Turk. J. Math. Comput. Sci.
17(1)(2025) 67–74
© MatDer
DOI : 10.47000/tjmcs.1535447

Vertex Removal on Perfect Italian Domination and γ_I^p-Stability of Graphs

Agnes Poovathingal[1,*], Joseph Varghese Kureethara[2]

[1]Department of Mathematics, Research Scholar of Christ University, 560029, Bengaluru, Karnataka , India.
[2]Department of Mathematics, Faculty of Christ University, 560029, Bengaluru, Karnataka , India.

Received: 19-08-2024 • Accepted: 23-03-2025

Abstract. Perfect Italian Domination (PID) is a domination concept where all vertices are assigned one of the labels among 0, 1 and 2 such that the sum of the labels in the neighbourhood of every vertex labelled 0 should be exactly 2. We examine a few graph classes of graphs and discuss the criticality of Perfect Italian Domination. We also define γ_I^p stable graphs and PID critical graphs. Following our definitions of γ_I^p-stable and PID critical graphs, we have grouped some graph classes. We characterise a family of trees that is γ_I^p-stable.

2020 AMS Classification: 05C69,05C75

Keywords: Leaf vertex, perfect Italian domination, Stem vertex.

1. Introduction

Let G be a simple connected graph with vertex set $V(G)$ and edge set $E(G)$. A subset $S \subseteq V(G)$ is a dominating set if every vertex in G is either in S or is adjacent to some vertex in S. Domination number of a graph G, $\gamma(G)$ is the cardinality of the smallest among the possible dominating sets of G [2].

Domination can be considered as a labelling problem where the vertices in the dominating set are labelled 1 and the remaining vertices are labelled 0. i.e., any vertex labelled 0 is adjacent to at least one vertex labelled 1. Numerous distinct cases of domination have been defined. Perfect domination is when each vertex labelled 0 is adjacent to exactly one vertex labelled 1. The k-fair domination is when each vertex labelled 0 is adjacent to exactly k vertices labelled 1.

Roman domination is a type of domination where there are three subsets V_0, V_1, V_2 for the vertex set $V(G)$ such that any vertex in V_0 should have a neighbour in V_2. The vertices in V_0 are labelled 0, vertices in V_1 are labelled 1 and the vertices in V_2 are labelled 2. Roman domination number $\gamma_R(G) = |V_1| + 2|V_2|$ where these are the sets which give the least value among all possible Roman dominating sets [7].

Italian domination is a generalisation of Roman domination. Here, the vertices in the set V_0 should be either adjacent to two vertices belonging to set V_1 or one vertex from the set V_2 [5]. Perfect Italian domination (PID) is an Italian domination with an additional constraint that, if there exists a vertex v_i in the set V_0, then exactly one among the following two cases should be true.

(1) $N(v_i) \cap V_1 = 2$ and $N(v_i) \cap V_2 = 0$.
(2) $N(v_i) \cap V_1 = 0$ and $N(v_i) \cap V_2 = 1$.

*Corresponding Author
Email addresses: agnes.poovathingal@res.christuniversity.in (A. Poovathingal), frjoseph@christuniversity.in (J.V. Kureethara)

Perfect Italian domination number, $\gamma_I^p(G) = |V_1| + 2|V_2|$ where these are the sets which give the smallest possible value among the ones satisfying the above condition. In [3] the authors found an upper bound for perfect Italian domination number of trees. They had also defined a constant $c_{\mathscr{G}}$ with which they proposed to determine an upper bound for perfect Italian domination as $c_{\mathscr{G}} \times n$ of various classes of graphs $\mathscr{G}$. Lauri and Mitillos [6] proved that $c_{\mathscr{G}} = 1$ for planar graphs and for cubic graphs $c_{\mathscr{G}} = 2/3$. They have also characterised graphs with $\gamma_I^p(G) = 2$ and $\gamma_I^p(G) = 3$.

In [12], the authors have found an upper bound for the perfect Italian domination number of cartesian product of two graphs. A relation between Roman domination number and perfect Italian domination number of a graph is determined. The perfect Italian domination problems in cographs are studied in [1] and Sierpinski graphs are determined in [13]. A comparative study between Domination and Perfect Italian Domination numbers is done in [8]. The complexity difference between finding the PID number and the Italian domination number is found in [10].

An essential part of the analysis of some graph property is studying the criticality concepts of that particular property [11]. Vertex removal significantly influences the Perfect Italian Domination concept in graphs, showcasing its important role in the network robustness and structure. When a vertex is removed from a graph, the PID number may or may not change.

Removal of a certain vertex from a graph can increase its PID whereas removal of another vertex from the same graph may decrease its PID number. There are cases in which removal of some vertices does not make any significant change in the Perfect Italian Domination property of that graph.

