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Research Article

A non-abelian tensor product of algebras with bracket

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Abstract

We introduce and study a non-abelian tensor product of two algebras with bracket with compatible actions on each other. We investigate its applications to the universal central extensions and the low-dimensional homology of perfect algebras with bracket.

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1. Introduction

The non-abelian tensor product in various algebraic categories, such as groups, (Hom)-Lie and (Hom)-Leibniz algebras, Lie superalgebras. etc., plays an essential role in the study of homotopy theory, low-dimensional homology or universal central extensions (see [3, 7, 13–15, 17, 20, 21]). It has also been used in the construction of low-dimensional non-abelian homology of groups, Lie algebras and Leibniz algebras, having interesting applications to the algebraic K-theory, cyclic homology and Hochschild homology, respectively [16–18].

In this paper, we choose to develop a non-abelian tensor product for a relatively new algebraic structure called algebra with bracket, introduced in [8] as a kind of generalization of the (non-commutative) Poisson algebra. It should be noted here that such a generalization of Poisson algebras originates in physics literature (see, e.g. [23]). Among other results in [8], Quillen cohomology of algebras with bracket is described via an explicit cochain complex. Further (co)homological investigations of algebras with bracket are carried out in [4, 6]. In particular, for our importance, we mention that in [4], a homology with trivial coefficients of algebras with bracket is developed with applications to universal central extensions. In [6], crossed modules for algebras with bracket are introduced, and the second cohomology is interpreted as the set of equivalence classes of crossed extensions. The eight-term exact cohomology sequence is also constructed.

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In the present paper, we continue the same line of homological study of algebras with bracket. We fit the homology with trivial coefficients [4] into the context of Quillen homology, introduce the non-abelian tensor product of algebras with bracket by generators and relations, and give applications in universal central extensions and low-dimensional homology.

The organization of this paper is as follows: after the introduction, in Section 2, we present some definitions and results for the development of the paper. We briefly recall the construction of homology for algebras with bracket from [4,8] and prove that it is consistent with the context of Quillen's homology theory (Theorem 2.6). In Section 3, we present all the ingredients for developing the non-abelian tensor product later. In particular, we define actions, semi-direct products and crossed modules of algebras with bracket. Additionally, we show that the category of crossed modules is equivalent to the category of cat^1-algebras with bracket (Theorem 3.12). Section 4 contains the main results of the paper. Here, we present the construction of the non-abelian tensor product of two algebras with bracket acting compatibly on each other (Proposition 4.3) and study its properties. Regarding trivial actions, we describe the non-abelian tensor product (Proposition 4.5). We establish a right-exactness property of the non-abelian tensor product of algebras with bracket (Theorem 4.6) and equip the non-abelian tensor product with crossed module structures (Proposition 4.7). Finally, as an application, for a given perfect algebra with bracket, we construct its universal central extension (Theorem 4.8) and a four-term exact homology sequence (Theorem 4.11).

2. Algebras with bracket

Throughout the paper we fix a ground field $\mathbb{K}$. All vector spaces and algebras are $\mathbb{K}$-vector spaces and $\mathbb{K}$-algebras, and linear maps are $\mathbb{K}$-linear maps as well. In what follows $\otimes$ means $\otimes_{\mathbb{K}}$.

2.1. Basic definitions

Definition 2.1 ([8])**.** An algebra with bracket, or an AWB for short, is an associative (not necessarily commutative) algebra A equipped with a bilinear map (bracket operation) $[-,-]\colon \mathsf{A}\times\mathsf{A}\to\mathsf{A}$, $(a,b)\mapsto[a,b]$ satisfying the following identity:

$$[ab,c]=[a,c]b+a[b,c] \tag{2.1}$$

for all $a,b,c\in\mathsf{A}$.

A homomorphism of AWBs is a homomorphism of associative algebras preserving the bracket operation. We denote by AWB the respective category of AWBs.

The category AWB is a variety of Ω-groups [19], and therefore it is a semi-abelian category [2, 22]: pointed, exact and protomodular with binary coproducts. So classical lemmas such as the Five Lemma [25] hold for AWBs which we will use later on. Below we give the definitions of ideal, center, commutator, action and semi-direct product of AWBs, and of course these notions agree with the corresponding general notions in the context of semi-abelian categories.

We now list some common examples of AWBs that will be discussed later. Other examples can be found in [5, 6, 8, 23].

Example 2.2.

(i) Any vector space A enriched with the trivial multiplication and bracket operation, i.e. $ab=0$ and $[a,b]=0$ for all $a,b\in\mathsf{A}$, is an AWB, called an abelian AWB. Hence, the category of vector spaces is a full subcategory of AWB and the respective inclusion functor $\mathsf{Vect}\hookrightarrow\mathsf{AWB}$ has a left adjoint, the so called abelianization functor $(-)^{\mathrm{ab}}\colon \mathsf{AWB}\to\mathsf{Vect}$, which will be described in Remark 2.3 (i) below.

(ii) Any associative algebra A together with the trivial bracket operation can be regarded as an AWB. This defines the inclusion functor $I\colon \mathsf{Ass} \to \mathsf{AWB}$, where Ass denotes the category of associative algebras. The functor I has a left adjoint $(-)^{\text{ass}}\colon \mathsf{AWB} \to \mathsf{Ass}$ described in Remark 2.3 (ii) below.

(iii) Another way of considering an associative algebra A as an AWB is to define the bracket operation by

$$[a,b] := ab - ba, \quad a, b \in \mathsf{A}\,.$$

This particular AWB is called the *tautological* AWB associated to the associative algebra A and will be denoted by $T(\mathsf{A})$.

Tautological AWBs constitute a full subcategory of AWB denoted by TAWB. The correspondence $T\colon \mathsf{Ass} \to \mathsf{TAWB}$, $\mathsf{A} \mapsto T(\mathsf{A})$, is functorial, and it establishes an isomorphism between the categories TAWB and Ass.

(iv) Any Poisson algebra is an AWB. In fact, the category Poiss of (non-commutative) Poisson algebras is a subcategory of AWB. The inclusion functor $\mathsf{Poiss} \hookrightarrow \mathsf{AWB}$ has as left adjoint the functor given by $\mathsf{A} \mapsto \mathsf{A}_{\mathsf{Poiss}}$, where $\mathsf{A}_{\mathsf{Poiss}}$ is the maximal quotient of A, such that the following relations hold: $[a,a] \sim 0$ and $[a,[b,c]] + [b,[c,a]] + [c,[a,b]] \sim 0$.

The following notions for AWBs are given in [4] and they agree with the corresponding notions in semi-abelian categories. *A subalgebra* B of an AWB A is a vector subspace which is closed under the product and the bracket operation, that is, $\mathsf{B}\ \mathsf{B} \subseteq \mathsf{B}$ and $[\mathsf{B},\mathsf{B}] \subseteq \mathsf{B}$. A subalgebra B is said to be a *left (respectively, right) ideal* if $\mathsf{A}\ \mathsf{B} \subseteq \mathsf{B}$, $[\mathsf{A},\mathsf{B}] \subseteq \mathsf{B}$ (respectively, $\mathsf{B}\ \mathsf{A} \subseteq \mathsf{B}$, $[\mathsf{B},\mathsf{A}] \subseteq \mathsf{B}$). If B is both left and right ideal, then it is said to be a *two-sided ideal*. In this case, the quotient A/B is endowed with an AWB structure naturally induced from the operations on A.

Let A be an AWB and B,C be two-sided ideals of A. We denote by $[[\mathsf{B},\mathsf{C}]]$ the subspace of A spanned by all elements of the form bc, cb, $[b,c]$, $[c,b]$, for all $b \in \mathsf{B}$ and $c \in \mathsf{C}$. It is easy to see that $[[\mathsf{B},\mathsf{C}]]$ is a two-sided ideal of A called *commutator ideal* of B and C. Obviously $[[\mathsf{B},\mathsf{C}]] \subseteq \mathsf{B} \bigcap \mathsf{C}$. In the particular case $\mathsf{B} = \mathsf{C} = \mathsf{A}$, one obtains the definition of derived algebra $[[\mathsf{A},\mathsf{A}]]$ of A. An AWB A is said to be *perfect* if $\mathsf{A} = [[\mathsf{A},\mathsf{A}]]$.

Remark 2.3.

