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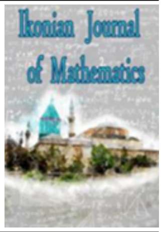

Action of Crossed Modules and Bar Construction

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Abstract — If a group N acts on a set X, a simplicial set $Bar(X,N)$ using the usual bar construction has been provided. In this construction, if the group N acts on a group G via a homomorphism $f : N \to G$, then $Bar(G,N)$ has a simplicial set structure. In the case of f has a crossed module structure, $Bar(G,N)$ has a normal simplicial group structure. In this work, by defining an action of a crossed module $\partial : N_1 \longrightarrow X_1$ on a homomorphism of groups $f : N_2 \longrightarrow X_2$ via a double map $\alpha : \partial \to f$, we will construct a bisimplicial set, using the 2-dimensional version of the usual Bar construction.

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1. Introduction

By considering a well-known equivalence between the category of crossed modules of groups introduced by Whitehead [13] and the category of simplicial groups with Moore complex of length 1 (cf. [1–4]), Farjoun and Segev, in [6], have reformulated this association in terms of homotopy co-limits. It is well-known that, if a group N acts on a set X, then, the usual bar construction gives a simplicial set $Bar(X,N)$. If the set X is any group G and the action of N on G is given by a homomorphism of groups $f : N \to G$, then $Bar(G,N)$ gives also a simplicial set. In this construction, an action of an element $n \in N$ on an element $g \in G$ is given by $g^n = gf(n)$. In the case, the homomorphism f has a normal map structure or a crossed module structure, $Bar(G,N)$ has a simplicial group structure. Thus, it has been associated to a crossed module an explicit simplicial group structure on the bar construction. More specifically, if there is a normal map structure or a crossed module structure on the homomorphism of groups $f : N \longrightarrow G$, then there is a normal simplicial group structure on the usual bar construction $Bar(G,N)$. For more details about normal map structure see also [9, 11].

In this work, we define an action of a crossed module of groups ∂ on a homomorphism of groups f and by using this action and considering the usual bar construction as a 2-dimensional version, we obtain a bisimplicial set. If $\partial : N_1 \longrightarrow X_1$ is a crossed module of groups and acts on a homomorphism of groups

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$f : N_2 \longrightarrow X_2$ via the double map α from the crossed module ∂ to the homomorphism f given pictorially by

$$\begin{array}{ccc} N_1 & \xrightarrow{\alpha_1} & N_2 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle f} \\ X_1 & \xrightarrow[\alpha_2]{} & X_2 \end{array} \tag{1.1}$$

then, the resulting Bar construction will give a bisimplicial set structure. In particular, in the case of this diagram is a crossed square defined by Guin-Walery and Loday [12] then, this bisimplicial set has a bisimplicial group structure.

2. Preliminaries

Simplicial objects are extremely useful in various algebraic settings corresponding to homotopy types of topological space [7, 8, 10]. Simplicial sets extend ideas of simplicial complexes in a neat way. They combine a resonably simple combinatorial definition with subtle algebraic properties. Their original construction was motivated in algebraic topology by the singular complex of a space. If X is a topological space, $Sing(X)$, denotes the collection of sets and mappings defined by $Sing(X)_n = top(\Delta^n, X), n \in \mathbb{N}$ where Δ^n is the usual topological n-simplex. There are inclusion maps $\delta_i : \Delta^{n-1} \longrightarrow \Delta^n$ and squashing maps $\sigma_j : \Delta^{n+1} \longrightarrow \Delta^n$ and these induce the face maps $d_i : Sing(X)_n \longrightarrow Sing(X)_{n-1} (0 \leq i < n)$ and degeneracy maps $s_j : Sing(X)_n \longrightarrow Sing(X)_{n+1} (0 \leq i < n)$ (cf. [8]). These maps satisfy the usual simplicial identities given by