If a vertex removal alters the connectivity of a graph, then there is high chance for the PID to rise to cover these seperated parts.

Understanding how vertex removal affects the PID number provides insights on finding the most reliable graph structure that is lesser vulnerable to disruptions in a network. Hence a study on the effect of vertex removal on PID number has a vital role to play in network designing and optimisation.

Stem vertices otherwise known as supporting vertices are vertices which are adjacent to a pendant vertex [4]. Strong stem vertices are stem vertices with at least two pendant vertices adjacent to each. In this paper we address a pendent vertex as a leaf vertex. Number of leaf vertices adjacent to a stem vertex x is denoted as $L(x)$ [4].

2. Vertex Deletion

In this work, we examine a few graph classes and discuss in detail the criticality concept of Perfect Italian Domination. Removing certain vertices can reduce the PID number, while other vertices may cause the PID number to increase. Additionally, there are some vertices whose removal does not affect the PID number. Graphs where the removal of any vertex does not change the PID number are called γ_I^p-stable graphs, whereas graphs where the removal of specific vertices causes a change in the PID number are known as PID-critical graphs.

Observation 2.1. For a complete graph K_n,

$$\gamma_I^p(K_n - v) = \begin{cases} \gamma_I^p(K_n) - 1 & n = 1 \text{ or } 2 \\ \gamma_I^p(K_n) & n \geq 3. \end{cases}$$

Observation 2.2. Let $K_{1,n}$ be a star, and let v be any random vertex of it. Then,

$$\gamma_I^p(K_{1,n} - v) = \begin{cases} \gamma_I^p(K_{1,n}) + (n-2), & \text{if v is the root vertex} \\ \gamma_I^p(K_{1,n}), & \text{otherwise.} \end{cases}$$

Theorem 2.3. *Let C_n be a cycle of order n, and let v be any vertex of it. Then, $\gamma_I^p(C_n - v) = \gamma_I^p(C_n)$.*

Proof. Removal of any random vertex from the cycle C_n makes it a path P_{n-1}. From [6], $\gamma_I^p(P_{n-1}) = \lceil \frac{(n-1)+1}{2} \rceil = \lceil \frac{n}{2} \rceil = \gamma_I^p(C_n)$. Hence, the proof. □

In order to study the vertex deletion in Path (P_n) graphs, we need to consider the following cases:

(i) The removed vertex v is a pendant vertex and n be odd.
(ii) The removed vertex v is a pendant vertex and n be even.
(iii) The removed vertex v is not a pendant vertex and n be even.
(iv) v is not a pendant vertex but n be odd and the two disconnected paths in $P_n - v$ are both odd paths.
(v) v is not a pendant vertex but n be odd and the two disconnected paths in $P_n - v$ are both even paths.

We will discuss the effect of the removal of the vertex v in each of the above cases and conclude the effect of vertex removal in case of a Path graph.

Lemma 2.4. *Let P_n be a path where $n \geq 3$, and let v be a vertex of it. Then, $\gamma_I^p(P_n - v) = \gamma_I^p(P_n)$ when any of the following conditions is satisfied.*

(i) *v is a pendant vertex and n is odd.*
(ii) *v is not a pendant vertex but n is odd and the two disconnected paths in $P_n - v$ are both odd paths.*
(iii) *v is not a pendant vertex and n is even.*

Proof. If v is a pendant vertex and n is odd, then the removal of a vertex $v \in P_n$ gives an even path P_{n-1}. From the [6], $\gamma_I^p(P_{n-1}) = \lceil \frac{(n-1)+1}{2} \rceil = \lceil \frac{n}{2} \rceil = \frac{n}{2} + \frac{1}{2}$ and $\gamma_I^p(P_n) = \lceil \frac{n+1}{2} \rceil = \frac{n+1}{2} = \gamma_I^p(P_{n-1})$.

If v is not a pendant vertex but n be odd and the two disconnected paths in $P_n - v$ are both odd paths P_{n_1}, P_{n_2}, then from [6] $\gamma_I^p(P_{n_1}) = \lceil \frac{n_1+1}{2} \rceil = \frac{n_1+1}{2}$. Similarly for P_{n_2}, $\gamma_I^p(P_{n_2}) = \frac{n_2+1}{2}$. From [6] we have the result, $\gamma_I^p(G) = \sum_{i=1}^k G_i$, where G_i's are components of the graph G hence, $\gamma_I^p(P_n - v) = \gamma_I^p(P_{n_1}) + \gamma_I^p(P_{n_2}) = \frac{n_1+1}{2} + \frac{n_2+1}{2} = \frac{n_1+n_2+2}{2} = \frac{n+1}{2} = \gamma_I^p(P_n)$.

