



A non-abelian tensor product of algebras with bracket

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Abstract

We introduce and study a non-abelian tensor product of two algebras with bracket with compatible actions on each other. We investigate its applications to the universal central extensions and the low-dimensional homology of perfect algebras with bracket.

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1. Introduction

The non-abelian tensor product in various algebraic categories, such as groups, (Hom)-Lie and (Hom)-Leibniz algebras, Lie superalgebras. etc., plays an essential role in the study of homotopy theory, low-dimensional homology or universal central extensions (see [3, 7, 13–15, 17, 20, 21]). It has also been used in the construction of low-dimensional non-abelian homology of groups, Lie algebras and Leibniz algebras, having interesting applications to the algebraic K -theory, cyclic homology and Hochschild homology, respectively [16–18].

In this paper, we choose to develop a non-abelian tensor product for a relatively new algebraic structure called algebra with bracket, introduced in [8] as a kind of generalization of the (non-commutative) Poisson algebra. It should be noted here that such a generalization of Poisson algebras originates in physics literature (see, e.g. [23]). Among other results in [8], Quillen cohomology of algebras with bracket is described via an explicit cochain complex. Further (co)homological investigations of algebras with bracket are carried out in [4, 6]. In particular, for our importance, we mention that in [4], a homology with trivial coefficients of algebras with bracket is developed with applications to universal central extensions. In [6], crossed modules for algebras with bracket are introduced, and the second cohomology is interpreted as the set of equivalence classes of crossed extensions. The eight-term exact cohomology sequence is also constructed.

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In the present paper, we continue the same line of homological study of algebras with bracket. We fit the homology with trivial coefficients [4] into the context of Quillen homology, introduce the non-abelian tensor product of algebras with bracket by generators and relations, and give applications in universal central extensions and low-dimensional homology.

The organization of this paper is as follows: after the introduction, in Section 2, we present some definitions and results for the development of the paper. We briefly recall the construction of homology for algebras with bracket from [4, 8] and prove that it is consistent with the context of Quillen's homology theory (Theorem 2.6). In Section 3, we present all the ingredients for developing the non-abelian tensor product later. In particular, we define actions, semi-direct products and crossed modules of algebras with bracket. Additionally, we show that the category of crossed modules is equivalent to the category of cat^1 -algebras with bracket (Theorem 3.12). Section 4 contains the main results of the paper. Here, we present the construction of the non-abelian tensor product of two algebras with bracket acting compatibly on each other (Proposition 4.3) and study its properties. Regarding trivial actions, we describe the non-abelian tensor product (Proposition 4.5). We establish a right-exactness property of the non-abelian tensor product of algebras with bracket (Theorem 4.6) and equip the non-abelian tensor product with crossed module structures (Proposition 4.7). Finally, as an application, for a given perfect algebra with bracket, we construct its universal central extension (Theorem 4.8) and a four-term exact homology sequence (Theorem 4.11).

2. Algebras with bracket

Throughout the paper we fix a ground field $\mathbb{K}$. All vector spaces and algebras are $\mathbb{K}$ -vector spaces and $\mathbb{K}$ -algebras, and linear maps are $\mathbb{K}$ -linear maps as well. In what follows $\otimes$ means $\otimes_{\mathbb{K}}$.

2.1. Basic definitions

Definition 2.1 ([8]). An algebra with bracket, or an AWB for short, is an associative (not necessarily commutative) algebra A equipped with a bilinear map (bracket operation) $[-, -]: A \times A \rightarrow A$, $(a, b) \mapsto [a, b]$ satisfying the following identity:

$$[ab, c] = [a, c]b + a[b, c] \quad (2.1)$$

for all $a, b, c \in A$.

A *homomorphism* of AWBs is a homomorphism of associative algebras preserving the bracket operation. We denote by AWB the respective category of AWBs.

The category AWB is a variety of Ω -groups [19], and therefore it is a semi-abelian category [2, 22]: pointed, exact and protomodular with binary coproducts. So classical lemmas such as the Five Lemma [25] hold for AWBs which we will use later on. Below we give the definitions of ideal, center, commutator, action and semi-direct product of AWBs, and of course these notions agree with the corresponding general notions in the context of semi-abelian categories.

We now list some common examples of AWBs that will be discussed later. Other examples can be found in [5, 6, 8, 23].

Example 2.2.

- (i) Any vector space A enriched with the trivial multiplication and bracket operation, i.e. $ab = 0$ and $[a, b] = 0$ for all $a, b \in A$, is an AWB, called an abelian AWB. Hence, the category of vector spaces is a full subcategory of AWB and the respective inclusion functor $\text{Vect} \hookrightarrow \text{AWB}$ has a left adjoint, the so called abelianization functor $(-)^{\text{ab}}: \text{AWB} \rightarrow \text{Vect}$, which will be described in Remark 2.3 (i) below.

- (ii) Any associative algebra A together with the trivial bracket operation can be regarded as an AWB. This defines the inclusion functor $I: \mathbf{Ass} \rightarrow \mathbf{AWB}$, where $\mathbf{Ass}$ denotes the category of associative algebras. The functor I has a left adjoint $(-)^{\mathbf{ass}}: \mathbf{AWB} \rightarrow \mathbf{Ass}$ described in Remark 2.3 (ii) below.
- (iii) Another way of considering an associative algebra A as an AWB is to define the bracket operation by

$$[a, b] := ab - ba, \quad a, b \in A.$$

This particular AWB is called the *tautological* AWB associated to the associative algebra A and will be denoted by $T(A)$.

Tautological AWBs constitute a full subcategory of $\mathbf{AWB}$ denoted by $\mathbf{TAWB}$. The correspondence $T: \mathbf{Ass} \rightarrow \mathbf{TAWB}$, $A \mapsto T(A)$, is functorial, and it establishes an isomorphism between the categories $\mathbf{TAWB}$ and $\mathbf{Ass}$.

- (iv) Any Poisson algebra is an AWB. In fact, the category $\mathbf{Pois}$ of (non-commutative) Poisson algebras is a subcategory of $\mathbf{AWB}$. The inclusion functor $\mathbf{Pois} \hookrightarrow \mathbf{AWB}$ has as left adjoint the functor given by $A \mapsto A_{\mathbf{Pois}}$, where $A_{\mathbf{Pois}}$ is the maximal quotient of A , such that the following relations hold: $[a, a] \sim 0$ and $[a, [b, c]] + [b, [c, a]] + [c, [a, b]] \sim 0$.

The following notions for AWBs are given in [4] and they agree with the corresponding notions in semi-abelian categories. A *subalgebra* B of an AWB A is a vector subspace which is closed under the product and the bracket operation, that is, $B \cdot B \subseteq B$ and $[B, B] \subseteq B$. A subalgebra B is said to be a *left (respectively, right) ideal* if $A \cdot B \subseteq B$, $[A, B] \subseteq B$ (respectively, $B \cdot A \subseteq B$, $[B, A] \subseteq B$). If B is both left and right ideal, then it is said to be a *two-sided ideal*. In this case, the quotient A/B is endowed with an AWB structure naturally induced from the operations on A .

Let A be an AWB and B, C be two-sided ideals of A . We denote by $[[B, C]]$ the subspace of A spanned by all elements of the form $bc, cb, [b, c], [c, b]$, for all $b \in B$ and $c \in C$. It is easy to see that $[[B, C]]$ is a two-sided ideal of A called *commutator ideal* of B and C . Obviously $[[B, C]] \subseteq B \cap C$. In the particular case $B = C = A$, one obtains the definition of derived algebra $[[A, A]]$ of A . An AWB A is said to be *perfect* if $A = [[A, A]]$.