(i) Given an AWB A, the quotient $\mathsf{A}/[[\mathsf{A},\mathsf{A}]]$ is always an abelian AWB and will be denoted by A^{ab}. Abelian AWBs (i.e. just vector spaces) are abelian group objects in the category AWB. The respective abelianization functor $(-)^{\text{ab}}\colon \mathsf{AWB} \to \mathsf{Vect}$, which is left adjoint to the inclusion functor $\mathsf{Vect} \hookrightarrow \mathsf{AWB}$, sends an AWB A into $\mathsf{A}^{\text{ab}} = \mathsf{A}/[[\mathsf{A},\mathsf{A}]]$.

(ii) To an AWB A, we associate the associative algebra A^{ass} defined as the maximal quotient of A such that the relation $[a,a'] \sim 0$ holds, for $a,a' \in \mathsf{A}$. This correspondence is functorial and satisfies the following universal property: given an associative algebra B, any homomorphism of AWBs $\mathsf{A} \to I(\mathsf{B})$ factors trough A^{ass}, where $I\colon \mathsf{Ass} \hookrightarrow \mathsf{AWB}$ is the inclusion functor as in Example 2.2 (iii). Thus, the functor $(-)^{\text{ass}}\colon \mathsf{AWB} \to \mathsf{Ass}$, $\mathsf{A} \mapsto \mathsf{A}^{\text{ass}}$, is left adjoint to I.

Given an AWB A, the set

$$\mathcal{Z}(\mathsf{A}) = \{a \in \mathsf{A} \mid ab = 0 = ba, [a,b] = 0 = [b,a],\ \text{for all } b \in \mathsf{A}\}$$

is a two-sided ideal of A, and it is called *the center* of A. Note that an AWB A is abelian if and only if $\mathsf{A} = \mathcal{Z}(\mathsf{A})$.

A central extension of an AWB A is an exact sequence of AWBs $0 \to \mathsf{M} \to \mathsf{B} \xrightarrow{\phi} \mathsf{A} \to 0$ such that $[[\mathsf{M},\mathsf{B}]] = 0$ (equivalently, $\mathsf{M} \subseteq \mathcal{Z}(\mathsf{B})$). It is said to be *universal central extension* if for every central extension $0 \to \mathsf{N} \to \mathsf{C} \xrightarrow{\psi} \mathsf{A} \to 0$ there is a unique homomorphism of AWBs $\alpha\colon \mathsf{B} \to \mathsf{C}$ such that $\psi \circ \alpha = \phi$.

The result immediately below is the analogue of classical results for universal central extensions in the categories of groups, Lie algebras, etc., and agrees with the similar result from [9] in the general framework of semi-abelian categories.

Theorem 2.4 ([4]). *An* AWB A *admits a universal central extension if and only if* A *is perfect. Moreover, the kernel of the universal central extension is isomorphic to the first homology of* A, $\mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A})$ *(see the definition below).*

2.2. Homology

In this subsection, we briefly review the homology of AWBs with trivial coefficients given in [4, 8].

Let V be a vector space. Let $R_1(V) = V$ and $R_n(V) = V^{\otimes n} \oplus V^{\otimes n}$, if $n \geq 2$. In order to distinguish elements from these tensor powers, we let $a_1 \otimes \cdots \otimes a_n$ be a typical element from the first component of $R_n(V)$, while $a_1 \circ \cdots \circ a_n$ from the second component of $R_n(V)$.

Given an AWB A, we let $\left(C_*^{\mathrm{AWB}}(\mathsf{A}), d_*\right)$ be the chain complex defined by

$$C_n^{\mathrm{AWB}}(\mathsf{A}) \coloneqq R_{n+1}(\mathsf{A}),\ \ n \geq 0,$$

with the boundary maps $d_n\colon C_n^{\mathrm{AWB}}(\mathsf{A}) \to C_{n-1}^{\mathrm{AWB}}(\mathsf{A})$, $n \geq 0$, given by

$$d_n(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^{n} (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1},$$

$$d_n(a_1 \circ \cdots \circ a_{n+1}) = \sum_{i=1}^{n} a_1 \otimes \cdots \otimes [a_i, a_{n+1}] \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_1 \circ \cdots \circ a_i a_{i+1} \circ \cdots \circ a_{n+1}$$

The homology of the complex $\left(C_*^{\mathrm{AWB}}(\mathsf{A}), d_*\right)$ is called the *homology with trivial coefficients* of the AWB A and it is denoted by $\mathsf{H}_*^{\mathrm{AWB}}(\mathsf{A})$.

Easy computations show that there is an isomorphism

$$\mathsf{H}_0^{\mathrm{AWB}}(\mathsf{A}) \cong \mathsf{A}/[[\mathsf{A}, \mathsf{A}]].$$

On the other hand, given a free presentation of A, that is, a short exact sequence of AWBs $0 \to \mathsf{R} \to \mathsf{F} \to \mathsf{A} \to 0$, where F is a free AWB, then there is an isomorphism

$$\mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A}) \cong (\mathsf{R} \cap [[\mathsf{F}, \mathsf{F}]])/[[\mathsf{R}, \mathsf{F}]]$$

(see [4, Corollary 2.14]).

Remark 2.5. Let A be an associative algebra and consider the ground field $\mathbb{K}$ as a trivial A-bimodule. Let $C_*^{\mathrm{Hoch}}(\mathsf{A}) = C_*^{\mathrm{Hoch}}(\mathsf{A}, \mathbb{K})$ and $\mathrm{Hoch}_*(\mathsf{A}) = \mathrm{Hoch}_*(\mathsf{A}, \mathbb{K})$ denote the Hochschild complex and the Hochschild homology of A with coefficients in $\mathbb{K}$ [27], respectively. Then $C_1^{\mathrm{Hoch}}(\mathsf{A}) = \mathsf{A} = C_0^{\mathrm{AWB}}(T(\mathsf{A}))$ and the natural injections

$$C_{n+1}^{\mathrm{Hoch}}(\mathsf{A}) = \mathsf{A}^{\otimes(n+1)} \hookrightarrow \mathsf{A}^{\otimes(n+1)} \oplus \mathsf{A}^{\otimes(n+1)} = C_n^{\mathrm{AWB}}(T(\mathsf{A})), \quad n \geq 1$$

gives rise to a morphism of chain complexes $C_{*+1}^{\mathrm{Hoch}}(\mathsf{A}) \hookrightarrow C_*^{\mathrm{AWB}}(T(\mathsf{A}))$. Thus, we have an induced homomorphism in homology $\mathrm{Hoch}_{n+1}(\mathsf{A}) \to \mathsf{H}_n^{\mathrm{AWB}}(T(\mathsf{A}))$ $(n \geq 0)$, which is clearly an epimorphism for $n = 1$ and an isomorphism for $n = 0$.

Now, we show that the homology of AWBs is fitted in the context of homology theory developed by Quillen in a very general framework [26] (see also the earlier work by Barr and Beck [1]). Let us recall that the Quillen homology of an object in an algebraic category $\mathcal{C}$ is defined via the derived functors of the abelianization functor $(-)^{\mathrm{ab}}\colon \mathcal{C} \to \mathcal{C}^{\mathrm{ab}}$ from $\mathcal{C}$

to the abelian category $\mathcal{C}^{\mathrm{ab}}$ of abelian group objects in $\mathcal{C}$. To specify this theory for AWBs, we proceed as follows.

Given an AWB A, choose any free simplicial resolution F_* of A, that is, an aspherical augmented simplicial AWB $\mathsf{F}_* \xrightarrow{\epsilon} \mathsf{A}$ (which means that all non-zero homotopies are trivial, $\pi_n(\mathsf{F}_*) = 0$ for $n \geq 0$, and ϵ induces an isomorphism $\pi_0(\mathsf{F}_*) \cong \mathsf{A}$) such that each component F_n, $n \geq 0$, is a free AWB. Then the n-th Quillen homology of A is defined by

$$H_n^Q(\mathsf{A}) = H_n(\mathsf{F}_*^{\mathrm{ab}}), \quad n \geq 0.$$

Here $\mathsf{F}_*^{\mathrm{ab}}$ is the simplicial vector space obtained by applying the functor $(-)^{\mathrm{ab}}$ dimension-wise to F_*.

In the proof of the theorem immediately below, we need to use the result from [8] that if F is a free AWB, then the homology of the complex $\left(C_*^{\mathrm{AWB}}(\mathsf{F}), d_*\right)$ vanishes in positive dimensions, that is,

$$\mathsf{H}_n^{\mathrm{AWB}}(\mathsf{F}) = 0, \quad \text{for} \quad n \geq 1.$$

Theorem 2.6. *Let A be an AWB. Then there is an isomorphism of vector spaces*

$$\mathsf{H}_n^{\mathrm{AWB}}(\mathsf{A}) \cong H_n^Q(\mathsf{A}), \quad n \geq 0.$$

Proof. First of all let us note that the homology chain complex C_*^{AWB} is functorial in the sense that a homomorphism $\mathsf{A} \to \mathsf{A}'$ gives rise the chain map $C_*^{\mathrm{AWB}}(\mathsf{A}) \to C_*^{\mathrm{AWB}}(\mathsf{A}')$ in the canonical way.