When n is even and v is not a pendant vertex, removal of v disconnects the path P_n to two smaller paths P_{n_1}, P_{n_2} where $n_1 + n_2 = n - 1$. Since $(n-1)$ is odd, one of n_1, n_2 is odd and the other is even. Without loss of generality let us assume n_1 be odd and n_2 is even [6]. $\gamma_I^p(P_{n_1}) = \lceil \frac{n_1+1}{2} \rceil = \frac{n_1+1}{2}$. $\gamma_I^p(P_{n_2}) = \lceil \frac{n_2+1}{2} \rceil = \frac{n_2+1}{2} + \frac{1}{2}$. From [6] we have the result, $\gamma_I^p(G) = \sum_{i=1}^k G_i$, where G_i's are components of the graph G. Hence, $\gamma_I^p(P_n - v) = \gamma_I^p(P_{n_1}) + \gamma_I^p(P_{n_2}) = \frac{n_1+1}{2} + \frac{n_2+1}{2} + \frac{1}{2} = \frac{n_1+n_2+2}{2} + \frac{1}{2} = \frac{n+1}{2} + \frac{1}{2} = \gamma_I^p(P_n)$ (since n is even). □

Lemma 2.5. *Let P_n be a path of order $n \geq 3$, and let n be even. If v is a pendant vertex, then $\gamma_I^p(P_n - v) = \gamma_I^p(P_n) - 1$.*

Proof. If v is a pendant vertex and n is even, then removal of $v \in P_n$ makes an odd path P_{n-1}. From the paper [6], $\gamma_I^p(P_{n-1}) = \lceil \frac{(n-1)+1}{2} \rceil = \lceil \frac{n}{2} \rceil = \frac{n}{2}$ and $\gamma_I^p(P_n) = \lceil \frac{n+1}{2} \rceil = \frac{n+1}{2} + \frac{1}{2} = \frac{n}{2} + 1 = \gamma_I^p(P_{n-1}) + 1$. Hence, the lemma. □

Lemma 2.6. *Let P_n be a path where $n \geq 3$ and let n be odd. If v is not a pendant vertex and the two disconnected paths in $P_n - v$ are both even paths, then $\gamma_I^p(P_n - v) = \gamma_I^p(P_n) + 1$.*

Proof. If v is not a pendant vertex, n is odd and the two disconnected paths formed after removal of v are even paths, P_{n_1}, P_{n_2} where $n_1 + n_2 + 1 = n$, then from [6] $\gamma_I^p(P_{n_1}) = \lceil \frac{n_1+1}{2} \rceil = \frac{n_1+1}{2} + \frac{1}{2} = \frac{n_1}{2} + 1$. Similarly for P_{n_2}, $\gamma_I^p(P_{n_2}) = \frac{n_2}{2} + 1$. From [6] we have the result, $\gamma_I^p(G) = \sum_{i=1}^k G_i$, where G_i's are components of the graph G hence, $\gamma_I^p(P_n - v) = \gamma_I^p(P_{n_1}) + \gamma_I^p(P_{n_2}) = \frac{n_1}{2} + \frac{n_2}{2} + 2 = \frac{n_1+n_2+4}{2} = \frac{n_1+n_2+1+3}{2} = \frac{n+3}{2} = \frac{n+1}{2} + 1 = \gamma_I^p(P_n) + 1$. □

Remark 2.7. It is easy to see that $\gamma_I^p(P_2 - v) = \gamma_I^p(P_2) - 1$.

Theorem 2.8. *Let P_n be a path where $n \geq 3$. Then, $\gamma_I^p(P_n - v)$ is any of the following.*

(i) $\gamma_I^p(P_n) - 1$, *if v is a pendant vertex and n is even.*
(ii) $\gamma_I^p(P_n)$,
  (a) *if v is a pendant vertex and n is odd or*
  (b) *if v is not a pendant vertex and n is even or*
  (c) *if v is not a pendant vertex, n is odd and the two disconnected paths in $P_n - v$ are both odd paths.*
(iii) $\gamma_I^p(P_n) + 1$, *if v is not a pendant vertex, n is odd and the two disconnected paths in $P_n - v$ are both even paths.*

Proof.
(i) If the removed vertex is an end vertex the graph remains connected and the path P_n becomes path P_{n-1}, then
  (a) when n is even, from Lemma 2.5 $\gamma_I^p(P_n - v) = \gamma_I^p(P_n) - 1$.
  (b) when n is odd, from Lemma 2.4 $\gamma_I^p(P_n - v) = \gamma_I^p(P_n)$.
(ii) If the removed vertex is any random vertex but not an end vertex, then the path P_n gets disconnected to two new paths P_{n_1}, P_{n_2} where $n_1 + n_2 = n - 1$.
  (a) When n is even we have Lemma 2.4 which says $\gamma_I^p(P_n - v) = \gamma_I^p(P_n)$.
  (b) When n is odd, $n_1 + n_2 = n - 1$ is even. In this case either of the two cases are possible depending on which vertex is removed-
    (i) if n_1, n_2 are both even then from Lemma 2.6, $\gamma_I^p(P_n - v) = \gamma_I^p(P_n) + 1$.
    (ii) if n_1, n_2 are both odd then from Lemma 2.4, $\gamma_I^p(P_n - v) = \gamma_I^p(P_n)$.