Remark 2.3.

- (i) Given an AWB A , the quotient $A/[[A, A]]$ is always an abelian AWB and will be denoted by $A^{\mathbf{ab}}$. Abelian AWBs (i.e. just vector spaces) are abelian group objects in the category $\mathbf{AWB}$. The respective abelianization functor $(-)^{\mathbf{ab}}: \mathbf{AWB} \rightarrow \mathbf{Vect}$, which is left adjoint to the inclusion functor $\mathbf{Vect} \hookrightarrow \mathbf{AWB}$, sends an AWB A into $A^{\mathbf{ab}} = A/[[A, A]]$.
- (ii) To an AWB A , we associate the associative algebra $A^{\mathbf{ass}}$ defined as the maximal quotient of A such that the relation $[a, a'] \sim 0$ holds, for $a, a' \in A$. This correspondence is functorial and satisfies the following universal property: given an associative algebra B , any homomorphism of AWBs $A \rightarrow I(B)$ factors through $A^{\mathbf{ass}}$, where $I: \mathbf{Ass} \hookrightarrow \mathbf{AWB}$ is the inclusion functor as in Example 2.2 (iii). Thus, the functor $(-)^{\mathbf{ass}}: \mathbf{AWB} \rightarrow \mathbf{Ass}$, $A \mapsto A^{\mathbf{ass}}$, is left adjoint to I .

Given an AWB A , the set

$$\mathcal{Z}(A) = \{a \in A \mid ab = 0 = ba, [a, b] = 0 = [b, a], \text{ for all } b \in A\}$$

is a two-sided ideal of A , and it is called *the center* of A . Note that an AWB A is abelian if and only if $A = \mathcal{Z}(A)$.

A *central extension* of an AWB A is an exact sequence of AWBs $0 \rightarrow M \rightarrow B \xrightarrow{\phi} A \rightarrow 0$ such that $[[M, B]] = 0$ (equivalently, $M \subseteq \mathcal{Z}(B)$). It is said to be *universal central extension*

if for every central extension $0 \rightarrow \mathbf{N} \rightarrow \mathbf{C} \xrightarrow{\psi} \mathbf{A} \rightarrow 0$ there is a unique homomorphism of AWBs $\alpha: \mathbf{B} \rightarrow \mathbf{C}$ such that $\psi \circ \alpha = \phi$.

The result immediately below is the analogue of classical results for universal central extensions in the categories of groups, Lie algebras, etc., and agrees with the similar result from [9] in the general framework of semi-abelian categories.

Theorem 2.4 ([4]). *An AWB $\mathbf{A}$ admits a universal central extension if and only if $\mathbf{A}$ is perfect. Moreover, the kernel of the universal central extension is isomorphic to the first homology of $\mathbf{A}$, $H_1^{\text{AWB}}(\mathbf{A})$ (see the definition below).*

2.2. Homology

In this subsection, we briefly review the homology of AWBs with trivial coefficients given in [4, 8].

Let V be a vector space. Let $R_1(V) = V$ and $R_n(V) = V^{\otimes n} \oplus V^{\otimes n}$, if $n \geq 2$. In order to distinguish elements from these tensor powers, we let $a_1 \otimes \cdots \otimes a_n$ be a typical element from the first component of $R_n(V)$, while $a_1 \circ \cdots \circ a_n$ from the second component of $R_n(V)$.

Given an AWB $\mathbf{A}$, we let $(C_*^{\text{AWB}}(\mathbf{A}), d_*)$ be the chain complex defined by

$$C_n^{\text{AWB}}(\mathbf{A}) := R_{n+1}(\mathbf{A}), \quad n \geq 0,$$

with the boundary maps $d_n: C_n^{\text{AWB}}(\mathbf{A}) \rightarrow C_{n-1}^{\text{AWB}}(\mathbf{A})$, $n \geq 0$, given by

$$d_n(a_1 \otimes \cdots \otimes a_{n+1}) = \sum_{i=1}^n (-1)^{i+1} a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_{n+1},$$

$$d_n(a_1 \circ \cdots \circ a_{n+1}) = \sum_{i=1}^n a_1 \otimes \cdots \otimes [a_i, a_{n+1}] \otimes \cdots \otimes a_n + \sum_{i=1}^{n-1} (-1)^i a_1 \circ \cdots \circ a_i a_{i+1} \circ \cdots \circ a_{n+1}$$

The homology of the complex $(C_*^{\text{AWB}}(\mathbf{A}), d_*)$ is called the *homology with trivial coefficients* of the AWB $\mathbf{A}$ and it is denoted by $H_*^{\text{AWB}}(\mathbf{A})$.

Easy computations show that there is an isomorphism

$$H_0^{\text{AWB}}(\mathbf{A}) \cong \mathbf{A}/[[\mathbf{A}, \mathbf{A}]].$$

On the other hand, given a free presentation of $\mathbf{A}$, that is, a short exact sequence of AWBs $0 \rightarrow \mathbf{R} \rightarrow \mathbf{F} \rightarrow \mathbf{A} \rightarrow 0$, where $\mathbf{F}$ is a free AWB, then there is an isomorphism

$$H_1^{\text{AWB}}(\mathbf{A}) \cong (\mathbf{R} \cap [[\mathbf{F}, \mathbf{F}]]) / [[\mathbf{R}, \mathbf{F}]]$$

(see [4, Corollary 2.14]).

Remark 2.5. Let $\mathbf{A}$ be an associative algebra and consider the ground field $\mathbb{K}$ as a trivial $\mathbf{A}$ -bimodule. Let $C_*^{\text{Hoch}}(\mathbf{A}) = C_*^{\text{Hoch}}(\mathbf{A}, \mathbb{K})$ and $\text{Hoch}_*(\mathbf{A}) = \text{Hoch}_*(\mathbf{A}, \mathbb{K})$ denote the Hochschild complex and the Hochschild homology of $\mathbf{A}$ with coefficients in $\mathbb{K}$ [27], respectively. Then $C_1^{\text{Hoch}}(\mathbf{A}) = \mathbf{A} = C_0^{\text{AWB}}(T(\mathbf{A}))$ and the natural injections

$$C_{n+1}^{\text{Hoch}}(\mathbf{A}) = \mathbf{A}^{\otimes(n+1)} \hookrightarrow \mathbf{A}^{\otimes(n+1)} \oplus \mathbf{A}^{\otimes(n+1)} = C_n^{\text{AWB}}(T(\mathbf{A})), \quad n \geq 1$$

gives rise to a morphism of chain complexes $C_{*+1}^{\text{Hoch}}(\mathbf{A}) \hookrightarrow C_*^{\text{AWB}}(T(\mathbf{A}))$. Thus, we have an induced homomorphism in homology $\text{Hoch}_{n+1}(\mathbf{A}) \rightarrow H_n^{\text{AWB}}(T(\mathbf{A}))$ ($n \geq 0$), which is clearly an epimorphism for $n = 1$ and an isomorphism for $n = 0$.

Now, we show that the homology of AWBs is fitted in the context of homology theory developed by Quillen in a very general framework [26] (see also the earlier work by Barr and Beck [1]). Let us recall that the Quillen homology of an object in an algebraic category $\mathcal{C}$ is defined via the derived functors of the abelianization functor $(-)^{\text{ab}}: \mathcal{C} \rightarrow \mathcal{C}^{\text{ab}}$ from $\mathcal{C}$

to the abelian category $\mathcal{C}^{\text{ab}}$ of abelian group objects in $\mathcal{C}$. To specify this theory for AWBs, we proceed as follows.