Now, given a free simplicial resolution F_* of A, by applying the functor C_n^{AWB} dimension-wise, and then taking the alternating sums of face homomorphisms, we get an augmented chain complex of vector spaces $C_n^{\mathrm{AWB}}(\mathsf{F}_*) \to C_n^{\mathrm{AWB}}(\mathsf{A})$. Since $\mathsf{F}_* \to \mathsf{A}$ is an aspherical simplicial AWB, we claim that $C_n^{\mathrm{AWB}}(\mathsf{F}_*) \to C_n^{\mathrm{AWB}}(\mathsf{A})$ is acyclic chain complex for any $n \geq 0$. This is easy to confirm, since by forgetting AWB structure in the simplicial AWB $\mathsf{F}_* \to \mathsf{A}$, we get a simplicial vector space having a linear left (right) contraction.

Then using the facts that $\mathsf{H}_n^{\mathrm{AWB}}(\mathsf{F}_m) = 0$ and $\mathsf{H}_0^{\mathrm{AWB}}(\mathsf{F}_m) = \mathsf{F}_m^{\mathrm{ab}}$ for any $n \geq 1$ and $m \geq 0$ it follows that both spectral sequences for the bicomplex $C_*^{\mathrm{AWB}}(\mathsf{F}_*)$ degenerate and give the required isomorphism. □

3. Crossed modules of AWBs

3.1. Actions and semi-direct product

Definition 3.1. Let A and M be two AWBs. An action of A on M consists of four bilinear maps

$$\begin{array}{ll} \mathsf{A} \times \mathsf{M} \to \mathsf{M}, \ (a,m) \mapsto {}^{a\cdot}m, & \mathsf{M} \times \mathsf{A} \to \mathsf{M}, \ (m,a) \mapsto m^{\cdot a}, \\ \mathsf{A} \times \mathsf{M} \to \mathsf{M}, \ (a,m) \mapsto {}^{a*}m, & \mathsf{M} \times \mathsf{A} \to \mathsf{M}, \ (m,a) \mapsto m^{*a}, \end{array}$$

such that the following conditions hold:

$$\begin{aligned} {}^{(a_1a_2)\cdot}m &= {}^{a_1\cdot}\left({}^{a_2\cdot}m\right), & \left({}^{a_1\cdot}m\right)^{*a_2} &= {}^{a_1\cdot}\left(m^{*a_2}\right) + {}^{[a_1,a_2]\cdot}m, \\ m^{\cdot(a_1a_2)} &= \left(m^{\cdot a_1}\right)^{\cdot a_2}, & \left(m^{\cdot a_1}\right)^{*a_2} &= \left(m^{*a_2}\right)^{\cdot a_1} + m^{\cdot[a_1,a_2]}, \\ \left({}^{a_1\cdot}m\right)^{\cdot a_2} &= {}^{a_1\cdot}\left(m^{\cdot a_2}\right), & {}^{(a_1a_2)*}m &= {}^{a_1\cdot}\left({}^{a_2*}m\right) + \left({}^{a_1*}m\right)^{\cdot a_2}, \\ (m_1m_2)^{\cdot a} &= m_1(m_2^{\cdot a}), & [m_1^{\cdot a}, m_2] &= m_1\left({}^{a*}m_2\right) + [m_1,m_2]^{\cdot a}, \\ {}^{a\cdot}\left(m_1m_2\right) &= \left({}^{a\cdot}m_1\right)m_2, & [{}^{a\cdot}m_1, m_2] &= {}^{a\cdot}[m_1,m_2] + \left({}^{a*}m_2\right)m_1, \\ \left(m_1^{\cdot a}\right)m_2 &= m_1\left({}^{a\cdot}m_2\right), & (m_1m_2)^{*a} &= m_1\left(m_2{}^{*a}\right) + \left(m_1{}^{*a}\right)m_2, \end{aligned} \tag{3.1}$$

for all $a, a_1, a_2 \in \mathsf{A}$, $m, m_1, m_2 \in \mathsf{M}$. The action is called trivial if all these bilinear maps are trivial, i.e. ${}^{a\cdot}m = m^{\cdot a} = {}^{a*}m = m^{*a} = 0$, for all $a \in \mathsf{A}$ and $m \in \mathsf{M}$.

Let us remark that if an action of an AWB A on an abelian AWB M is given, then all six equations in the last three lines of (3.1) vanish. Among the remaining six equations, the first three equations in the first column say that M is a bimodule over A, and then the first three equations in the second column say that we get the definition of a *representation* M of A (see [8]).

Example 3.2.

(i) If M is a representation of an AWB A thought as an abelian AWB, then there is an action of A on the abelian AWB M.

(ii) If A is a subalgebra of some AWB B (maybe $\mathsf{A} = \mathsf{B}$) and if M is a two-sided ideal of B, then the operations in B yield an action of A on M given by ${}^{a\cdot}m = am$, $m^{\cdot a} = ma$, ${}^{a*}m = [a, m]$, $m^{*a} = [m, a]$, for all $m \in \mathsf{M}$ and $a \in \mathsf{A}$.

(iii) If $0 \to \mathsf{M} \xrightarrow{i} \mathsf{B} \xrightarrow{\pi} \mathsf{A} \to 0$ is a split short exact sequence of AWBs, that is, there exists a homomorphism $s\colon \mathsf{A} \to \mathsf{B}$ of AWBs such that $\pi \circ s = \mathsf{Id}_{\mathsf{A}}$, then there is an action of A on M, given by:

$$
\begin{aligned}
{}^{a\cdot}m &= i^{-1}\big(s(a)i(m)\big), & m^{\cdot a} &= i^{-1}\big(i(m)s(a)\big),\\
{}^{a*}m &= i^{-1}\big(\,[s(a), i(m)]\,\big), & m^{*a} &= i^{-1}\big(\,[i(m), s(a)]\,\big),
\end{aligned}
$$

for any $a \in \mathsf{A}$, $m \in \mathsf{M}$.

(iv) Any homomorphism of AWBs $f\colon \mathsf{A} \to \mathsf{M}$ induces an action of A on M in the standard way by taking images of elements of A and operations in M, i.e. ${}^{a\cdot}m = f(a)m$, $m^{\cdot a} = mf(a)$, ${}^{a*}m = [f(a), m]$ and $m^{*a} = [m, f(a)]$, for $m \in \mathsf{M}$, $a \in \mathsf{A}$.

(v) If $\mu\colon \mathsf{M} \to \mathsf{A}$ is a surjective homomorphism of AWBs and the kernel of μ is contained in the center of M, i.e. $\mathsf{Ker}(\mu) \subseteq \mathcal{Z}(\mathsf{M})$, then there is an action of A on M, defined in the standard way, i.e. by choosing pre-images of elements of A and taking operations in M.

Definition 3.3. Let A and M be AWBs with an action of A on M. The semi-direct product of M and A, denoted by $\mathsf{M} \rtimes \mathsf{A}$, is the AWB whose underlying vector space is $\mathsf{M} \oplus \mathsf{A}$ endowed with the operations

$$
\begin{aligned}
(m_1, a_1)(m_2, a_2) &= \big(m_1m_2 + {}^{a_1\cdot}m_2 + m_1{}^{\cdot a_2}, a_1a_2\big),\\
[(m_1, a_1), (m_2, a_2)] &= \big([m_1, m_2] + {}^{a_1*}m_2 + m_1{}^{*a_2}, [a_1, a_2]\big)
\end{aligned}
$$

for all $m_1, m_2 \in \mathsf{M}$, $a_1, a_2 \in \mathsf{A}$.

Given an action of an AWB A on M, straightforward calculations show that the sequence of AWBs

$$0 \longrightarrow \mathsf{M} \xrightarrow{\ i\ } \mathsf{M} \rtimes \mathsf{A} \xrightarrow{\ \pi\ } \mathsf{A} \longrightarrow 0$$

where $i(m) = (m, 0), \pi(m, a) = a$, is exact. Moreover M is a two-sided ideal of $\mathsf{M} \rtimes \mathsf{A}$ and this sequence splits by $s\colon \mathsf{A} \to \mathsf{M} \rtimes \mathsf{A}, s(a) = (0, a)$. Then, as in Example 3.2 (iii), the above sequence induces another action of A on M given by

$$
\begin{aligned}
{}^{a\cdot}m &= i^{-1}\big((0, a)(m, 0)\big), & m^{\cdot a} &= i^{-1}\big((m, 0)(0, a)\big),\\
{}^{a*}m &= i^{-1}\big[(0, a), (m, 0)\big], & m^{*a} &= i^{-1}\big[(m, 0), (0, a)\big],
\end{aligned}
$$

which actually matches the given one.