Hence, the theorem. □

3. Perfect Italian Domination Stability

A graph G, is said to be Perfect Italian Domination Stable or γ_I^p-stable, when $\gamma_I^p(G) = \gamma_I^p(G - v)$, where v is any vertex belonging to G. Hence, graphs which are not γ_I^p-stable are considered as PID critical graphs.

From the results of the above section we can conclude that-

(i) Cycles, C_n are γ_I^p-stable.
(ii) Complete graphs, $K_n, n \geq 3$ are γ_I^p-stable.
(iii) Stars $K_{1,n}$ where $n \geq 3$ are not γ_I^p-stable.
(iv) P_3 is γ_I^p-stable whereas the paths P_n where $n = 2$ or $n \geq 4$ are not γ_I^p-stable.

3.1. γ_I^p-stability on Generalised Stars.

Theorem 3.1. *Let $S_{n_1,n_2...n_k}$ be a generalisation of star. Then, it is γ_I^p-stable if and only if the graph is either S_2 or the graph $S_{n_1,n_2...,n_k}$ has at most an even n_i and the remaining $n_i's$ are equal to* 3.

Proof. $S_{n_1,n_2,...n_k}$ is a graph constructed by joining one of the end vertices of each paths $P_{n_1}, P_{n_2}, ..., P_{n_k}$ to a vertex v by an edge. That is each of the paths is extended by a vertex v hence an even path becomes an odd path and an odd path becomes an even path. We know that paths have a minimum PID labelling by $0's$ and $1's$ [6].

(i) If v is labelled 1, then labelling each of the even paths by $(0-1-...0-1)$, the odd paths by $(0-1-...-1-1)$ gives a minimum PID labelling for $S_{n_1,n_2,...n_k}$. There are $\lceil \frac{n_i-1+1}{2} \rceil = \lceil \frac{n_i}{2} \rceil$ 1's labelled in each of the paths [6].
  (a) Let $k > 3$ and there exist at least two even paths Then removing the root vertex decomposes the graph to kP_{n_i} and each of the paths will have $\lceil \frac{n_i+1}{2} \rceil$ vertices labelled 1. If n_i is odd, then $\lceil \frac{n_i}{2} \rceil = \lceil \frac{n_i+1}{2} \rceil$. Hence, removal of root vertex does not effect the odd paths. If n_i is even, then $\lceil \frac{n_i}{2} \rceil = \lceil \frac{n_i+1}{2} \rceil + 1$. Hence removal of the root vertex means adding a new label 1 to the even paths. This implies that if there exist at least two even paths, then the graph $S_{n_1,n_2,...n_k}$ is not γ_I^p-stable.
  (b) Let $k > 3$, and there exists at least an odd path of length greater than 3. Then, removal of a pendant vertex from an odd path decreases the PID number by 1. Hence, $S_{n_1,n_2,...,n_k}$ is not γ_I^p-stable if at least one of the $n_i > 3$ is odd.

(ii) Let $k > 3$ and $n_1 = n_2 = ... = n_k = 3$. Then labelling the root vertex v by 0, two $P_3's$ are labelled by $1-0-1$, and the remaining $k-2$ $P_3's$ are labelled by $0-2-0$. Removing the root vertex decomposes the graph to kP_3 and each of them can retain the same labelling. Since a P_3 is a γ_I^p-stable graph, removing a vertex of any P_3 do not effect the PID of the graph $S_{3,3,...,3}$. Hence, $S_{3,3,...,3}$ is γ_I^p-stable.