Given an AWB A , choose any free simplicial resolution F_* of A , that is, an aspherical augmented simplicial AWB $F_* \xrightarrow{\epsilon} A$ (which means that all non-zero homotopies are trivial, $\pi_n(F_*) = 0$ for $n \geq 0$, and ϵ induces an isomorphism $\pi_0(F_*) \cong A$) such that each component F_n , $n \geq 0$, is a free AWB. Then the n -th Quillen homology of A is defined by

$$H_n^Q(A) = H_n(F_*^{\text{ab}}), \quad n \geq 0.$$

Here F_*^{ab} is the simplicial vector space obtained by applying the functor $(-)^{\text{ab}}$ dimension-wise to F_* .

In the proof of the theorem immediately below, we need to use the result from [8] that if F is a free AWB, then the homology of the complex $(C_*^{\text{AWB}}(F), d_*)$ vanishes in positive dimensions, that is,

$$H_n^{\text{AWB}}(F) = 0, \quad \text{for } n \geq 1.$$

Theorem 2.6. *Let A be an AWB. Then there is an isomorphism of vector spaces*

$$H_n^{\text{AWB}}(A) \cong H_n^Q(A), \quad n \geq 0.$$

Proof. First of all let us note that the homology chain complex C_*^{AWB} is functorial in the sense that a homomorphism $A \rightarrow A'$ gives rise the chain map $C_*^{\text{AWB}}(A) \rightarrow C_*^{\text{AWB}}(A')$ in the canonical way.

Now, given a free simplicial resolution F_* of A , by applying the functor C_n^{AWB} dimension-wise, and then taking the alternating sums of face homomorphisms, we get an augmented chain complex of vector spaces $C_n^{\text{AWB}}(F_*) \rightarrow C_n^{\text{AWB}}(A)$. Since $F_* \rightarrow A$ is an aspherical simplicial AWB, we claim that $C_n^{\text{AWB}}(F_*) \rightarrow C_n^{\text{AWB}}(A)$ is acyclic chain complex for any $n \geq 0$. This is easy to confirm, since by forgetting AWB structure in the simplicial AWB $F_* \rightarrow A$, we get a simplicial vector space having a linear left (right) contraction.

Then using the facts that $H_n^{\text{AWB}}(F_m) = 0$ and $H_0^{\text{AWB}}(F_m) = F_m^{\text{ab}}$ for any $n \geq 1$ and $m \geq 0$ it follows that both spectral sequences for the bicomplex $C_*^{\text{AWB}}(F_*)$ degenerate and give the required isomorphism. $\square$

3. Crossed modules of AWBs

3.1. Actions and semi-direct product

Definition 3.1. Let A and M be two AWBs. An action of A on M consists of four bilinear maps

$$\begin{aligned} A \times M &\rightarrow M, & (a, m) &\mapsto {}^a m, & M \times A &\rightarrow M, & (m, a) &\mapsto m^{\cdot a}, \\ A \times M &\rightarrow M, & (a, m) &\mapsto {}^{a*} m, & M \times A &\rightarrow M, & (m, a) &\mapsto m^{*a}, \end{aligned}$$

such that the following conditions hold:

$$\begin{aligned} (a_1 a_2)^{\cdot} m &= a_1^{\cdot} (a_2^{\cdot} m), & (a_1^{\cdot} m)^{*a_2} &= a_1^{\cdot} (m^{*a_2}) + [a_1, a_2]^{\cdot} m, \\ m^{\cdot (a_1 a_2)} &= (m^{\cdot a_1})^{\cdot a_2}, & (m^{\cdot a_1})^{*a_2} &= (m^{*a_2})^{\cdot a_1} + m^{\cdot [a_1, a_2]}, \\ (a_1^{\cdot} m)^{\cdot a_2} &= a_1^{\cdot} (m^{\cdot a_2}), & (a_1 a_2)^{*} m &= a_1^{\cdot} (a_2^* m) + (a_1^* m)^{\cdot a_2}, \\ (m_1 m_2)^{\cdot a} &= m_1^{\cdot} (m_2^{\cdot a}), & [m_1^{\cdot a}, m_2] &= m_1^{\cdot} (m_2^{\cdot a}) + [m_1, m_2]^{\cdot a}, \\ {}^a (m_1 m_2) &= ({}^a m_1) m_2, & [{}^a m_1, m_2] &= {}^a [m_1, m_2] + ({}^{a*} m_2) m_1, \\ (m_1^{\cdot a}) m_2 &= m_1^{\cdot} (m_2^{\cdot a}), & (m_1 m_2)^{*a} &= m_1^{\cdot} (m_2^{*a}) + (m_1^{*a}) m_2, \end{aligned} \tag{3.1}$$

for all $a, a_1, a_2 \in A$, $m, m_1, m_2 \in M$. The action is called trivial if all these bilinear maps are trivial, i.e. ${}^a m = m^{\cdot a} = {}^{a*} m = m^{*a} = 0$, for all $a \in A$ and $m \in M$.

Let us remark that if an action of an AWB A on an abelian AWB M is given, then all six equations in the last three lines of (3.1) vanish. Among the remaining six equations, the first three equations in the first column say that M is a bimodule over A , and then the first three equations in the second column say that we get the definition of a *representation* M of A (see [8]).

Example 3.2.

- (i) If M is a representation of an AWB A thought as an abelian AWB, then there is an action of A on the abelian AWB M .
- (ii) If A is a subalgebra of some AWB B (maybe $A = B$) and if M is a two-sided ideal of B , then the operations in B yield an action of A on M given by ${}^a m = am$, $m^a = ma$, ${}^{a*} m = [a, m]$, $m^{*a} = [m, a]$, for all $m \in M$ and $a \in A$.
- (iii) If $0 \rightarrow M \xrightarrow{i} B \xrightarrow{\pi} A \rightarrow 0$ is a split short exact sequence of AWBs, that is, there exists a homomorphism $s: A \rightarrow B$ of AWBs such that $\pi \circ s = \text{Id}_A$, then there is an action of A on M , given by:

$$\begin{aligned} {}^a m &= i^{-1}(s(a)i(m)), & m^a &= i^{-1}(i(m)s(a)), \\ {}^{a*} m &= i^{-1}([s(a), i(m)]), & m^{*a} &= i^{-1}([i(m), s(a)]), \end{aligned}$$

for any $a \in A$, $m \in M$.

- (iv) Any homomorphism of AWBs $f: A \rightarrow M$ induces an action of A on M in the standard way by taking images of elements of A and operations in M , i.e. ${}^a m = f(a)m$, $m^a = mf(a)$, ${}^{a*} m = [f(a), m]$ and $m^{*a} = [m, f(a)]$, for $m \in M$, $a \in A$.
- (v) If $\mu: M \rightarrow A$ is a surjective homomorphism of AWBs and the kernel of μ is contained in the center of M , i.e. $\text{Ker}(\mu) \subseteq \mathcal{Z}(M)$, then there is an action of A on M , defined in the standard way, i.e. by choosing pre-images of elements of A and taking operations in M .

Definition 3.3. Let A and M be AWBs with an action of A on M . The semi-direct product of M and A , denoted by $M \rtimes A$, is the AWB whose underlying vector space is $M \oplus A$ endowed with the operations

$$\begin{aligned} (m_1, a_1)(m_2, a_2) &= (m_1 m_2 + {}^{a_1} m_2 + m_1 {}^{a_2}, a_1 a_2), \\ [(m_1, a_1), (m_2, a_2)] &= ([m_1, m_2] + {}^{a_1} m_2 + m_1 {}^{a_2}, [a_1, a_2]) \end{aligned}$$

for all $m_1, m_2 \in M$, $a_1, a_2 \in A$.