3.2. Crossed modules

Definition 3.4. A crossed module of AWBs is a homomorphism of AWBs $\mu\colon \mathsf{M} \to \mathsf{A}$ together with an action of A on M such that the following identities hold:

(CM1)

$$\mu(m^{\cdot a}) = \mu(m)a, \qquad \mu({}^{a\cdot}m) = a\mu(m),$$
$$\mu(m^{*a}) = [\mu(m), a], \qquad \mu({}^{a*}m) = [a, \mu(m)];$$

(CM2)

$$^{\mu(m)\cdot}m' = mm' \quad = m^{\cdot\mu(m')},$$
$$^{\mu(m)*}m' = [m, m'] = m^{*\mu(m')}$$

for all $m, m' \in \mathsf{M}$, $a \in \mathsf{A}$.

Definition 3.5. A morphism of crossed modules $\left(\mathsf{M} \xrightarrow{\mu} \mathsf{A}\right) \to \left(\mathsf{M}' \xrightarrow{\mu'} \mathsf{A}'\right)$ is a pair (α, β), where $\alpha\colon \mathsf{M} \to \mathsf{M}'$ and $\beta\colon \mathsf{A} \to \mathsf{A}'$ are homomorphisms of AWBs satisfying:

(a) $\beta \circ \mu = \mu' \circ \alpha$.

(b)

$$\alpha({}^{a\cdot}m) = {}^{\beta(a)\cdot}\alpha(m), \qquad \alpha(m^{\cdot a}) = \alpha(m)^{\cdot\beta(a)},$$
$$\alpha({}^{a*}m) = {}^{\beta(a)*}\alpha(m), \qquad \alpha(m^{*a}) = \alpha(m)^{*\beta(a)}$$

for all $a \in \mathsf{A}$, $m \in \mathsf{M}$.

It is clear that crossed modules of AWBs constitute a category, denoted by XAWB.

The following lemma is an easy consequence of Definition 3.4.

Lemma 3.6. *Let $\mu\colon \mathsf{M} \to \mathsf{A}$ be a crossed module of AWBs. Then the following statements are satisfied:*

(i) $\mathsf{Ker}(\mu) \subseteq \mathcal{Z}(\mathsf{M})$.

(ii) $\mathsf{Im}(\mu)$ *is a two-sided ideal of* A.

(iii) $\mathsf{Im}(\mu)$ *acts trivially on* $\mathcal{Z}(\mathsf{M})$*, and so trivially on* $\mathsf{Ker}(\mu)$*. Hence* $\mathsf{Ker}(\mu)$ *inherits an action of* $\mathsf{A}/\mathsf{Im}(\mu)$ *making* $\mathsf{Ker}(\mu)$ *a representation of the* AWB $\mathsf{A}/\mathsf{Im}(\mu)$.

Example 3.7.

(i) Let A be an AWB and B be a two-sided ideal of A, then the inclusion $\mathsf{B} \hookrightarrow \mathsf{A}$ is a crossed module, where the action of A on B is given by the operations in A (see Example 3.2 (ii)). Conversely, if $\mu\colon \mathsf{B} \to \mathsf{A}$ is a crossed module of AWBs and μ is an injective map, then B is isomorphic to a two-sided ideal of A by Lemma 3.6 (ii).

(ii) For any representation M of an AWB A, the trivial map $0\colon \mathsf{M} \to \mathsf{A}$ is a crossed module with the action of A on the abelian AWB M described in Example 3.2 (i).
Conversely, if $0\colon \mathsf{M} \to \mathsf{A}$ is a crossed module of AWBs, then M is necessarily an abelian AWB and the action of A on M is equivalent to M being a representation of A.

(iii) Any homomorphism of AWBs $\mu\colon \mathsf{M} \to \mathsf{A}$, with M abelian and $\mathsf{Im}(\mu) \subseteq \mathcal{Z}(\mathsf{A})$, provides a crossed module with A acting trivially on M.

(iv) If $0 \to \mathsf{N} \to \mathsf{M} \xrightarrow{\mu} \mathsf{A} \to 0$ is a central extension of AWBs, then μ is a crossed module with the induced action of A on M (see Example 3.2 (v)).

Proposition 3.8. *Let $\mu\colon \mathsf{M} \to \mathsf{A}$ be a crossed module of AWBs. Then the maps*

(i) $(\mu, \mathsf{Id}_\mathsf{A})\colon \mathsf{M} \rtimes \mathsf{A} \to \mathsf{A} \rtimes \mathsf{A}$,

(ii) $(\mathsf{Id}_\mathsf{M}, \mu)\colon \mathsf{M} \rtimes \mathsf{M} \to \mathsf{M} \rtimes \mathsf{A}$,

(iii) $\varphi\colon \mathsf{M} \rtimes \mathsf{A} \to \mathsf{M} \rtimes \mathsf{A}$ *given by* $\varphi(m, a) = (-m, \mu(m) + a)$,

are homomorphisms of AWBs.

Proof. (i) is a direct consequence of equalities in (CM1) of Definition 3.4, (ii) follows from equalities in (CM2), whilst (iii) requires both (CM1) and (CM2). □

Remark 3.9. The functors I and T given in Example 2.2 (ii) and (iii) preserve actions and crossed modules in the sense of the following assertions:

(i) Any action of an associative algebra A on another associative algebra M, $\mathsf{A} \times \mathsf{M} \to \mathsf{M}$, $(a, m) \mapsto a \cdot m$ and $M \times A \to M$, $(m, a) \mapsto m \cdot a$ (see [10,12]) defines an action of the AWB $I(\mathsf{A})$ on $I(\mathsf{M})$ (resp. of $T(\mathsf{A})$ on $T(\mathsf{M})$), by letting

$$^{a\cdot}m = a \cdot m, \;\; m^{\cdot a} = m \cdot a, \;\; ^{a*}m = 0, \;\; m^{*a} = 0,$$

$$\Big(\text{resp.} \quad ^{a\cdot}m = a \cdot m, \;\; m^{\cdot a} = m \cdot a, \;\; ^{a*}m = a \cdot m - m \cdot a, \;\; m^{*a} = m \cdot a - a \cdot m\Big)$$

for all $a \in \mathsf{A}$ and $m \in \mathsf{M}$.

(ii) If $\mu\colon \mathsf{M} \to \mathsf{A}$ is a crossed module of associative algebras (see again [10,12]), then the homomorphisms of AWBs $I(\mu)\colon I(\mathsf{M}) \to I(\mathsf{A})$ and $T(\mu)\colon T(\mathsf{M}) \to T(\mathsf{A})$, together with the actions of $I(\mathsf{A})$ on $I(\mathsf{M})$ and of $T(\mathsf{A})$ on $T(\mathsf{M})$, are crossed modules of AWBs.

In [6] we proved equivalence of crossed modules of AWBs with internal categories in the category of AWBs. Now we show their equivalence with cat^1-AWBs. The following definition of cat^1-AWB is given in complete analogy with Loday's original notion of cat^1-groups [24].

Definition 3.10. A cat^1-AWB $(\mathsf{R}, \mathsf{P}, s, t)$ consists of an AWB R, together with a subalgebra P and two homomorphisms $s, t\colon \mathsf{R} \to \mathsf{P}$ of AWBs satisfying the following conditions:

(a) $s|_{\mathsf{P}} = t|_{\mathsf{P}} = \mathsf{Id}_{\mathsf{P}}$.
(b) $\mathsf{Ker}(s)\ \mathsf{Ker}(t) = 0 = \mathsf{Ker}(t)\ \mathsf{Ker}(s)$.
(c) $[\mathsf{Ker}(s), \mathsf{Ker}(t)] = 0 = [\mathsf{Ker}(t), \mathsf{Ker}(s)]$.

Definition 3.11. A morphism of cat^1-AWBs $(\mathsf{R}, \mathsf{P}, s, t) \to (\mathsf{R}', \mathsf{P}', s', t')$ is a homomorphism of AWBs $f\colon \mathsf{R} \to \mathsf{R}'$ such that $f(\mathsf{P}) \subseteq \mathsf{P}'$ and $s' \circ f = f|_{\mathsf{P}} \circ s$, $t' \circ f = f|_{\mathsf{P}} \circ t$.