(iii) If all the $n_i's$ are equal to 3 except one which is an even number, then labelling the vertices as mentioned in 1 gives a minimum PID number for the given graph. As mentioned in the case 1a, removal of the root vertex v is same as addition of a label 1 to the even path. Since there exists only one even path this vertex removal does not affect the PID number. As discussed earlier, n_i is even implies that the even path P_{n_i} along with v is an odd path of order $n_i + 1$. Hence, removal of the pendant vertex from the path P_{n_i+1}, where n_i is even turns it to an even path of order $n_i + 1 - 1 = n_i$. Since $\lceil \frac{n_i+1+1}{2} \rceil = \lceil \frac{n_i+1}{2} \rceil$ implies that PID number is not altered. If any vertex other than the pendant vertex or root vertex is removed from the path P_{n_i+1} where n_i is even, then this path of odd order $n_i + 1$ is decomposed to either two odd paths or two even paths say $P_r, P_{n_i+1-1-r}$. Since $\lceil \frac{n_i+1+1}{2} \rceil = \lceil \frac{r+1}{2} \rceil + \lceil \frac{n_i-r+1}{2} \rceil$, we can conclude that even on removal of any vertex the PID number is not altered.

(iv) If $k = 1$ or $k = 2$, then graph is a path of length $n_1 + 1$ or $n_1 + n_2 + 1$. This implies that S_2 or $S_{1,1}$ which are isomorphic to P_3 are the only γ_I^p-stable graphs among all the cases when $k < 3$.

Hence, $S_2, S_{n_1,n_2,...,n_k}$ where at most one n_i is an even number and all the remaining $n_i's$ are equal to 3 for $i = 1, 2, ..., k$ are the only generalised star graphs which are γ_I^p-stable. □

3.2. γ_I^p-stability on Corona Product of Graphs.

Proposition 3.2. *Let G be a connected graph. Then, $G \circ K_1$ is a PID critical graph.*

Proof. There exists a minimum PID labelling for $G \circ K_1$, where pendant vertices are not labelled 0 [9]. This implies that the pendant vertices are either labelled by 2 or 1. If a pendant vertex is labelled by 2, then obviously its stem vertex should be zero labelled. Hence, removal of the stem vertex disconnects the graph and relabelling the isolated vertex by 1 decreases the PID number. A pendant vertex v labelled by 1 has its stem vertex either labelled 0 or by 1. If its stem vertex is labelled by 1, then removal of v decreases the PID number. Let the stem vertex of v say v' is zero labelled, then there exists a neighbour u' for $v' \in G$ labelled 1 and the pendant vertex u adjacent to the vertex u' is labelled by 1. Removal of the vertex u' decreases the PID number. Hence we can conclude that $G \circ K_1$ is not γ_I^p-stable but it is PID critical graph. □

Proposition 3.3. *Let G be a connected graph. Then, $G \circ 2K_1$ is γ_I^p-stable.*

Proof. Since each vertex in the graph G is a stem vertex to two pendant vertices, each of them can either be labelled 0 or by 2 in the minimum PID labelling. If a vertex $v \in G$ is labelled 0, then its pendant vertices are labelled by 1 each and all its neighbouring vertices in the graph G are labelled by 0. Since G is a connected graph, this leads to all the vertices of G zero labelled and the pendant vertices are all labelled by 1. If a vertex $v \in G$ is labelled by 2, then its pendant vertices are labelled by 0 and none of the neighbours of v can be zero labelled. Since G is a connected graph no vertex of G except v will be labelled 0.

This implies that for a $G \circ \overline{K}_2$ there exist only two possible minimum PID labellings. Either all the vertices of G are labelled by 2 and all the pendant vertices are labelled by 0 or all the vertices of G are labelled by 0 and the pendant vertices are labelled by 1.

The vertices in the graph (except one) is either a pendant vertex or a stem vertex with two pendant vertices, hence each of them should have at least a total label of 2 along with each of their pendant vertices. Hence, if any stem vertex is removed, then the pendant vertices turn to isolated vertices and take the labels of 1. This implies that PID is not altered. If a pendant vertex x is removed, then the stem vertex of the removed pendant vertex v is relabelled by 1 and its remaining one pendant vertex is labelled by 1. The remaining stem vertices are labelled by 2 and their respective pendant vertices are labelled by 0. This relabelling does not make any change in the PID number. This proves that the PID is not altered on removal of any vertex from $G \circ \overline{K}_2$. □

Proposition 3.4. *Let G be a γ_I^p-stable graph, and let v be a stem vertex of G. Then, $|L(v)| \leq 2$.*

Proof. Assume that there exist a strong stem vertex v with $|L(v)| \geq 3$. Then in the minimum possible PID labelling of G, the leaf vertices adjacent to a vertex v can only be given the label 0 and v can only be labelled by 2. Hence, on removal of the vertex v, G becomes disconnected and the leaf vertices become isolated vertices increasing the PID number by at least 1. □

Corollary 3.5. *Any graph with a stem vertex v such that $|L(v)| \geq 3$ is not γ_I^p-stable.*

Remark 3.6. The graph $G \circ nK_1$, where $n \geq 3$ is not γ_I^p-stable.