Given an action of an AWB A on M , straightforward calculations show that the sequence of AWBs

$$0 \longrightarrow M \xrightarrow{i} M \rtimes A \xrightarrow{\pi} A \longrightarrow 0$$

where $i(m) = (m, 0)$, $\pi(m, a) = a$, is exact. Moreover M is a two-sided ideal of $M \rtimes A$ and this sequence splits by $s: A \rightarrow M \rtimes A$, $s(a) = (0, a)$. Then, as in Example 3.2 (iii), the above sequence induces another action of A on M given by

$$\begin{aligned} {}^a m &= i^{-1}((0, a)(m, 0)), & m^a &= i^{-1}((m, 0)(0, a)), \\ {}^{a*} m &= i^{-1}([0, a], (m, 0)], & m^{*a} &= i^{-1}([m, 0], (0, a)], \end{aligned}$$

which actually matches the given one.

3.2. Crossed modules

Definition 3.4. A crossed module of AWBs is a homomorphism of AWBs $\mu: M \rightarrow A$ together with an action of A on M such that the following identities hold:

(CM1)

$$\begin{aligned}\mu(m \cdot^a) &= \mu(m)a, & \mu({}^a m) &= a\mu(m), \\ \mu(m^{*a}) &= [\mu(m), a], & \mu({}^{a*}m) &= [a, \mu(m)];\end{aligned}$$

(CM2)

$$\begin{aligned}\mu(m) \cdot m' &= mm' = m \cdot \mu(m'), \\ \mu(m)^* m' &= [m, m'] = m^* \mu(m')\end{aligned}$$

for all $m, m' \in M$, $a \in A$.

Definition 3.5. A morphism of crossed modules $(M \xrightarrow{\mu} A) \rightarrow (M' \xrightarrow{\mu'} A')$ is a pair (α, β) , where $\alpha: M \rightarrow M'$ and $\beta: A \rightarrow A'$ are homomorphisms of AWBs satisfying:

- (a) $\beta \circ \mu = \mu' \circ \alpha$.
- (b)

$$\begin{aligned}\alpha({}^a m) &= \beta(a) \cdot \alpha(m), & \alpha(m \cdot^a) &= \alpha(m) \cdot \beta(a), \\ \alpha({}^{a*}m) &= \beta(a)^* \alpha(m), & \alpha(m^{*a}) &= \alpha(m)^* \beta(a)\end{aligned}$$

for all $a \in A$, $m \in M$.

It is clear that crossed modules of AWBs constitute a category, denoted by $\mathbf{XAWB}$.

The following lemma is an easy consequence of Definition 3.4.

Lemma 3.6. Let $\mu: M \rightarrow A$ be a crossed module of AWBs. Then the following statements are satisfied:

- (i) $\text{Ker}(\mu) \subseteq \mathcal{Z}(M)$.
- (ii) $\text{Im}(\mu)$ is a two-sided ideal of A .
- (iii) $\text{Im}(\mu)$ acts trivially on $\mathcal{Z}(M)$, and so trivially on $\text{Ker}(\mu)$. Hence $\text{Ker}(\mu)$ inherits an action of $A/\text{Im}(\mu)$ making $\text{Ker}(\mu)$ a representation of the AWB $A/\text{Im}(\mu)$.

Example 3.7.

- (i) Let A be an AWB and B be a two-sided ideal of A , then the inclusion $B \hookrightarrow A$ is a crossed module, where the action of A on B is given by the operations in A (see Example 3.2 (ii)). Conversely, if $\mu: B \rightarrow A$ is a crossed module of AWBs and μ is an injective map, then B is isomorphic to a two-sided ideal of A by Lemma 3.6 (ii).
- (ii) For any representation M of an AWB A , the trivial map $0: M \rightarrow A$ is a crossed module with the action of A on the abelian AWB M described in Example 3.2 (i). Conversely, if $0: M \rightarrow A$ is a crossed module of AWBs, then M is necessarily an abelian AWB and the action of A on M is equivalent to M being a representation of A .
- (iii) Any homomorphism of AWBs $\mu: M \rightarrow A$, with M abelian and $\text{Im}(\mu) \subseteq \mathcal{Z}(A)$, provides a crossed module with A acting trivially on M .
- (iv) If $0 \rightarrow N \rightarrow M \xrightarrow{\mu} A \rightarrow 0$ is a central extension of AWBs, then μ is a crossed module with the induced action of A on M (see Example 3.2 (v)).

Proposition 3.8. Let $\mu: M \rightarrow A$ be a crossed module of AWBs. Then the maps

- (i) $(\mu, \text{Id}_A): M \rtimes A \rightarrow A \rtimes A$,
- (ii) $(\text{Id}_M, \mu): M \rtimes M \rightarrow M \rtimes A$,
- (iii) $\varphi: M \rtimes A \rightarrow M \rtimes A$ given by $\varphi(m, a) = (-m, \mu(m) + a)$,

are homomorphisms of AWBs.

Proof. (i) is a direct consequence of equalities in (CM1) of Definition 3.4, (ii) follows from equalities in (CM2), whilst (iii) requires both (CM1) and (CM2). $\square$

Remark 3.9. The functors I and T given in Example 2.2 (ii) and (iii) preserve actions and crossed modules in the sense of the following assertions:

- (i) Any action of an associative algebra A on another associative algebra M , $A \times M \rightarrow M$, $(a, m) \mapsto a \cdot m$ and $M \times A \rightarrow M$, $(m, a) \mapsto m \cdot a$ (see [10, 12]) defines an action of the AWB $I(A)$ on $I(M)$ (resp. of $T(A)$ on $T(M)$), by letting

$${}^a \cdot m = a \cdot m, \quad m \cdot^a = m \cdot a, \quad {}^{a*} m = 0, \quad m^{*a} = 0,$$

$$(\text{resp. } {}^a \cdot m = a \cdot m, \quad m \cdot^a = m \cdot a, \quad {}^{a*} m = a \cdot m - m \cdot a, \quad m^{*a} = m \cdot a - a \cdot m)$$

for all $a \in A$ and $m \in M$.

- (ii) If $\mu: M \rightarrow A$ is a crossed module of associative algebras (see again [10, 12]), then the homomorphisms of AWBs $I(\mu): I(M) \rightarrow I(A)$ and $T(\mu): T(M) \rightarrow T(A)$, together with the actions of $I(A)$ on $I(M)$ and of $T(A)$ on $T(M)$, are crossed modules of AWBs.

In [6] we proved equivalence of crossed modules of AWBs with internal categories in the category of AWBs. Now we show their equivalence with cat^1 -AWBs. The following definition of cat^1 -AWB is given in complete analogy with Loday's original notion of cat^1 -groups [24].

Definition 3.10. A cat^1 -AWB (R, P, s, t) consists of an AWB R , together with a subalgebra P and two homomorphisms $s, t: R \rightarrow P$ of AWBs satisfying the following conditions:

- (a) $s|_P = t|_P = \text{Id}_P$.
- (b) $\text{Ker}(s) \text{Ker}(t) = 0 = \text{Ker}(t) \text{Ker}(s)$.
- (c) $[\text{Ker}(s), \text{Ker}(t)] = 0 = [\text{Ker}(t), \text{Ker}(s)]$.