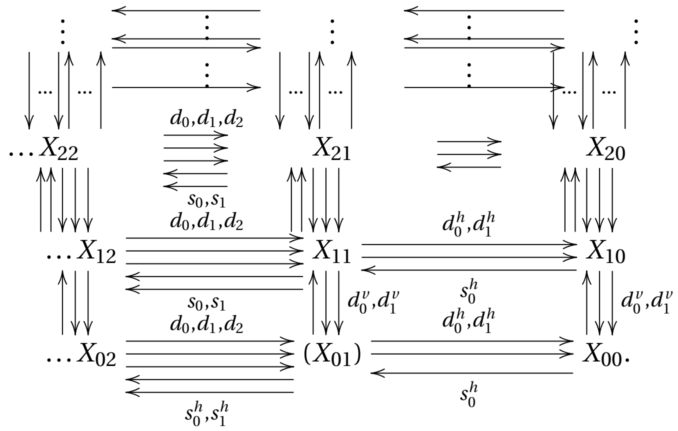
1) $d_i d_j = d_{j-1} d_i$, if $i < j$
2) $d_i s_j = s_{j-1} d_i$, if $i < j$
3) $d_j s_j = id = d_{j+1} s_j$
4) $d_i s_j = s_j d_{i-1}$, if $i > j+1$
5) $s_i s_j = s_{j+1} s_i$, if $i \leq j$

Generally, this structure is abstracted to give a family of sets $\{K_n : n \geq 0\}$, face maps $d_i : K_n \longrightarrow K_{n-1}$, and degeneracy maps $s_j : K_n \longrightarrow K_{n+1}$ satisfying these simplicial identities.

Thus, if $\mathscr{C}$ is any category, a simplicial object in $\mathscr{C}$ is given by a family of objects of $\mathscr{C}$ together with the maps d_i and s_j satisfying the usual simplicial identities. Therefore, a simplicial group is a simplicial object in the category of groups.

2.1. Bisimplicial Sets and Bisimplicial Groups

In this section, we recall the 2-dimensional simplical structure in the categories of sets and groups. A bisimplicial set $\boldsymbol{X}$ consists of a collection of sets X_{ij} , $i, j = 0, 1, 2, ...$ together with the horizontal and vertical face maps $d^h_{ij} : X_{ij} \longrightarrow X_{ij-1}$, $d^v_{ij} : X_{ij} \longrightarrow X_{i-1j}$ and degeneracy maps $s^h_{ij} : X_{ij} \longrightarrow X_{i+1j}$, $s^v_{ij} : X_{ij} \longrightarrow X_{ij+1}$, such that the simplicial identities hold. We can demonstrate it by the diagram bellow:



(2.1)

If each X_{ij} is a group and the faces and degeneracies are homomorphisms of groups, then $\boldsymbol{X}$ is called a bisimplicial group.

3. Bar Construction via the Group Actions

In this section, we recall the usual Bar construction for a group action on a set. Suppose that a group N acts on a set X. We denote this action by x^n for all $n \in N$ and $x \in X$. This action satisfies the usual conditions given by

i) $(x^n)^m = x^{n \cdot m}$, for all $x \in X$ and $n, m \in N$

ii) $x^e = x$ for all $x \in X$ and $e \in N$.

The Bar construction $\mathscr{B} = Bar(X, N)$ is a simplicial set which consists of the following data;

1) For $k \in \mathbb{Z}^+ \cup \{0\}$, a set $\mathscr{B}_k$ which is defined by
$\mathscr{B}_0 = X$ and $\mathscr{B}_k = X \times N^k$, for $k \geq 1$, together with

2) the face maps ${d_i}^k \equiv d_i : \mathscr{B}_k \longrightarrow \mathscr{B}_{k-1}$, for all $k \geq 1$ and $0 \leq i \leq k$, defined by:

   i) $d_0(x, n_1, n_2, \ldots, n_k) = (x^{n_1}, n_2, \ldots, n_k)$

   ii) $d_i(x, n_1, n_2, \ldots, n_k) = (x, n_1, n_2, \ldots, n_i \cdot n_{i+1}, \ldots, n_k)$ for $1 \leq i < k$,

   iii) $d_k(x, n_1, n_2, \ldots, n_k) = (x, n_1, n_2, \ldots, n_{k-1})$

3) and together with degeneracy maps ${s_j}^k \equiv s_j : \mathscr{B}_k \longrightarrow \mathscr{B}_{k+1}$, defined by
$$s_j(x, n_1, n_2, \ldots, n_k) = (x, n_1, n_2, \ldots, n_j, 1, n_{j+1}, \ldots, n_k)$$

