)^m(uw)^n(uv)^p}{m!n!p!}du\right)$$

By (8.5) and continuing in the same process as in Theorem 8.1, (8.9) is obtained. □

9. Conclusion

In conclusion, the findings presented in this paper introduce new results that can potentially extend other special matrix functions. We have developed an extension of the BMF and investigated the GHMF and KHMF, exploring their key relationships and properties. Additionally, we extended the AMF and LMF and derived their integral representations using the beta matrix function. We also highlighted significant statistical applications of the EBMF. These generalized matrix functions have wide-ranging applications, including quantum mechanics, describing the time evolution of quantum systems, multivariate statistics, modeling multivariate distributions and hypothesis testing, control theory, analyzing the stability and response of dynamic systems, and mathematical physics, solving systems of differential equations with matrix arguments. The results from this study open several promising avenues for future research. Potential directions include extending other special matrix functions, such as the Whittaker, Wright, and Fox-H matrix functions, as well as Jacobi and Laguerre matrix polynomials. Researchers could also explore special integral transforms of these extended matrix functions, including the Euler-Beta, Laplace, and k-transforms. With its exponential terms, the generalized beta distribution provides additional flexibility and could be useful in machine learning, especially in regularization and Bayesian frameworks. Researchers could explore using matrix-variate beta distributions in deep learning models for regularization, uncertainty quantification, and matrix-variate variational autoencoders.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

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