Definition 3.11. A morphism of cat^1 -AWBs $(R, P, s, t) \rightarrow (R', P', s', t')$ is a homomorphism of AWBs $f: R \rightarrow R'$ such that $f(P) \subseteq P'$ and $s' \circ f = f|_P \circ s$, $t' \circ f = f|_P \circ t$.

We let $\text{cat}^1\text{-AWB}$ denote the category of cat^1 -AWBs. Then we have the following theorem.

Theorem 3.12. *The categories $\text{cat}^1\text{-AWB}$ and XAWB are equivalent.*

Proof. To a given cat^1 -AWB (R, P, s, t) we associate a crossed module $\mu = t|_M: M \rightarrow P$, where $M = \text{Ker}(s)$ and the action of P on M is given by the operations in R (see Example 3.2 (ii)). It is easy to see that $\mu: M \rightarrow P$ is a crossed module of AWBs and the assignment defines a functor $\Phi: \text{cat}^1\text{-AWB} \rightarrow \text{XAWB}$.

Conversely, let $\mu: M \rightarrow P$ be a crossed module of AWBs, then the associated cat^1 -AWB is given by $s, t: M \rtimes P \rightarrow P$, where $s(m, p) = p$, $t(m, p) = \mu(m) + p$, $m \in M$, $p \in P$. It is straightforward to see that this assignment is functorial and provides a quasi-inverse functor for Φ . $\square$

4. Non-abelian tensor product of AWBs

Definition 4.1. Let M and N be AWBs with mutual actions on each other. The actions are said to be compatible if

$$\begin{aligned} m^{(m' \cdot n')} &= m(m' \cdot n'), & m^{(n' \cdot m')} &= m(n' \cdot m'), \\ m^{(m' * n')} &= m(m' * n'), & m^{(n' * m')} &= m(n' * m'), \\ m^{*(m' \cdot n')} &= [m, m' \cdot n'], & m^{*(n' \cdot m')} &= [m, n' \cdot m'], \\ m^{*(m' * n')} &= [m, m' * n'], & m^{*(n' * m')} &= [m, n' * m'], \\ (m^n) \cdot n' &= (m \cdot n)n', & (n^m) \cdot n' &= (n \cdot m)n', \\ (m^{*n}) \cdot n' &= (m * n)n', & (n^{*m}) \cdot n' &= (n * m)n', \end{aligned} \tag{4.1}$$

$$\begin{aligned}
(m^{\cdot n})^* n' &= [m^{\cdot} n, n'], & (n^{\cdot m})^* n' &= [n^{\cdot} m, n'], \\
(m^{*n})^* n' &= [m^* n, n'], & (n^{*m})^* n' &= [n^{*m}, n'],
\end{aligned}$$

and moreover, another 16 equations obtained by exchanging the roles of elements of M and N in (4.1) are also valid.

Example 4.2.

- (a) If M and N are two-sided ideals of an AWB A , then the mutual actions on each other considered in Example 3.2 (ii) are compatible.
- (b) Let $\mu: M \rightarrow P$ and $\nu: N \rightarrow P$ be two crossed modules of AWBs. Then the mutual actions of M on N via μ and of N on M via ν are compatible.

Let M and N be AWBs with mutually compatible actions on each other. We denote by $M \odot N$ the vector space spanned by all symbols $m \odot n$, $n \odot m$ and by $M \otimes N$ the vector space spanned by all symbols $m \otimes n$, $n \otimes m$, for $m \in M$, $n \in N$. Let $M \boxtimes N$ denotes the quotient of $(M \odot N) \oplus (M \otimes N)$ by the following relations:

$$\begin{aligned}
\lambda(m \star n) &= (\lambda m) \star n = m \star (\lambda n), \\
(m + m') \star n &= m \star n + m' \star n, & m \star (n + n') &= m \star n + m \star n', \\
m^{\cdot n} \star m'^{\cdot} n' &= m^{\cdot} n \star m'^{\cdot} n', & m^{\cdot n} \star n'^{\cdot} m' &= m^{\cdot} n \star n'^{\cdot} m', \\
n^{\cdot} m \star n'^{\cdot} m' &= n^{\cdot} m \star n'^{\cdot} m', & n^{\cdot} m \star m'^{\cdot} n' &= n^{\cdot} m \star m'^{\cdot} n', \\
m^{\cdot n} \star m'^* n' &= m^{\cdot} n \star m'^* n', & m^{\cdot n} \star n'^* m' &= m^{\cdot} n \star n'^* m', \\
n^{\cdot} m \star m'^* n' &= n^{\cdot} m \star m'^* n', & n^{\cdot} m \star n'^* m' &= n^{\cdot} m \star n'^* m', \\
m^{*n} \star m'^{\cdot} n' &= m^{*n} \star m'^{\cdot} n', & m^{*n} \star n'^{\cdot} m' &= m^{*n} \star n'^{\cdot} m', \\
n^{*m} \star m'^{\cdot} n' &= n^{*m} \star m'^{\cdot} n', & n^{*m} \star n'^{\cdot} m' &= n^{*m} \star n'^{\cdot} m', \\
m^{*n} \star m'^* n' &= m^{*n} \star m'^* n', & m^{*n} \star n'^* m' &= m^{*n} \star n'^* m', \\
n^{*m} \star m'^* n' &= n^{*m} \star m'^* n', & n^{*m} \star n'^* m' &= n^{*m} \star n'^* m', \\
(m_1 m_2) \odot n &= m_1 \odot (m_2^{\cdot} n), & n \odot (m_1 m_2) &= (n^{\cdot} m_1) \odot m_2, \\
(m_1 m_2) \otimes n &= m_1 \otimes (m_2^{*n}) + (m_1^{*n}) \odot m_2, \\
m_1^{\cdot} n \otimes m_2 &= m_1 \odot n^{*m_2} + [m_1, m_2] \odot n, \\
n^{\cdot} m_1 \otimes m_2 &= n^{*m_2} \odot m_1 + n \odot [m_1, m_2], \\
m_1^{\cdot} n \odot m_2 &= m_1 \odot n^{*m_2},
\end{aligned} \tag{4.2}$$

and another 25 relations obtained by exchanging the roles of elements of M and N in (4.2), where the symbol $\star$ stands for either $\odot$ or $\otimes$.

Proposition 4.3. *The vector space $M \boxtimes N$ endowed with the product and bracket operations given on the generators by*

$$\begin{aligned}
(m \odot n)(m' \odot n') &= (m^{\cdot n}) \odot (m'^{\cdot} n'), & (m \odot n)(n' \odot m') &= (m^{\cdot n}) \odot (n'^{\cdot} m'), \\
(n \odot m)(m' \odot n') &= (n^{\cdot} m) \odot (m'^{\cdot} n'), & (n \odot m)(n' \odot m') &= (n^{\cdot} m) \odot (n'^{\cdot} m'), \\
(m \odot n)(m' \otimes n') &= (m^{\cdot n}) \odot (m'^* n'), & (m \odot n)(n' \otimes m') &= (m^{\cdot n}) \odot (n'^* m'), \\
(n \odot m)(m' \otimes n') &= (n^{\cdot} m) \odot (m'^* n'), & (n \odot m)(n' \otimes m') &= (n^{\cdot} m) \odot (n'^* m'), \\
(m \otimes n)(m' \odot n') &= (m^{*n}) \odot (m'^{\cdot} n'), & (m \otimes n)(n' \odot m') &= (m^{*n}) \odot (n'^{\cdot} m'), \\
(n \otimes m)(m' \odot n') &= (n^{*m}) \odot (m'^{\cdot} n'), & (n \otimes m)(n' \odot m') &= (n^{*m}) \odot (n'^{\cdot} m'), \\
(m \otimes n)(m' \otimes n') &= (m^{*n}) \odot (m'^* n'), & (m \otimes n)(n' \otimes m') &= (m^{*n}) \odot (n'^* m'),
\end{aligned}$$