4. A Particular Family of γ_I^p-stable Trees

In this subsection we are trying to characterize trees which are γ_I^p-stable. We have found a family of trees $\mathscr{T}$ which are γ_I^p-stable and also have found few conditions which trees will satisfy if they are γ_I^p-stable.

There exist trees where PID remains unchanged, decreases and increases depending on which vertex is removed.

Example 4.1. Consider a Coconut tree graph $CT_{3,3}$ as shown in the below Figure 1. On removal of the vertex u, the PID remains unchanged. On removal of the vertex f, the PID decreases by 2. On removal of the vertex e the PID decreases by 1. When the vertex d is removed the star get disconnected and PID increases by 1.

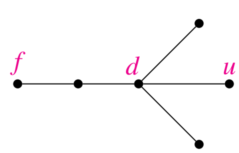

Figure 1. PID labelling of a Coconut tree graph

We will define a family of trees $\mathscr{T}$.

Definition 4.2. $T \in \mathscr{T}$ are trees constructed from a sequence $T_0, T_1, ..., T_n$ of graphs where $T_0 = K_{1,2}, T = T_n$. If $n \geq 1$, then T_{i+1} is obtained from T_i by doing any one of the operations O_1 or O_2, for $i = 0, 1, ..., n-1$.

Operation O_1: Let $T_i \in \mathscr{T}$ and $u \in V(T_i)$ such that $|L(u)| = 2$. Then, add a $K_{1,2}$ say, $x_1x_2x_3$ (where x_2 is a stem vertex) and an edge ux_2 in order to get T_{i+1}.

Operation O_2: Let $T_i \in \mathscr{T}$ and $u \in V(T_i)$ such that $|L(u)| = 2$. Then, add a $K_{1,3}$, $x_1x_2x_3x_4$, (where x_2 is a stem vertex) and an edge ux_3 to get T_{i+1}.

Lemma 4.3. *Let T be any tree, and let $u \in V(T)$. If T' is a tree obtained from T by the operation O_1, then $\gamma_I^p(T') = \gamma_I^p(T) + 2$.*

Proof. (i) If u is labelled 0, then the only possible labels that can be given to x_1, x_2, x_3 is x_1, x_3 labelled 1 each and x_2 labelled 0.

(ii) If u is labelled 2, then the only possible labels that can be given to x_1, x_2, x_3 is x_1, x_3 labelled 0 each and x_2 labelled 2.

(iii) If u is labelled 1, then the only possible labels that can be given to x_2 is 2 and x_1, x_3 by $0's$ (since x_1, x_3 are pendant vertices to x_2).

In both the cases $\gamma_I^p(T') \leq \gamma_I^p(T) + 2$.

If you are adding a new strong stem vertex at least an increase of value 2 is required in the PID number $\Longrightarrow$ $\gamma_I^p(T') \geq \gamma_I^p(T) + 2$. From above two results, $\gamma_I^p(T') = \gamma_I^p(T) + 2$. □

Lemma 4.4. *Let T be any tree and $u \in V(T)$ such that $|L(u)| = 2$. If T' is a tree obtained from T by adding an edge x_1x_2 such that u is adjacent to x_1, then $\gamma_I^p(T') = \gamma_I^p(T) + 2$.*

Proof. Since $|L(u)| = 2$, u can only be labelled by 0 or by 2.

If u is labelled 2 in T, then the only possible values x_1, x_2 can take is 1 each. Hence, $\gamma_I^p(T') = \gamma_I^p(T) + 2$.

If u is labelled 0 in T, then the only possible labelling for x_1x_2 is 0-2. In this case also $\gamma_I^p(T') = \gamma_I^p(T) + 2$. Hence, proved. □

Lemma 4.5. *Let T_i be a γ_I^p-stable tree, and let T_{i+1} be a tree obtained from T_i by the operation O_1. Then, T_{i+1} is also γ_I^p-stable.*

Proof. If T_i is a γ_I^p-stable tree, then for any $v \in V(T_i)$, $\gamma_I^p(T_i - v) = \gamma_I^p(T_i) \longrightarrow (a)$
From Lemma 4.3, $\gamma_I^p(T_{i+1}) = \gamma_I^p(T_i) + 2 \longrightarrow (b)$
To prove T_{i+1} is γ_I^p-stable we need to show $\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_{i+1})$

(i) For any $v \in V(T_i) - \{u\}$, ($u \in T_i$ is the vertex to which $K_{1,2}$ is attached by O_1)
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i - v) + 2$ (since (b) is true for any tree)
$= \gamma_I^p(T_i) + 2$ (from (a))
$= \gamma_I^p(T_{i+1})$.

(ii) If $v = u$, then on removal of v we get two disconnected trees $T_i - u$ and $K_{1,2}$. $\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i - v) + \gamma_i^p(K_{1,2}) = \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$ (from (a) and (b)).