We let $\mathsf{cat}^1\!-\!\mathsf{AWB}$ denote the category of cat^1-AWBs. Then we have the following theorem.

Theorem 3.12. *The categories* $\mathsf{cat}^1\!-\!\mathsf{AWB}$ *and* XAWB *are equivalent.*

Proof. To a given cat^1-AWB $(\mathsf{R}, \mathsf{P}, s, t)$ we associate a crossed module $\mu = t|_{\mathsf{M}}\colon \mathsf{M} \to \mathsf{P}$, where $\mathsf{M} = \mathsf{Ker}(s)$ and the action of P on M is given by the operations in R (see Example 3.2 (ii)). It is easy to see that $\mu\colon \mathsf{M} \to \mathsf{P}$ is a crossed module of AWBs and the assignment defines a functor $\Phi\colon \mathsf{cat}^1\!-\!\mathsf{AWB} \longrightarrow \mathsf{XAWB}$.

Conversely, let $\mu\colon \mathsf{M} \to \mathsf{P}$ be a crossed module of AWBs, then the associated cat^1-AWB is given by $s, t\colon \mathsf{M} \rtimes \mathsf{P} \to \mathsf{P}$, where $s(m, p) = p$, $t(m, p) = \mu(m) + p$, $m \in \mathsf{M}$, $p \in \mathsf{P}$. It is straightforward to see that this assignment is functorial and provides a quasi-inverse functor for Φ. □

4. Non-abelian tensor product of AWBs

Definition 4.1. Let M and N be AWBs with mutual actions on each other. The actions are said to be compatible if

$$\begin{aligned}
m^{\cdot(m'^{\cdot n'})} &= m(m'^{\cdot n'}), & m^{\cdot(n'^{\cdot m'})} &= m(^{n'\cdot}m'),\\
m^{\cdot(m'^{*n'})} &= m(m'^{*n'}), & m^{\cdot(n'^{*m'})} &= m(^{n'*}m'),\\
m^{*(m'^{\cdot n'})} &= [m, m'^{\cdot n'}], & m^{*(n'^{\cdot m'})} &= [m, ^{n'\cdot}m'],\\
m^{*(m'^{*n'})} &= [m, m'^{*n'}], & m^{*(n'^{*m'})} &= [m, ^{n'*}m'],\\
^{(m^{\cdot n})\cdot}n' &= (^{m\cdot}n)n', & ^{(^{n\cdot}m)\cdot}n' &= (n^{\cdot m})n',\\
^{(m^{*n})\cdot}n' &= (^{m*}n)n', & ^{(^{n*}m)\cdot}n' &= (n^{*m})n',
\end{aligned} \tag{4.1}$$

$$
{}^{(m^{\cdot n})*}n' = [{}^{m\cdot}n, n'], \qquad {}^{({}^{n\cdot}m)*}n' = [n^{\cdot m}, n'],
$$
$$
{}^{(m^{*n})*}n' = [{}^{m*}n, n'], \qquad {}^{({}^{n*}m)*}n' = [n^{*m}, n'],
$$

and moreover, another 16 equations obtained by exchanging the roles of elements of M and N in (4.1) are also valid.

Example 4.2.

(a) If M and N are two-sided ideals of an AWB A, then the mutual actions on each other considered in Example 3.2 (ii) are compatible.

(b) Let $\mu\colon \mathsf{M} \to \mathsf{P}$ and $\nu\colon \mathsf{N} \to \mathsf{P}$ be two crossed modules of AWBs. Then the mutual actions of M on N via μ and of N on M via ν are compatible.

Let M and N be AWBs with mutually compatible actions on each other. We denote by $\mathsf{M} \odot \mathsf{N}$ the vector space spanned by all symbols $m \odot n$, $n \odot m$ and by $\mathsf{M} \circledast \mathsf{N}$ the vector space spanned by all symbols $m \circledast n$, $n \circledast m$, for $m \in \mathsf{M}$, $n \in \mathsf{N}$. Let $\mathsf{M} \boxtimes \mathsf{N}$ denotes the quotient of $(\mathsf{M} \odot \mathsf{N}) \oplus (\mathsf{M} \circledast \mathsf{N})$ by the following relations:

$$
\begin{array}{ll}
\lambda(m \star n) = (\lambda m) \star n = m \star (\lambda n), & \\
(m + m') \star n = m \star n + m' \star n, & m \star (n + n') = m \star n + m \star n', \\
m^{\cdot n} \star {}^{m'\cdot}n' = {}^{m\cdot}n \star m'^{\cdot n'}, & m^{\cdot n} \star n'^{\cdot m'} = {}^{m\cdot}n \star {}^{n'\cdot}m', \\
{}^{n\cdot}m \star n'^{\cdot m'} = n^{\cdot m} \star {}^{n'\cdot} m', & {}^{n\cdot}m \star {}^{m'\cdot}n' = n^{\cdot m} \star m'^{\cdot n'}, \\
m^{\cdot n} \star {}^{m'*}n' = {}^{m\cdot}n \star m'^{*n'}, & m^{\cdot n} \star n'^{*m'} = {}^{m\cdot}n \star {}^{n'*}m', \\
{}^{n\cdot}m \star {}^{m'*}n' = n^{\cdot m} \star m'^{*n'}, & {}^{n\cdot}m \star n'^{*m'} = n^{\cdot m} \star {}^{n'*}m', \\
m^{*n} \star {}^{m'\cdot}n' = {}^{m*}n \star m'^{\cdot n'}, & m^{*n} \star n'^{\cdot m'} = {}^{m*}n \star {}^{n'\cdot}m', \\
{}^{n*}m \star {}^{m'\cdot}n' = n^{*m} \star m'^{\cdot n'}, & {}^{n*}m \star n'^{\cdot m'} = n^{*m} \star {}^{n'\cdot}m', \\
m^{*n} \star {}^{m'*}n' = {}^{m*}n \star m'^{*n'}, & m^{*n} \star n'^{*m'} = {}^{m*}n \star {}^{n'*}m', \\
{}^{n*}m \star {}^{m'*}n' = n^{*m} \star m'^{*n'}, & {}^{n*}m \star n'^{*m'} = n^{*m} \star {}^{n'*}m', \\
(m_1m_2) \odot n = m_1 \odot ({}^{m_2\cdot}n), & n \odot (m_1m_2) = (n^{\cdot m_1}) \odot m_2, \\
(m_1m_2) \circledast n = m_1 \odot ({}^{m_2*}n) + ({}^{m_1*}n) \odot m_2, & \\
{}^{m_1\cdot}n \circledast m_2 = m_1 \odot n^{*m_2} + [m_1, m_2] \odot n, & \\
n^{\cdot m_1} \circledast m_2 = n^{*m_2} \odot m_1 + n \odot [m_1, m_2], & \\
{}^{m_1\cdot}n \odot m_2 = m_1 \odot n^{\cdot m_2}, &
\end{array}
\tag{4.2}
$$

and another 25 relations obtained by exchanging the roles of elements of M and N in (4.2), where the symbol $\star$ stands for either $\odot$ or $\circledast$.

Proposition 4.3. *The vector space* $\mathsf{M}\boxtimes\mathsf{N}$ *endowed with the product and bracket operations given on the generators by*

$$
\begin{array}{ll}
(m \odot n)(m' \odot n') = (m^{\cdot n}) \odot ({}^{m'\cdot}n'), & (m \odot n)(n' \odot m') = (m^{\cdot n}) \odot (n'^{\cdot m'}), \\
(n \odot m)(m' \odot n') = ({}^{n\cdot}m) \odot ({}^{m'\cdot}n'), & (n \odot m)(n' \odot m') = ({}^{n\cdot}m) \odot (n'^{\cdot m'}), \\
(m \odot n)(m' \circledast n') = (m^{\cdot n}) \odot ({}^{m'*}n'), & (m \odot n)(n' \circledast m') = (m^{\cdot n}) \odot (n'^{*m'}), \\
(n \odot m)(m' \circledast n') = ({}^{n\cdot}m) \odot (m'^{*n'}), & (n \odot m)(n' \circledast m') = ({}^{n\cdot}m) \odot (n'^{*m'}), \\
(m \circledast n)(m' \odot n') = (m^{*n}) \odot ({}^{m'\cdot}n'), & (m \circledast n)(n' \odot m') = (m^{*n}) \odot (n'^{\cdot m'}), \\
(n \circledast m)(m' \odot n') = ({}^{n*}m) \odot ({}^{m'\cdot}n'), & (n \circledast m)(n' \odot m') = ({}^{n*}m) \odot (n'^{\cdot m'}), \\
(m \circledast n)(m' \circledast n') = (m^{*n}) \odot ({}^{m'*}n'), & (m \circledast n)(n' \circledast m') = (m^{*n}) \odot (n'^{*m'}),
\end{array}
$$