$$\begin{aligned}
(n \otimes m)(m' \otimes n') &= (n^* m) \odot (m'^* n'), & (n \otimes m)(n' \otimes m') &= (n^* m) \odot (n'^* m'), \\
[m \odot n, m' \odot n'] &= (m^{\cdot n}) \otimes (m'^{\cdot n'}), & [m \odot n, n' \odot m'] &= (m^{\cdot n}) \otimes (n'^{\cdot m'}), \\
[n \odot m, m' \odot n'] &= (n^{\cdot m}) \otimes (m'^{\cdot n'}), & [n \odot m, n' \odot m'] &= (n^{\cdot m}) \otimes (n'^{\cdot m'}), \\
[m \odot n, m' \otimes n'] &= (m^{\cdot n}) \otimes (m'^* n'), & [m \odot n, n' \otimes m'] &= (m^{\cdot n}) \otimes (n'^* m'), \\
[n \odot m, m' \otimes n'] &= (n^{\cdot m}) \otimes (m'^* n'), & [n \odot m, n' \otimes m'] &= (n^{\cdot m}) \otimes (n'^* m'), \\
[m \otimes n, m' \odot n'] &= (m^* n) \otimes (m'^{\cdot n'}), & [m \otimes n, n' \odot m'] &= (m^* n) \otimes (n'^{\cdot m'}), \\
[n \otimes m, m' \odot n'] &= (n^* m) \otimes (m'^{\cdot n'}), & [n \otimes m, n' \odot m'] &= (n^* m) \otimes (n'^{\cdot m'}), \\
[m \otimes n, m' \otimes n'] &= (m^* n) \otimes (m'^* n'), & [m \otimes n, n' \otimes m'] &= (m^* n) \otimes (n'^* m'), \\
[n \otimes m, m' \otimes n'] &= (n^* m) \otimes (m'^* n'), & [n \otimes m, n' \otimes m'] &= (n^* m) \otimes (n'^* m'),
\end{aligned}$$

has the structure of an AWB.

Proof. Straightforward calculations show that, under the conditions of compatible actions (4.1), by using the relations in (4.2), the described operations on $M \boxtimes N$ satisfy the fundamental identity (2.1). $\square$

Definition 4.4. The structure of AWB on $M \boxtimes N$ provided by Proposition 4.3 is called the non-abelian tensor product of the AWBs M and N .

In particular, if the actions are trivial, the non-abelian tensor product can be described as follows.

Proposition 4.5. If M and N are two AWBs with trivial actions on each other, then there is an isomorphism of abelian AWBs

$$M \boxtimes N \cong (M^{ab} \otimes_{\mathbb{K}} N^{ab}) \oplus (N^{ab} \otimes_{\mathbb{K}} M^{ab}) \oplus (M^{ab} \otimes_{\mathbb{K}} N^{ab}) \oplus (N^{ab} \otimes_{\mathbb{K}} M^{ab}).$$

Proof. Equations in Proposition 4.3 show us easily that $M \boxtimes N$ is abelian in the case of trivial actions. The defining relations (4.2) of the non-abelian tensor product say that the vector space $M \boxtimes N$ is the quotient of $(M \otimes_{\mathbb{K}} N) \oplus (N \otimes_{\mathbb{K}} M) \oplus (M \otimes_{\mathbb{K}} N) \oplus (N \otimes_{\mathbb{K}} M)$ by the relations

$$\begin{aligned}
0 &= (m_1 m_2) \otimes n = [m_1, m_2] \otimes n \\
&= n \otimes (m_1 m_2) = n \otimes [m_1, m_2] \\
&= m \otimes (n_1 n_2) = m \otimes [n_1, n_2] \\
&= (n_1 n_2) \otimes m = [n_1, n_2] \otimes m
\end{aligned}$$

for all $m, m_1, m_2 \in M, n, n_1, n_2 \in N$. This provides the required isomorphism. $\square$

The non-abelian tensor product of AWBs is functorial in the following sense: let $f: M \rightarrow M'$ and $g: N \rightarrow N'$ be homomorphisms of AWBs together with mutually compatible actions of M and N , also M' and N' on each other such that f, g preserve these actions, i.e.

$$\begin{aligned}
f(n^{\cdot m}) &= g(n)^{\cdot} f(m), \quad f(m^{\cdot n}) = f(m)^{\cdot} g(n), \quad f(n^* m) = g(n)^* f(m), \quad f(n^{\cdot m}) = g(n)^{\cdot} f(m), \\
g(m^{\cdot n}) &= f(m)^{\cdot} g(n), \quad g(n^{\cdot m}) = g(n)^{\cdot} f(m), \quad g(m^* n) = f(m)^* g(n), \quad g(m^{\cdot n}) = f(m)^{\cdot} g(n).
\end{aligned}$$

for all $m \in M, n \in N$, then there is a homomorphism of AWBs

$$f \boxtimes g: M \boxtimes N \rightarrow M' \boxtimes N'$$

defined by

$$\begin{aligned}
(f \boxtimes g)(m \odot n) &= f(m) \odot g(n), & (f \boxtimes g)(n \odot m) &= g(n) \odot f(m), \\
(f \boxtimes g)(m \otimes n) &= f(m) \otimes g(n), & (f \boxtimes g)(n \otimes m) &= g(n) \otimes f(m).
\end{aligned}$$

The non-abelian tensor product of AWBs has a kind of right-exactness property presented in the following theorem.

Theorem 4.6. *Let $0 \rightarrow M_1 \xrightarrow{f} M_2 \xrightarrow{g} M_3 \rightarrow 0$ be a short exact sequence of AWBs. Let N be an AWB together with compatible actions of N and M_i ($i = 1, 2, 3$) on each other and f, g preserve these actions. Then there is an exact sequence of AWBs*

$$M_1 \boxtimes N \xrightarrow{f \boxtimes \text{Id}_N} M_2 \boxtimes N \xrightarrow{g \boxtimes \text{Id}_N} M_3 \boxtimes N \rightarrow 0.$$

Proof. It is clear that the composition $(g \boxtimes \text{Id}_N)(f \boxtimes \text{Id}_N)$ is the trivial map, i.e. $\text{Im}(f \boxtimes \text{Id}_N) \subseteq \text{Ker}(g \boxtimes \text{Id}_N)$ and at the same time $f \boxtimes \text{Id}_N$ is an epimorphism.