(iii) If $v = x_2$, then on removal of v we get T_i and two isolated vertices x_1, x_3.
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i) + \gamma_I^p(2K_1) = \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$.

(iv) If $v = x_1$ or x_2, (WLG let $v = x_1$) then on removal of v we get $T_i \cup x_2x_3$.
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i \cup x_2x_3)$ (from Lemma 4.4)
$= \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$.

Hence, we have the result. □

Lemma 4.6. *Let T be any tree with a stem vertex u such that $|L(u)| = 2$ and let T' be a tree obtained from T by O_2. Then, $\gamma_I^p(T') = \gamma_I^p(T) + 2$.*

Proof. Let T be a tree with PID number $\gamma_I^p(T)$. Since $K_{1,3}$ is joined to a strong stem vertex $u \in T$, u can only be labelled by 0 or 2. Let us define the possible PID labellings for T'.

(i) If u is labelled 0, then x_3, x_1, x_4 are labelled 0 and x_2 is labelled 2. Remaining vertices are given the same labels as in the minimum PID labelling of T.

(ii) If u is labelled 2, then x_3, x_2 are labelled 0 and x_1, x_4 are labelled 1. Remaining vertices are given the same labels as in the minimum PID labelling of T.

In both the cases $\gamma_I^p(T') \leq \gamma_I^p(T) + 2$.

If you are adding a new strong stem vertex at least an increase of value 2 is required in the PID number $\Longrightarrow$ $\gamma_I^p(T') \geq \gamma_I^p(T) + 2$. From above two results, $\gamma_I^p(T') = \gamma_I^p(T) + 2$. □

Lemma 4.7. *Let $T \in \mathscr{T}$ and $u \in V(T)$ be a stem vertex where $|L(u)| = 2$. If T' is a tree obtained by adding a vertex x such that ux is an edge, then $\gamma_I^p(T') = \gamma_I^p(T)$.*

Proof. From the construction of the trees in $\mathscr{T}$, its clear that a vertex belonging to T is a pendant vertex, a strong stem vertex or a 2−degree non-stem vertex adjacent to exactly two stem vertices. The 2-degree vertices can be labelled by 0 and one among each of its neighbouring strong stem vertex is labelled by 2 whereas the other strong stem vertex by 0. The pendant vertices adjacent to the 2 labelled stem vertices are labelled 0 each whereas the pendant vertices adjacent to the stem vertices labelled 0 are labelled by 1 each. This is a minimum perfect Italian domination labelling for T because perfect Italian domination is greater than or equal to twice the number of strong stem vertices in a graph.

Since u is a stem vertex with $|L(u)| = 2$ in T, addition of a vertex x and adding an edge ux increases the $|L(u)|$ to 3 in T'. This implies u can only be labelled by 2. If there exist a two degree vertex v adjacent to u, label it 0 and the other strong stem vertex adjacent to v by 0. Remaining vertices can be labelled as mentioned in the labelling of T. This implies that perfect Italian domination number of T and T' are equal. □

Lemma 4.8. *Let T be a tree and $u \in V(T)$ be a stem vertex where $|L(u)| = 2$. If T' is a tree obtained by adding a P_3-$v_1v_2v_3$ such that u is adjacent to v_1, then $\gamma_I^p(T') = \gamma_I^p(T) + 2$.*

Proof. Since $|L(u)| = 2$, u is labelled by 0 or 2. Let us define a PID labelling for T'.

If u is labelled 2, then P_3 can only take the labelling $0 - 0 - 2$ or $1 - 0 - 1$.

If u is labelled 0 then P_3 can only take the labelling $0 - 2 - 0$. In both the cases remaining vertices in T' are given the same labels as in T. Hence $\gamma_I^p(T') = \gamma_I^p(T) + 2$. □

Lemma 4.9. *Let T_{i+1} be a tree obtained from T_i by the operation O_2. If T_i is a γ_I^p-stable tree, then T_{i+1} is also γ_I^p-stable.*

Proof. If T_i is a γ_I^p-stable tree, then for any $v \in V(T_i)$, $\gamma_I^p(T_i - v) = \gamma_I^p(T_i) \longrightarrow (a)$
From Lemma 4.6, $\gamma_I^p(T_{i+1}) = \gamma_I^p(T_i) + 2 \longrightarrow (b)$
To prove T_{i+1} is γ_I^p-stable we need to show $\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_{i+1})$

(i) For any $v \in V(T_i) - \{u\}$,
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i - v) + 2$ (since (b) is true for any tree)
$= \gamma_I^p(T_i) + 2$ (from (a))
$= \gamma_I^p(T_{i+1})$.