$$(n \circledast m)(m' \circledast n') = ({}^{n*}m) \odot ({}^{m'*}n'), \qquad (n \circledast m)(n' \circledast m') = ({}^{n*}m) \odot (n'^{*m'}),$$
$$[m \odot n, m' \odot n'] = (m^{\cdot n}) \circledast ({}^{m'\cdot}n'), \qquad [m \odot n, n' \odot m'] = (m^{\cdot n}) \circledast (n'^{\cdot m'}),$$
$$[n \odot m, m' \odot n'] = ({}^{n\cdot}m) \circledast ({}^{m'\cdot}n'), \qquad [n \odot m, n' \odot m'] = ({}^{n\cdot}m) \circledast (n'^{\cdot m'}),$$
$$[m \odot n, m' \circledast n'] = (m^{\cdot n}) \circledast ({}^{m'*}n'), \qquad [m \odot n, n' \circledast m'] = (m^{\cdot n}) \circledast (n'^{*m'}),$$
$$[n \odot m, m' \circledast n'] = ({}^{n\cdot}m) \circledast ({}^{m'*}n'), \qquad [n \odot m, n' \circledast m'] = ({}^{n\cdot}m) \circledast (n'^{*m'}),$$
$$[m \circledast n, m' \odot n'] = (m^{*n}) \circledast ({}^{m'\cdot}n'), \qquad [m \circledast n, n' \odot m'] = (m^{*n}) \circledast (n'^{\cdot m'}),$$
$$[n \circledast m, m' \odot n'] = ({}^{n*}m) \circledast ({}^{m'\cdot}n'), \qquad [n \circledast m, n' \odot m'] = ({}^{n*}m) \circledast (n'^{\cdot m'}),$$
$$[m \circledast n, m' \circledast n'] = (m^{*n}) \circledast ({}^{m'*}n'), \qquad [m \circledast n, n' \circledast m'] = (m^{*n}) \circledast (n'^{*m'}),$$
$$[n \circledast m, m' \circledast n'] = ({}^{n*}m) \circledast ({}^{m'*}n'), \qquad [n \circledast m, n' \circledast m'] = ({}^{n*}m) \circledast (n'^{*m'}),$$

has the structure of an AWB.

Proof. Straightforward calculations show that, under the conditions of compatible actions (4.1), by using the relations in (4.2), the described operations on $\mathsf{M} \boxtimes \mathsf{N}$ satisfy the fundamental identity (2.1). □

Definition 4.4. The structure of AWB on $\mathsf{M} \boxtimes \mathsf{N}$ provided by Proposition 4.3 is called the non-abelian tensor product of the AWBs M and N.

In particular, if the actions are trivial, the non-abelian tensor product can be described as follows.

Proposition 4.5. *If* M *and* N *are two AWBs with trivial actions on each other, then there is an isomorphism of abelian AWBs*

$$\mathsf{M} \boxtimes \mathsf{N} \cong \left(\mathsf{M}^{ab} \otimes_{\mathbb{K}} \mathsf{N}^{ab}\right) \oplus \left(\mathsf{N}^{ab} \otimes_{\mathbb{K}} \mathsf{M}^{ab}\right) \oplus \left(\mathsf{M}^{ab} \otimes_{\mathbb{K}} \mathsf{N}^{ab}\right) \oplus \left(\mathsf{N}^{ab} \otimes_{\mathbb{K}} \mathsf{M}^{ab}\right).$$

Proof. Equations in Proposition 4.3 show us easily that $\mathsf{M} \boxtimes \mathsf{N}$ is abelian in the case of trivial actions. The defining relations (4.2) of the non-abelian tensor product say that the vector space $\mathsf{M} \boxtimes \mathsf{N}$ is the quotient of $(\mathsf{M} \otimes_{\mathbb{K}} \mathsf{N}) \oplus (\mathsf{N} \otimes_{\mathbb{K}} \mathsf{M}) \oplus (\mathsf{M} \otimes_{\mathbb{K}} \mathsf{N}) \oplus (\mathsf{N} \otimes_{\mathbb{K}} \mathsf{M})$ by the relations

$$\begin{aligned} 0 = (m_1 m_2) \otimes n &= [m_1, m_2] \otimes n \\ = n \otimes (m_1 m_2) &= n \otimes [m_1, m_2] \\ = m \otimes (n_1 n_2) &= m \otimes [n_1, n_2] \\ = (n_1 n_2) \otimes m &= [n_1, n_2] \otimes m \end{aligned}$$

for all $m, m_1, m_2 \in \mathsf{M}$, $n, n_1, n_2 \in \mathsf{N}$. This provides the required isomorphism. □

The non-abelian tensor product of AWBs is functorial in the following sense: let $f \colon \mathsf{M} \to \mathsf{M}'$ and $g \colon \mathsf{N} \to \mathsf{N}'$ be homomorphisms of AWBs together with mutually compatible actions of M and N, also M' and N' on each other such that f, g preserve these actions, i.e.

$$f({}^{n\cdot}m) = {}^{g(n)\cdot}f(m),\ f(m^{\cdot n}) = f(m)^{\cdot g(n)},\ f({}^{n*}m) = {}^{g(n)*}f(m),\ f({}^{n*}m) = {}^{g(n)*}f(m),$$
$$g({}^{m\cdot}n) = {}^{f(m)\cdot}g(n),\ g(n^{\cdot m}) = g(n)^{\cdot f(m)},\quad g({}^{m*}n) = {}^{f(m)*}g(n),\ g({}^{m*}n) = {}^{f(m)*}g(n).$$

for all $m \in \mathsf{M}, n \in \mathsf{N}$, then there is a homomorphism of AWBs

$$f \boxtimes g \colon \mathsf{M} \boxtimes \mathsf{N} \longrightarrow \mathsf{M}' \boxtimes \mathsf{N}'$$

defined by

$$(f \boxtimes g)(m \odot n) = f(m) \odot g(n), \quad (f \boxtimes g)(n \odot m) = g(n) \odot f(m),$$
$$(f \boxtimes g)(m \circledast n) = f(m) \circledast g(n), \quad (f \boxtimes g)(n \circledast m) = g(n) \circledast f(m).$$

The non-abelian tensor product of AWBs has a kind of right-exactness property presented in the following theorem.

Theorem 4.6. *Let* $0 \to \mathsf{M}_1 \xrightarrow{f} \mathsf{M}_2 \xrightarrow{g} \mathsf{M}_3 \to 0$ *be a short exact sequence of AWBs. Let* N *be an* AWB *together with compatible actions of* N *and* M_i $(i = 1, 2, 3)$ *on each other and* f, g *preserve these actions. Then there is an exact sequence of AWBs*

$$\mathsf{M}_1 \boxtimes \mathsf{N} \xrightarrow{f \boxtimes \mathsf{Id}_\mathsf{N}} \mathsf{M}_2 \boxtimes \mathsf{N} \xrightarrow{g \boxtimes \mathsf{Id}_\mathsf{N}} \mathsf{M}_3 \boxtimes \mathsf{N} \to 0.$$

Proof. It is clear that the composition $(g \boxtimes \mathsf{Id}_\mathsf{N})\,(f \boxtimes \mathsf{Id}_\mathsf{N})$ is the trivial map, i.e. $\mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N}) \subseteq \mathsf{Ker}\,(f \boxtimes \mathsf{Id}_\mathsf{N})$ and at the same time $f \boxtimes \mathsf{Id}_\mathsf{N}$ is an epimorphism.