Farjoun and Segev in [6] by taking $X = G$ a group and the group action of N on G via a homomorphism $\eta : N \longrightarrow G$; i.e. the action is $g^n = g\eta(n)$ for all $g \in G$, $n \in N$, studied over the resulting simplicial set given by $Bar(G, N)$. This action satisfies the following conditions:

i) $(g^{n_1})^{n_2} = (g\eta(n_1))^{n_2} = g\eta(n_1)\eta(n_2) = g\eta(n_1 \cdot n_2)$

ii) $g^{e_N} = g\eta(e_N) = g \cdot e_G = g, \qquad$ for all $n_1, n_2 \in N$ and $g \in G$.

In this context, using the action of the group N on the group G by the homomorphism $\eta : N \longrightarrow G$ given above, the simplicial set $Bar(G,N)$ is as follows:

1) For $k \in \mathbb{Z}^+ \cup \{0\}$, a set $\mathscr{B}_k$ which is defined by
   $\mathscr{B}_0 = G$ and $\mathscr{B}_k = G \times N^k$, for $k \geq 1$, together with

2) the face maps $d_i{}^k \equiv d_i : \mathscr{B}_k \longrightarrow \mathscr{B}_{k-1}$, for all $k \geq 1$ and $0 \leq i \leq k$, defined by:

   i) $d_0(g, n_1, n_2, \ldots, n_k) = (g\eta(n_1), n_2, \ldots, n_k)$

   ii) $d_i(g, n_1, n_2, \ldots, n_k) = (g, n_1, n_2, \ldots, n_i \cdot n_{i+1}, \ldots, n_k)$ for $1 \leq i < k$,

   iii) $d_k(g, n_1, n_2, \ldots, n_k) = (g, n_1, n_2, \ldots, n_{k-1})$

3) and together with degeneracy maps $s_j{}^k \equiv s_j : \mathscr{B}_k \longrightarrow \mathscr{B}_{k+1}$, defined by
$$s_j(g, n_1, n_2, \ldots, n_k) = (g, n_1, n_2, \ldots, n_j, 1, n_{j+1}, \ldots, n_k)$$
   for all $k \geq 0$ and $0 \leq j \leq k$, $\quad g \in G$, $n_j \in N$.

By using the homomorphism $\eta : N \longrightarrow G$, it can be easily shown that $Bar(G,N)$ is a simplicial set.

4. Action of Crossed Module

In this section, we will define the notion of an action of a crossed module on a homomorphism of groups. Using this notion, we will construct a bisimplicial set as a 2-dimensional version of the usual Bar construction. We will consider the action of a crossed module on a homomorphism via a double map α.

Recall that a crossed module defined by Whitehead [13] consists of a homomorphism of groups $\eta : N \longrightarrow G$ which is called a normal map in [6] together with a homomorphism $\ell : G \longrightarrow Aut(N)$ from G to the automorphism groups of N, which is called a normal structure on η satisfying the following conditions;

NM1. $\eta(\ell_g(n)) = g^{-1}\eta(n)g$, for all $g \in G$ and $n \in N$

NM2. $\ell_{\eta(n')}(n) = n'^{-1}nn'$ for all $n, n' \in N$,

and where ℓ is given by $g \longmapsto \ell_g : N \longrightarrow N$ and where ℓ_g is an automorphism of N and the action of G on N is given by $n^g = \ell_g(n)$ for all $g \in G$ and $n \in N$. Using this notation, we can write the above conditions briefly as usual:

1. $\eta(n^g) = g^{-1}\eta(n)g$
2. $n^{\eta(n')} = n'^{-1}nn'$

for all $g \in G$ and $n, n' \in N$.