(ii) If $v = u$, then on removal of v we get two disconnected trees $T_i - u$ and $K_{1,3}$. $\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i - v) + \gamma_I^p(K_{1,3}) = \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$.

(iii) If $v = x_2$, then on removal of v we get two isolated vertices x_1, x_4 and a tree $T_i \cup x_3$ (T_i with an extra edge ux_3).
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i \cup x_3) + \gamma_I^p(2K_1) = \gamma_I^p(T_i) + 2$ (from Lemma 4.7)
$= \gamma_I^p(T_{i+1})$.

(iv) If $v = x_1$ or x_4 (WLG let $v = x_1$), then on removal of the vertex v we get $T_i \cup x_3x_2x_4$. $\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i \cup x_3x_2x_4) = \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$ (from Lemma 4.8).

(v) If $v = x_3$, then on removal of v we get two disconnected trees T_i and $K_{1,3}$.
$\gamma_I^p(T_{i+1} - v) = \gamma_I^p(T_i) + \gamma_i^p(K_{1,3}) = \gamma_I^p(T_i) + 2 = \gamma_I^p(T_{i+1})$.

From (i), (ii), (iii), (iv), (v) we have the result. □

Theorem 4.10. *A tree $T \in \mathscr{T}$ is γ_I^p-stable.*

Proof. We prove this by induction on number of operations done on the base graph to obtain the resultant graph. $T_0 = K_{1,2}$ is γ_I^p-stable. Let us assume that graph obtained by doing $k \leq i$ operations are γ_I^p-stable. Let T_{i+1} be a tree obtained by doing $i + 1$ operations. i.e, T_{i+1} is obtained from T_i by doing one of the operations O_1 or O_2, where T_i is γ_I^p-stable. From Lemma 4.5 and Lemma 4.9, T_{i+1} is γ_I^p-stable. Thus, we can conclude that any tree $T \in \mathscr{T}$ is γ_I^p−stable. □

5. Conclusion

Whether by increasing or decreasing the Perfect Italian Domination number, the vertex removals reveal insights into structural vulnerability and the change in the minimum number of vertices required to have complete control over the graph. Only a few graph classes are discussed in detail. Hence, a study exploring the vertex removal on many other graph classes will deepen the understanding of network dynamics. In this paper, a few γ_I^p-stable graph classes are also found. A γ_I^p-stable family of trees is characterised. More γ_I^p-stable graph classes are open for further research.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

All authors jointly worked on the results and they have read and agreed to the published version of the manuscript.

References

[1] Banerjee, S., Henning, M.A., Pradhan, D., *Perfect Italian domination in cographs*, Applied Mathematics and Computation **391**(2021).
[2] Haynes, T.W., Hedetniemi, S., Slater, P., Fundamentals of Domination in Graphs, CRC Press, 1998.
[3] Haynes, T.W., Henning, M.A. *Perfect Italian domination in trees*, Discrete Applied Mathematics **260**(2019), 164–177.
[4] Haynes, T.W., Hedetniemi, S.T., Henning, M.A., Structures of Domination in Graphs, 2021.
[5] Henning, M.A., Klostermeyer, W.F., *Italian domination in trees*, Discrete Applied Mathematics, **217**(2017), 557–564.
[6] Lauri, J., Mitillos, C., *Perfect Italian domination on planar and regular graphs*, Discrete Applied Mathematics **285**(2020), 676–687.
[7] Mojdeh, D.A., Parsian, A., Masoumi, I., *Strong Roman domination number of complementary prism graphs*, Turkish Journal of Mathematics and Computer Science **11**(1)(2019), 40–47.
[8] Poovathingal, A., Kureethara, J. V., *On some graphs whose domination number is the Perfect Italian domination number*, Congress on Intelligent Systems, (2023), 191–200.
[9] Poovathingal, A., Kureethara J.V., *Modelling networks with attached storage using Perfect Italian domination*, International Conference on MAchine inTelligence for Research & Innovations, (2023), 23–33.
[10] Pradhan, D., Banerjee, S., Liu, J.B., *Perfect Italian domination in graphs: Complexity and algorithms*, Discrete Applied Mathematics **319**(2022), 271–295.
[11] Sumner, D.P., *Critical concepts in domination*, Annals of Discrete Mathematics, **48**(1991), 33–46.
[12] Varghese, J., Aparna Lakshmanan, S., *Perfect Italian domination number of graphs*, arXiv preprint arXiv:1910.12260, (2019).
[13] Varghese, J., Anu, V., Aparna Lakshmanan, S., *Italian domination and perfect Italian domination on Sierpiński graphs*, Journal of Discrete Mathematical Sciences and Cryptography, **24**(2021), 1885–1894.