$\text{Im}(f \boxtimes \text{Id}_N)$ is generated by the elements of the form $f(m_1) \odot n, n \odot f(m_1), f(m_1) \otimes n$ and $n \otimes f(m_1)$, for all $m_1 \in M_1, n \in N$. Since f preserves actions of N , by the relations given in Proposition 4.3, it is easily verified that $\text{Im}(f \boxtimes \text{Id}_N)$ is a two-sided ideal of $M_2 \boxtimes N$. For instance, taking a generator of the form $m_2 \odot n'$ in $M_2 \boxtimes N$ we have

$$\begin{aligned} (f(m_1) \odot n)(m_2 \odot n') &= f(m_1)^{\cdot n} \odot {}^{m_2}n' = f(m_1^{\cdot n}) \odot {}^{m_2}n' \in \text{Im}(f \boxtimes \text{Id}_N), \\ (f(m_1) \otimes n)(m_2 \odot n') &= f(m_1)^{*n} \odot {}^{m_2}n' = f(m_1^{*n}) \odot {}^{m_2}n' \in \text{Im}(f \boxtimes \text{Id}_N). \end{aligned}$$

Then, there is a homomorphism of AWBs

$$\alpha: (M_2 \boxtimes N) / \text{Im}(f \boxtimes \text{Id}_N) \longrightarrow (M_3 \boxtimes N)$$

induced by $g \boxtimes \text{Id}_N$, that is, defined on generators by

$$\begin{aligned} \alpha(\overline{m_2 \odot n}) &= g(m_2) \odot n, & \alpha(\overline{n \odot m_2}) &= n \odot g(m_2), \\ \alpha(\overline{m_2 \otimes n}) &= g(m_2) \otimes n, & \alpha(\overline{n \otimes m_2}) &= n \otimes g(m_2). \end{aligned}$$

where the overdrawn generator denotes the coset of the corresponding element. On the other hand, we have well-defined homomorphism of AWBs

$$\alpha': (M_3 \boxtimes N) \longrightarrow (M_2 \boxtimes N) / \text{Im}(f \boxtimes \text{Id}_N)$$

given on generators by

$$\begin{aligned} \alpha'(m_3 \odot n) &= \overline{m_2 \odot n}, & \alpha'(\overline{n \odot m_3}) &= \overline{n \odot m_2}, \\ \alpha'(m_3 \otimes n) &= \overline{m_2 \otimes n}, & \alpha'(\overline{n \otimes m_3}) &= \overline{n \otimes m_2}, \end{aligned}$$

where $m_2 \in M_2$ is any element such that $g(m_2) = m_3$. Obviously α and α' are inverse to each other, i.e. α is an isomorphism. Then the required exactness follows. $\square$

Proposition 4.7. *Let M and N be AWBs with compatible actions on each other.*

(a) *There are homomorphisms of AWBs*

$$\begin{aligned} \psi_M: M \boxtimes N &\rightarrow M \text{ given by, } \psi_M(m \odot n) = m^{\cdot n}, \psi_M(n \odot m) = {}^n m, \\ &\psi_M(m \otimes n) = m^{*n}, \psi_M(n \otimes m) = {}^{n*} m; \\ \text{and } \psi_N: M \boxtimes N &\rightarrow N \text{ given by, } \psi_N(m \odot n) = {}^m n, \psi_N(n \odot m) = n^{\cdot m}, \\ &\psi_N(m \otimes n) = {}^{m*} n, \psi_N(n \otimes m) = n^{*m}. \end{aligned}$$

- (b) *There are actions of $\mathbf{M}$ and $\mathbf{N}$ on the non-abelian tensor product $\mathbf{M} \boxtimes \mathbf{N}$ given, for all $m, m' \in \mathbf{M}, n, n' \in \mathbf{N}$, by*

$$\begin{aligned}
 m \cdot (m' \odot n') &= m \odot (m' \cdot n'), & m \cdot (n' \odot m') &= m \odot (n' \cdot m'), \\
 (m' \odot n') \cdot m &= (m' \cdot n') \odot m, & (n' \odot m') \cdot m &= (n' \cdot m') \odot m, \\
 m^* (m' \odot n') &= m \otimes (m' \cdot n'), & m^* (n' \odot m') &= m \otimes (n' \cdot m'), \\
 (m' \odot n')^* m &= (m' \cdot n') \otimes m, & (n' \odot m')^* m &= (n' \cdot m') \otimes m, \\
 m \cdot (m' \otimes n') &= m \odot (m' \cdot n'), & m \cdot (n' \otimes m') &= m \odot (n' \cdot m'), \\
 (m' \otimes n') \cdot m &= (m' \cdot n') \odot m, & (n' \otimes m') \cdot m &= (n' \cdot m') \odot m, \\
 m^* (m' \otimes n') &= m \otimes (m' \cdot n'), & m^* (n' \otimes m') &= m \otimes (n' \cdot m'), \\
 (m' \otimes n')^* m &= (m' \cdot n') \otimes m, & (n' \otimes m')^* m &= (n' \cdot m') \otimes m,
 \end{aligned}$$

and

$$\begin{aligned}
 n \cdot (m' \odot n') &= n \odot (m' \cdot n'), & n \cdot (n' \odot m') &= n \odot (n' \cdot m'), \\
 (m' \odot n') \cdot n &= (m' \cdot n') \odot n, & (n' \odot m') \cdot n &= (n' \cdot m') \odot n, \\
 n^* (m' \odot n') &= n \otimes (m' \cdot n'), & n^* (n' \odot m') &= n \otimes (n' \cdot m'), \\
 (m' \odot n')^* n &= (m' \cdot n') \otimes n, & (n' \odot m')^* n &= (n' \cdot m') \otimes n, \\
 n \cdot (m' \otimes n') &= n \odot (m' \cdot n'), & n \cdot (n' \otimes m') &= n \odot (n' \cdot m'), \\
 (m' \otimes n') \cdot n &= (m' \cdot n') \odot n, & (n' \otimes m') \cdot n &= (n' \cdot m') \odot n, \\
 n^* (m' \otimes n') &= n \otimes (m' \cdot n'), & n^* (n' \otimes m') &= n \otimes (n' \cdot m'), \\
 (m' \otimes n')^* n &= (m' \cdot n') \otimes n, & (n' \otimes m')^* n &= (n' \cdot m') \otimes n.
 \end{aligned}$$

- (c) *The homomorphisms $\psi_{\mathbf{M}}$ and $\psi_{\mathbf{N}}$ together with the actions described in the statement (b) are crossed modules of AWBs.*

Proof. This is straightforward but tedious verification. $\square$

Theorem 4.8. *If $\mathbf{A}$ is a perfect AWB, then $\psi_{\mathbf{A}}: \mathbf{A} \boxtimes \mathbf{A} \rightarrow \mathbf{A}$ is the universal central extension of $\mathbf{A}$.*

Proof. Clearly $\psi_{\mathbf{A}}: \mathbf{A} \boxtimes \mathbf{A} \rightarrow \mathbf{A}$ is an epimorphism if $\mathbf{A}$ is perfect. Moreover, it is a crossed module of AWBs by Proposition 4.7 (c). Then Lemma 3.6 (i) says that it is a central extension.

To show the universal property, consider any central extension $0 \rightarrow \mathbf{M} \rightarrow \mathbf{B} \xrightarrow{\phi} \mathbf{A} \rightarrow 0$. Since $\text{Ker}(\phi) \subseteq \mathcal{Z}(\mathbf{B})$, we get a well-defined homomorphism of AWBs $\varphi: \mathbf{A} \boxtimes \mathbf{A} \rightarrow \mathbf{B}$ given on generators by $\varphi(a \odot a') = b b'$ and $\varphi(a \otimes a') = [b, b']$, where b and b' are any elements in $\phi^{-1}(a)$ and $\phi^{-1}(a')$, respectively. Obviously $\phi\varphi = \psi_{\mathbf{A}}$. Moreover, since $\mathbf{A}$ is perfect, it follows by the equalities in Proposition 4.3 that $\mathbf{A} \boxtimes \mathbf{A}$ is a perfect AWB as well. Then [4, Lemma 3.1] implies that φ is the unique homomorphism satisfying the required conditions. $\square$

Bearing in mind that the universal central extension of a perfect AWB is unique up to isomorphism, by [4, Theorem 3.5] we conclude that

$$H_1^{\text{AWB}}(\mathbf{A}) \cong \text{Ker}(\psi_{\mathbf{A}}: \mathbf{A} \boxtimes \mathbf{A} \rightarrow \mathbf{A}).$$

Moreover, if $0 \rightarrow \mathbf{R} \rightarrow \mathbf{F} \xrightarrow{\rho} \mathbf{A} \rightarrow 0$ is a free presentation of a perfect AWB $\mathbf{A}$, then its universal central extension is

$$0 \longrightarrow \frac{\mathbf{R} \cap [[\mathbf{F}, \mathbf{F}]]}{[[\mathbf{F}, \mathbf{R}]]} \longrightarrow \frac{[[\mathbf{F}, \mathbf{F}]]}{[[\mathbf{F}, \mathbf{R}]]} \xrightarrow{\rho^*} \mathbf{A} \longrightarrow 0$$

(see [4]), hence

$$A \boxtimes A \cong \frac{[[F, F]]}{[[F, R]]}$$

due to the uniqueness (up to isomorphisms) of the universal central extension.