Example 4.1. Suppose that $N \trianglelefteq G$ is a normal subgroup of G. Then G acts on N by conjugation. In this case, we have

$$\ell : G \longrightarrow Aut(N)$$
$$g \longmapsto \ell_g : N \longrightarrow N$$

and where $\ell_g(n) = n^g = g^{-1}ng \in N$ since $N \trianglelefteq G$, for all $g \in G$. Then, the inclusion map $i : N \longrightarrow G$ gives a crossed module together with this usual action.

4.1. The Action of a Crossed Module on a Homomorphism of Groups

In this section, we will define a double action of a crossed module on a homomorphism of groups. Suppose that $f : N_2 \longrightarrow X_2$ is a homomorphism of groups and $\partial : N_1 \longrightarrow X_1$ is a crossed module. The action of ∂ on f consists of

(i) anaction of the group N_1 on the group N_2, denoted by ${n_2}^{n_1}$,

(ii) an action of the group X_1 on the group X_2, denoted by ${x_2}^{x_1}$,

(iii) an action of the group N_1 on the group X_1, denoted by ${x_1}^{n_1}$,

(iv) a group action of N_2 on X_2, denoted by ${x_2}^{n_2}$,

(v) $f({n_2}^{n_1}) = f(n_2)^{\partial(n_1)}$,

(vi) $N_1 \times X_1$ acts on $N_2 \times X_2$, denoted by $(n_2, x_2)^{(n_1,x_1)} = ({n_2}^{n_1}, {x_2}^{x_1})$

for all $x_1 \in X_1$, $x_2 \in X_2$, $n_1 \in N_1$ and $n_2 \in N_2$

Now, consider the pair of homomorphisms of groups $\alpha_1 : N_1 \longrightarrow N_2$ and $\alpha_2 : X_1 \longrightarrow X_2$ and the following square

$$\alpha := \left\{ \begin{array}{ccc} N_1 & \xrightarrow{\alpha_1} & N_2 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle f} \\ X_1 & \xrightarrow[\alpha_2]{} & X_2 \end{array} \right.$$

in which ∂ is a crossed module and f is a homomorphisms of groups. If this square is a commutative diagram, i.e, $f\alpha_1 = \alpha_2\partial$, then we call $\alpha := (\alpha_1, \alpha_2)$ is a double map.

Assume that the crossed module $\partial : N_1 \longrightarrow X_1$ acts on the homomorphism of groups $f : N_2 \longrightarrow X_2$ via the double map α. In this case, for the action of ∂ on f via α, we can write;

i) the action of N_1 on N_2 is given by ${n_2}^{n_1} = n_2\alpha_1(n_1)$,

ii) the action of X_1 on X_2 is given by ${x_2}^{x_1} = x_2\alpha_2(x_1)$,

iii) the action of N_1 on X_1 is given by ${x_1}^{n_1} = x_1\partial(n_1)$,

iv) the action of N_2 on X_2 is given by ${x_2}^{n_2} = x_2 f(n_2)$,

v)

$$\begin{aligned} f({n_2}^{n_1}) &= f(n_2\alpha_1(n_1)) \\ &= f(n_2) f\alpha_1(n_1) \\ &= f(n_2)\alpha_2\partial(n_1) \\ &= f(n_2)^{\partial(n_1)} \end{aligned}$$

for all $x_i \in X_i$, $n_i \in N_i$ and $i = 1, 2$,

vi) the action of $N_1 \times X_1$ on $N_2 \times X_2$, is given by
$(n_2, x_2)^{(n_1,x_1)} = (n_2, x_2)\alpha((n_1, x_1)) = (n_2, x_2)(\alpha_1(n_1), \alpha_2(x_1)) = (n_2\alpha_1(n_1), x_2\alpha_2(x_1))$,

for all $x_1 \in X_1$, $x_2 \in X_2$, $n_1 \in N_1$ and $n_2 \in N_2$.