$\mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N})$ is generated by the elements of the form $f(m_1) \odot n$, $n \odot f(m_1)$, $f(m_1) \circledast n$ and $n \circledast f(m_1)$, for all $m_1 \in \mathsf{M}_1$, $n \in \mathsf{N}$. Since f preserves actions of N, by the relations given in Proposition 4.3, it is easily verified that $\mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N})$ is a two-sided ideal of $\mathsf{M}_2 \boxtimes \mathsf{N}$. For instance, taking a generator of the form $m_2 \odot n'$ in $\mathsf{M}_2 \boxtimes \mathsf{N}$ we have

$$\begin{aligned}(f(m_1) \odot n)\,(m_2 \odot n') &= f(m_1)^{\cdot n} \odot {}^{m_2 \cdot} n' = f(m_1^{\cdot n}) \odot {}^{m_2 \cdot} n' \in \mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N})\,,\\ (f(m_1) \circledast n)\,(m_2 \odot n') &= f(m_1)^{* n} \odot {}^{m_2 \cdot} n' = f(m_1^{* n}) \odot {}^{m_2 \cdot} n' \in \mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N})\,.\end{aligned}$$

Then, there is a homomorphism of AWBs

$$\alpha\colon\ (\mathsf{M}_2 \boxtimes \mathsf{N})\,/\mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N}) \longrightarrow (\mathsf{M}_3 \boxtimes \mathsf{N})$$

induced by $g \boxtimes \mathsf{Id}_\mathsf{N}$, that is, defined on generators by

$$\begin{aligned}\alpha\left(\overline{m_2 \odot n}\right) &= g(m_2) \odot n, \quad & \alpha\left(\overline{n \odot m_2}\right) &= n \odot g(m_2),\\ \alpha\left(\overline{m_2 \circledast n}\right) &= g(m_2) \circledast n, \quad & \alpha\left(\overline{n \circledast m_2}\right) &= n \circledast g(m_2).\end{aligned}$$

where the overdrawn generator denotes the coset of the corresponding element. On the other hand, we have well-defined homomorphism of AWBs

$$\alpha'\colon\ (\mathsf{M}_3 \boxtimes \mathsf{N}) \longrightarrow (\mathsf{M}_2 \boxtimes \mathsf{N})\,/\mathsf{Im}\,(f \boxtimes \mathsf{Id}_\mathsf{N})$$

given on generators by

$$\begin{aligned}\alpha'\,(m_3 \odot n) &= \left(\overline{m_2 \odot n}\right), \quad & \alpha'\,\left(\overline{n \odot m_3}\right) &= \overline{n \odot m_2},\\ \alpha'\,(m_3 \circledast n) &= \overline{m_2 \circledast n}, \quad & \alpha\,(n \circledast m_3) &= \overline{n \circledast m_2},\end{aligned}$$

where $m_2 \in \mathsf{M}_2$ is any element such that $g(m_2) = m_3$. Obviously α and α' are inverse to each other, i.e. α is an isomorphism. Then the required exactness follows. □

Proposition 4.7. *Let* M *and* N *be AWBs with compatible actions on each other.*

(a) *There are homomorphisms of AWBs*

$$\begin{aligned}&\psi_\mathsf{M}\colon \mathsf{M} \boxtimes \mathsf{N} \to \mathsf{M} \text{ given by, } && \psi_\mathsf{M}(m \odot n) = m^{\cdot n},\ \psi_\mathsf{M}(n \odot m) = {}^{n \cdot} m,\\ & && \psi_\mathsf{M}(m \circledast n) = m^{* n},\ \psi_\mathsf{M}(n \circledast m) = {}^{n *} m;\\ &\text{and } \psi_\mathsf{N}\colon \mathsf{M} \boxtimes \mathsf{N} \to \mathsf{N} \text{ given by, } && \psi_\mathsf{N}(m \odot n) = {}^{m \cdot} n,\ \psi_\mathsf{N}(n \odot m) = n^{\cdot m},\\ & && \psi_\mathsf{N}(m \circledast n) = {}^{m *} n,\ \psi_\mathsf{N}(n \circledast m) = n^{* m}.\end{aligned}$$

(b) *There are actions of* M *and* N *on the non-abelian tensor product* $\mathsf{M} \boxtimes \mathsf{N}$ *given, for all* $m, m' \in \mathsf{M}, n, n' \in \mathsf{N}$, *by*

$$
\begin{aligned}
{}^{m\cdot}(m' \odot n') &= m \odot ({}^{m'\cdot}n'), & {}^{m\cdot}(n' \odot m') &= m \odot (n'^{\cdot m'}),\\
(m' \odot n')^{\cdot m} &= ({}^{m'\cdot}n') \odot m, & (n' \odot m')^{\cdot m} &= (n'^{\cdot m'}) \odot m,\\
{}^{m*}(m' \odot n') &= m \circledast ({}^{m'\cdot}n'), & {}^{m*}(n' \odot m') &= m \circledast (n'^{\cdot m'}),\\
(m' \odot n')^{*m} &= ({}^{m'\cdot}n') \circledast m, & (n' \odot m')^{*m} &= (n'^{\cdot m'}) \circledast m,\\
{}^{m\cdot}(m' \circledast n') &= m \odot ({}^{m'*}n'), & {}^{m\cdot}(n' \circledast m') &= m \odot (n'^{*m'}),\\
(m' \circledast n')^{\cdot m} &= ({}^{m'*}n') \odot m, & (n' \circledast m')^{\cdot m} &= (n'^{*m'}) \odot m,\\
{}^{m*}(m' \circledast n') &= m \circledast ({}^{m'*}n'), & {}^{m*}(n' \circledast m') &= m \circledast (n'^{*m'}),\\
(m' \circledast n')^{*m} &= ({}^{m'*}n') \circledast m, & (n' \circledast m')^{*m} &= (n'^{*m'}) \circledast m,
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{n\cdot}(m' \odot n') &= n \odot (m'^{\cdot n'}), & {}^{n\cdot}(n' \odot m') &= n \odot ({}^{n'\cdot}m'),\\
(m' \odot n')^{\cdot n} &= (m'^{\cdot n'}) \odot n, & (n' \odot m')^{\cdot n} &= ({}^{n'\cdot}m') \odot n,\\
{}^{n*}(m' \odot n') &= n \circledast (m'^{\cdot n'}), & {}^{n*}(n' \odot m') &= n \circledast ({}^{n'\cdot}m'),\\
(m' \odot n')^{*n} &= (m'^{\cdot n'}) \circledast n, & (n' \odot m')^{*n} &= ({}^{n'\cdot}m') \circledast n,\\
{}^{n\cdot}(m' \circledast n') &= n \odot (m'^{*n'}), & {}^{n\cdot}(n' \circledast m') &= n \odot ({}^{n'*}m'),\\
(m' \circledast n')^{\cdot n} &= (m'^{*n'}) \odot n, & (n' \circledast m')^{\cdot n} &= ({}^{n'*}m') \odot n,\\
{}^{n*}(m' \circledast n') &= n \circledast (m'^{*n'}), & {}^{n*}(n' \circledast m') &= n \circledast ({}^{n'*}m'),\\
(m' \circledast n')^{*n} &= (m'^{*n'}) \circledast n, & (n' \circledast m')^{*n} &= ({}^{n'*}m') \circledast n.
\end{aligned}
$$

(c) *The homomorphisms* ψ_{M} *and* ψ_{N} *together with the actions described in the statement (b) are crossed modules of* AWBs.

Proof. This is straightforward but tedious verification. □

Theorem 4.8. *If* A *is a perfect* AWB, *then* $\psi_{\mathsf{A}} \colon \mathsf{A} \boxtimes \mathsf{A} \to \mathsf{A}$ *is the universal central extension of* A.

Proof. Clearly $\psi_{\mathsf{A}} \colon \mathsf{A} \boxtimes \mathsf{A} \to \mathsf{A}$ is an epimorphism if A is perfect. Moreover, it is a crossed module of AWBs by Proposition 4.7 (c). Then Lemma 3.6 (i) says that it is a central extension.

To show the universal property, consider any central extension $0 \to \mathsf{M} \to \mathsf{B} \xrightarrow{\phi} \mathsf{A} \to 0$. Since $\mathsf{Ker}(\phi) \subseteq \mathcal{Z}(\mathsf{B})$, we get a well-defined homomorphism of AWBs $\varphi \colon \mathsf{A} \boxtimes \mathsf{A} \to \mathsf{B}$ given on generators by $\varphi(a \odot a') = b\,b'$ and $\varphi(a \circledast a') = [b, b']$, where b and b' are any elements in $\phi^{-1}(a)$ and $\phi^{-1}(a')$, respectively. Obviously $\phi\varphi = \psi_{\mathsf{A}}$. Moreover, since A is perfect, it follows by the equalities in Proposition 4.3 that $\mathsf{A} \boxtimes \mathsf{A}$ is a perfect AWB as well. Then [4, Lemma 3.1] implies that φ is the unique homomorphism satisfying the required conditions. □

Bearing in mind that the universal central extension of a perfect AWB is unique up to isomorphism, by [4, Theorem 3.5] we conclude that

$$\mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A}) \cong \mathsf{Ker}(\psi_{\mathsf{A}} \colon \mathsf{A} \boxtimes \mathsf{A} \twoheadrightarrow \mathsf{A}).$$

Moreover, if $0 \to \mathsf{R} \to \mathsf{F} \xrightarrow{\rho} \mathsf{A} \to 0$ is a free presentation of a perfect AWB A, then its universal central extension is

$$0 \longrightarrow \frac{\mathsf{R} \cap [[\mathsf{F}, \mathsf{F}]]}{[[\mathsf{F}, \mathsf{R}]]} \longrightarrow \frac{[[\mathsf{F}, \mathsf{F}]]}{[[\mathsf{F}, \mathsf{R}]]} \xrightarrow{\rho^*} \mathsf{A} \longrightarrow 0$$

(see [4]), hence

$$\mathsf{A} \boxtimes \mathsf{A} \cong \frac{[[\mathsf{F},\mathsf{F}]]}{[[\mathsf{F},\mathsf{R}]]}$$

due to the uniqueness (up to isomorphisms) of the universal central extension.