Remark 4.9. The article [4] provides another description of the universal central extension of a perfect AWB A . In particular, it is shown that, given an AWB A , the quotient $\frac{A^{\otimes 2} \oplus A^{\otimes 2}}{I_A}$ has an AWB structure, where I_A is the image of the map $d_2: A^{\otimes 3} \oplus A^{\otimes 3} \rightarrow A^{\otimes 2} \oplus A^{\otimes 2}$ in the homology chain complex $(C_*^{\text{AWB}}(A), d_*)$, that is, I_A is the subspace of $A^{\otimes 2} \oplus A^{\otimes 2}$ spanned by the elements of the form

$$(a_1 a_2) \otimes a_3 - a_1 \otimes (a_2 a_3),$$

$$[a_1, a_2] \otimes a_3 + a_1 \otimes [a_2, a_3] - (a_1 a_2) \circ a_3,$$

for any $a_1, a_2, a_3 \in A$. Moreover, if A is a perfect AWB, then it gives the construction of the universal central extension of A . As a consequence, we have the following isomorphism of AWBs

$$\frac{A^{\otimes 2} \oplus A^{\otimes 2}}{I_A} \xrightarrow{\cong} A \boxtimes A,$$

given by $a_1 \otimes a_2 \mapsto a_1 \odot a_2$ and $a_1 \circ a_2 \mapsto a_1 \otimes a_2$.

Proposition 4.10. *If M is a two-sided ideal of an AWB A , then there is the exact sequence of AWBs*

$$(M \boxtimes A) \rtimes (A \boxtimes M) \xrightarrow{\sigma} A \boxtimes A \xrightarrow{\tau} A/M \boxtimes A/M \rightarrow 0.$$

Proof. The functorial property of the non-abelian tensor product applied to the projection $A \twoheadrightarrow A/M$ induces the surjective homomorphism τ , and applied to $\text{inc}: M \rightarrow A$ and $\text{Id}: A \rightarrow A$ provides the homomorphisms $\sigma': M \boxtimes A \rightarrow A \boxtimes A$ and $\sigma'': A \boxtimes M \rightarrow A \boxtimes A$.

Define $\sigma(x, y) = \sigma'(x) + \sigma''(y)$, for all $x \in M \boxtimes A$, $y \in A \boxtimes M$. $\text{Im}(\sigma)$ is a two sided ideal of $A \boxtimes A$ spanned by the elements of the form $m \odot a$, $a \odot m$, $m \otimes a$, $a \otimes m$ for all $a \in A$ and $m \in M$.

By the identities in Proposition 4.3 and the relations (4.2) of the non-abelian tensor product, τ induces a homomorphism of AWBs $\bar{\tau}: \frac{A \boxtimes A}{\text{Im}(\sigma)} \rightarrow A/M \boxtimes A/M$. Define $\tau': A/M \boxtimes A/M \rightarrow \frac{A \boxtimes A}{\text{Im}(\sigma)}$ by $\tau'((a_1 + M) \odot (a_2 + M)) = a_1 \odot a_2 + \text{Im}(\sigma)$, $\tau'((a_1 + M) \otimes (a_2 + M)) = a_1 \otimes a_2 + \text{Im}(\sigma)$. It is easy to check that τ' is a well-defined homomorphism that is inverse to $\bar{\tau}$. $\square$

Theorem 4.11. *Let M be a two-sided ideal of a perfect AWB A . Then there is an exact sequence of vector spaces*

$$\text{Ker}(M \boxtimes A \xrightarrow{\psi_M} M) \rightarrow H_1^{\text{AWB}}(A) \rightarrow H_1^{\text{AWB}}(A/M) \rightarrow \frac{M}{[[A, M]]} \rightarrow 0$$

Proof. Due to Proposition 4.10 there is the following commutative diagram of AWBs with exact rows

$$\begin{array}{ccccccc} (M \boxtimes A) \rtimes (A \boxtimes M) & \xrightarrow{\sigma} & A \boxtimes A & \xrightarrow{\tau} & A/M \boxtimes A/M & \longrightarrow & 0 \\ \downarrow \psi & & \downarrow \psi_A & & \downarrow \psi_{A/M} & & \\ 0 & \longrightarrow & M & \xrightarrow{\pi} & A & \longrightarrow & 0 \end{array}$$

where

$$\begin{aligned} \psi(m_1 \odot a_1, a_2 \odot m_2) &= m_1 a_1 + a_2 m_2, & \psi(m_1 \odot a_1, a_2 \otimes m_2) &= m_1 a_1 + [a_2, m_2], \\ \psi(m_1 \otimes a_1, a_2 \odot m_2) &= [m_1, a_1] + a_2 m_2, & \psi(m_1 \otimes a_1, a_2 \otimes m_2) &= [m_1, a_1] + [a_2, m_2]. \end{aligned}$$

The Snake lemma provides the exact sequence

$$\mathrm{Ker}(\psi) \rightarrow \mathrm{Ker}(\psi_A) \rightarrow \mathrm{Ker}(\psi_{A/M}) \rightarrow \mathrm{Coker}(\psi) \rightarrow 0.$$

By Theorem 4.8 $\mathrm{Ker}(\psi_A) = H_1^{\mathrm{AWB}}(A)$, and since A/M is a perfect AWB as well, we also have $\mathrm{Ker}(\psi_{A/M}) = H_1^{\mathrm{AWB}}(A/M)$. Obviously $\mathrm{Coker}(\psi) = \frac{M}{[[A, M]]}$. Then the fact that there is a surjective map $\mathrm{Ker}(\psi_M) \rightarrow \mathrm{Ker}(\psi)$ completes the proof. $\square$

4.1. Further investigation

As mentioned above, the category of AWBs is an example of a semi-abelian category. In the paper [11], it is explained how, in the context of a semi-abelian category, internal crossed squares can be used to set up an intrinsic approach to the non-abelian tensor product. Namely, it is shown that the non-abelian tensor products form the internal crossed squares in a semi-abelian category, in which the so-called “Smith is Huq” condition [11] is fulfilled. This condition is sufficient to construct the non-abelian tensor product of two objects acting compatibly on each other.

In future work, we plan to investigate the consistency of our constructions with the categorical definitions from [11] and ultimately prove the following:

Conjecture 4.12. *Let M and N be AWBs with mutually compatible actions on each other. The definition of the non-abelian tensor product $M \boxtimes N$ as in Definition 4.4 agrees with the categorical one (see [11, Definition 6.6]).*

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