5. Construction of Bisimplicial Set Via α

Using the action of $\partial : N_1 \longrightarrow X_1$ on $f : N_2 \longrightarrow X_2$ via α , we use the Bar construction to obtain bisimplicial sets analogously to that given by simplicial set.

First of all, using the action of N_2 on X_2 given by ${x_2}^{n_2} = x_2 f(n_2)$, we obtain a simplicial set as follows

$$\ldots X_2 \times {N_2}^k \;\substack{\longrightarrow\\ \vdots \\ \longrightarrow \\ \longleftarrow \\ \vdots \\ \longleftarrow}\; \ldots \qquad X_2 \times {N_2}^3 \;\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow\\ \longleftarrow}\; X_2 \times N_2 \times N_2 \;\underset{s_0,s_1}{\overset{d_0,d_1,d_2}{\substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow\\ \longleftarrow\\ \longleftarrow}}}\; X_2 \times N_2 \;\underset{s_0}{\overset{d_0,d_1}{\substack{\longrightarrow\\ \longrightarrow\\ \longleftarrow}}}\; X_2$$

together with face and degeneracy maps given by respectively

1) i) $d_0(x_2, n_{21}, n_{22}, ..., n_{2k}) = ({x_2}^{n_{21}}, n_{22}, ..., n_{2k}) = (x_2 f(n_{21}), n_{22}, ..., n_{2k})$

   ii) $d_i(x_2, n_{21}, n_{22}, ..., n_{2k}) = (x_2, n_{21}, n_{22}, ..., n_{2i} n_{2i+1}, ..., n_{2k})$

   iii) $d_k(x_2, n_{21}, n_{22}, ..., n_{2k}) = (x_2, n_{21}, n_{22}, ..., n_{2k-1})$

2) $s_i(x_2, n_{21}, n_{22}, ..., n_{2k}) = (x_2, n_{21}, n_{22}, ..., n_{2i}, 1, n_{2i+1}, ..., n_{2k})$

for all $x_2 \in X_2$ and $n_{2i} \in N_2$. We denote this simplicial set by $\boldsymbol{X_2//N_2}$. In this structure, $\boldsymbol{(X_2//N_2)_0}$=$X_2$, $\boldsymbol{(X_2//N_2)_1}$=$X_2 \times N_2$, and $\boldsymbol{(X_2//N_2)_k}$=$X_2 \times (N_2)^k$ for all k.

Similarly, using the action of N_1 on X_1 given by ${x_1}^{n_1} = x_1 \partial(n_1)$, we can create the simplicial set $\boldsymbol{(X_1//N_1}$, where $\boldsymbol{(X_1//N_1)_0{=}X_1}$, $\boldsymbol{(X_1//N_1)_1{=}X_1 \times N_1}$, and $\boldsymbol{(X_1//N_1)_k{=}X_1 \times (N_1)^k}$ for all $\boldsymbol{k}$ together with

1) the face operators

   i) $\boldsymbol{d_0(x_1, n_{11}, n_{12}, ..., n_{1k}) = ({x_1}^{n_{11}}, n_{12}, ..., n_{1k}) = (x_2 \partial(n_{11}), n_{12}, ..., n_{1k})}$,

   ii) $\boldsymbol{d_i(x_1, n_{11}, n_{12}, ..., n_{1k}) = (x_1, n_{11}, n_{12}, ..., n_{1i} n_{1i+1}, ..., n_{1k})}$,

   iii) $\boldsymbol{d_k(x_1, n_{11}, n_{12}, ..., n_{1k}) = (x_1, n_{11}, n_{12}, ..., n_{1k-1})}$

2) and the degeneracy operators
   $\boldsymbol{s_i(x_1, n_{11}, n_{12}, ..., n_{1k}) = (x_1, n_{11}, n_{12}, ..., n_{1i}, 1, n_{1i+1}, ..., n_{1k})}$.