Remark 4.9. The article [4] provides another description of the universal central extension of a perfect AWB A. In particular, it is shown that, given an AWB A, the quotient $\frac{\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}}{I_{\mathsf{A}}}$ has an AWB structure, where I_{A} is the image of the map $d_2 \colon \mathsf{A}^{\otimes 3} \oplus \mathsf{A}^{\otimes 3} \to \mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}$ in the homology chain complex $\left(C_*^{\mathrm{AWB}}(\mathsf{A}), d_*\right)$, that is, I_{A} is the subspace of $\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}$ spanned by the elements of the form

$$(a_1a_2) \otimes a_3 - a_1 \otimes (a_2a_3),$$
$$[a_1,a_2] \otimes a_3 + a_1 \otimes [a_2,a_3] - (a_1a_2) \circ a_3,$$

for any $a_1, a_2, a_3 \in \mathsf{A}$. Moreover, if A is a perfect AWB, then it gives the construction of the universal central extension of A. As a consequence, we have the following isomorphism of AWBs

$$\frac{\mathsf{A}^{\otimes 2} \oplus \mathsf{A}^{\otimes 2}}{I_{\mathsf{A}}} \xrightarrow{\cong} \mathsf{A} \boxtimes \mathsf{A},$$

given by $a_1 \otimes a_2 \mapsto a_1 \odot a_2$ and $a_1 \circ a_2 \mapsto a_1 \circledast a_2$.

Proposition 4.10. *If* M *is a two-sided ideal of an* AWB A*, then there is the exact sequence of AWBs*

$$(\mathsf{M} \boxtimes \mathsf{A}) \rtimes (\mathsf{A} \boxtimes \mathsf{M}) \xrightarrow{\sigma} \mathsf{A} \boxtimes \mathsf{A} \xrightarrow{\tau} \mathsf{A}/\mathsf{M} \boxtimes \mathsf{A}/\mathsf{M} \to 0.$$

Proof. The functorial property of the non-abelian tensor product applied to the projection $\mathsf{A} \twoheadrightarrow \mathsf{A}/\mathsf{M}$ induces the surjective homomorphism τ, and applied to $\mathsf{inc} \colon \mathsf{M} \to \mathsf{A}$ and $\mathsf{Id} \colon \mathsf{A} \to \mathsf{A}$ provides the homomorphisms $\sigma' \colon \mathsf{M} \boxtimes \mathsf{A} \to \mathsf{A} \boxtimes \mathsf{A}$ and $\sigma'' \colon \mathsf{A} \boxtimes \mathsf{M} \to \mathsf{A} \boxtimes \mathsf{A}$.

Define $\sigma(x,y) = \sigma'(x) + \sigma''(y)$, for all $x \in \mathsf{M} \boxtimes \mathsf{A}$, $y \in \mathsf{A} \boxtimes \mathsf{M}$. $\mathsf{Im}(\sigma)$ is a two sided ideal of $\mathsf{A} \boxtimes \mathsf{A}$ spanned by the elements of the form $m \odot a$, $a \odot m$, $m \circledast a$, $a \circledast m$ for all $a \in \mathsf{A}$ and $m \in \mathsf{M}$.

By the identities in Proposition 4.3 and the relations (4.2) of the non-abelian tensor product, τ induces a homomorphism of AWBs $\bar{\tau} \colon \frac{\mathsf{A} \boxtimes \mathsf{A}}{\mathsf{Im}(\sigma)} \to \mathsf{A}/\mathsf{M} \boxtimes \mathsf{A}/\mathsf{M}$. Define $\tau' \colon \mathsf{A}/\mathsf{M} \boxtimes \mathsf{A}/\mathsf{M} \to \frac{\mathsf{A} \boxtimes \mathsf{A}}{\mathsf{Im}(\sigma)}$ by $\tau'\big((a_1 + \mathsf{M}) \odot (a_2 + \mathsf{M})\big) = a_1 \odot a_2 + \mathsf{Im}(\sigma)$, $\tau'\big((a_1 + \mathsf{M}) \circledast (a_2 + \mathsf{M})\big) = a_1 \circledast a_2 + \mathsf{Im}(\sigma)$. It is easy to check that τ' is a well-defined homomorphism that is inverse to $\bar{\tau}$. ☐

Theorem 4.11. *Let* M *be a two-sided ideal of a perfect* AWB A*. Then there is an exact sequence of vector spaces*

$$\mathsf{Ker}(\mathsf{M} \boxtimes \mathsf{A} \xrightarrow{\psi_{\mathsf{M}}} \mathsf{M}) \to \mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A}) \to \mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A}/\mathsf{M}) \to \frac{\mathsf{M}}{[[\mathsf{A},\mathsf{M}]]} \to 0$$

Proof. Due to Proposition 4.10 there is the following commutative diagram of AWBs with exact rows

$$\begin{array}{ccccccccc}
 & & (\mathsf{M} \boxtimes \mathsf{A}) \rtimes (\mathsf{A} \boxtimes \mathsf{M}) & \xrightarrow{\sigma} & \mathsf{A} \boxtimes \mathsf{A} & \xrightarrow{\tau} & \mathsf{A}/\mathsf{M} \boxtimes \mathsf{A}/\mathsf{M} & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi_{\mathsf{A}}} & & \downarrow{\scriptstyle \psi_{\mathsf{A}/\mathsf{M}}} & & \\
0 & \longrightarrow & \mathsf{M} & \longrightarrow & \mathsf{A} & \xrightarrow{\pi} & \mathsf{A}/\mathsf{M} & \longrightarrow & 0
\end{array}$$

where

$$\psi(m_1 \odot a_1, a_2 \odot m_2) = m_1a_1 + a_2m_2, \qquad \psi(m_1 \odot a_1, a_2 \circledast m_2) = m_1a_1 + [a_2,m_2],$$
$$\psi(m_1 \circledast a_1, a_2 \odot m_2) = [m_1,a_1] + a_2m_2, \qquad \psi(m_1 \circledast a_1, a_2 \circledast m_2) = [m_1,a_1] + [a_2,m_2].$$

The Snake lemma provides the exact sequence

$$\mathsf{Ker}(\psi) \to \mathsf{Ker}(\psi_{\mathsf{A}}) \to \mathsf{Ker}(\psi_{\mathsf{A/M}}) \to \mathsf{Coker}(\psi) \to 0.$$

By Theorem 4.8 $\mathsf{Ker}(\psi_{\mathsf{A}}) = \mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A})$, and since $\mathsf{A}\,/\,\mathsf{M}$ is a perfect AWB as well, we also have $\mathsf{Ker}(\psi_{\mathsf{A}\,/\,\mathsf{M}}) = \mathsf{H}_1^{\mathrm{AWB}}(\mathsf{A}\,/\,\mathsf{M})$. Obviously $\mathsf{Coker}(\psi) = \frac{\mathsf{M}}{[[\mathsf{A},\mathsf{M}]]}$. Then the fact that there is a surjective map $\mathsf{Ker}(\psi_{\mathsf{M}}) \to \mathsf{Ker}(\psi)$ completes the proof. □

4.1. Further investigation

As mentioned above, the category of AWBs is an example of a semi-abelian category. In the paper [11], it is explained how, in the context of a semi-abelian category, internal crossed squares can be used to set up an intrinsic approach to the non-abelian tensor product. Namely, it is shown that the non-abelian tensor products form the internal crossed squares in a semi-abelian category, in which the so-called "Smith is Huq" condition [11] is fulfilled. This condition is sufficient to construct the non-abelian tensor product of two objects acting compatibly on each other.

In future work, we plan to investigate the consistency of our constructions with the categorical definitions from [11] and ultimately prove the following:

Conjecture 4.12. *Let* M *and* N *be AWBs with mutually compatible actions on each other. The definition of the non-abelian tensor product* $\mathsf{M} \boxtimes \mathsf{N}$ *as in Definition 4.4 agrees with the categorical one (see [11, Definition 6.6]).*

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