We can define a map

$$\boldsymbol{\Phi : (X_1//N_1) \longrightarrow (X_2//N_2)}$$

on each step defined by

$$\boldsymbol{\Phi : X_1 \times {N_1}^k \longrightarrow X_2 \times (N_2)^k}$$

$$\boldsymbol{\Phi_k(x_1, n_{11}, n_{12}, ..., n_{1k}) = (\alpha_2(x_1), \alpha_1(n_{11}), \alpha_1(n_{12}), ..., \alpha_1(n_{1k}))}$$

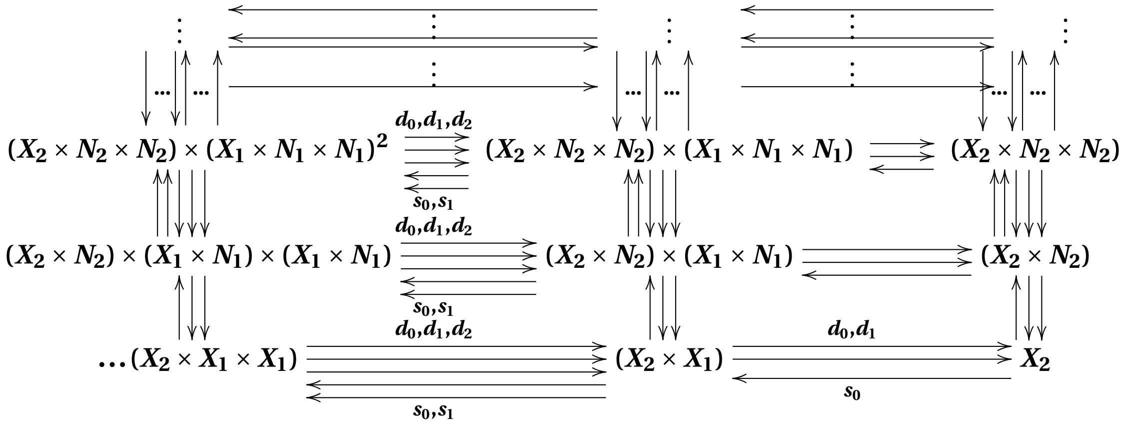
Assume that $\boldsymbol{X_1 \times {N_1}^k}$ acts on $\boldsymbol{X_2 \times {N_2}^k}$ via $\boldsymbol{\Phi_k}$, namely ;

$$\begin{aligned}\boldsymbol{(x_2, n_{21}, n_{22}, ..., n_{2k})^{(x_1, n_{11}, n_{12}, ..., n_{1k})}} &= \boldsymbol{(x_2, n_{21}, n_{22}, ..., n_{2k}) \cdot \Phi_k(x_1, n_{11}, n_{12}, ..., n_{1k})} \\ &= \boldsymbol{(x_2 \alpha_2(x_1), n_{21} \alpha_1(n_{11}), n_{22} \alpha_1(n_{12}), ..., n_{2k} \alpha_1(n_{1k}))}.\end{aligned}$$

Using this action, we obtain a bisimplicial set $\mathscr{B} = (X_2//N_2)//(X_1//N_1)$ for each $i, j \in \mathbb{Z}^+ \cup \{0\}$;

$$\mathscr{B}_{ij} = (X_2 \times {N_2}^{i}) \times (X_1 \times {N_1}^{i})^{j}.$$

For example: $\mathscr{B}_{00} = X_2$, $\mathscr{B}_{01} = X_2 \times X_1$, $\mathscr{B}_{10} = X_2 \times N_2$, $\mathscr{B}_{11} = (X_2 \times N_2) \times (X_1 \times N_1)$ and so on. We can illustrate this bisimplicial set picturially as:



Now we define the horizontal faces and degeneracy maps as follows:

i) ${d_0}^{ij^h} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{ij-1}$ $\quad i \geq 0$ and $j \geq 1$

$${d_0}^{ij^h}((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2, n_{21}, \ldots, n_{2i})^{(x_1, {n_{11}}^1, \ldots, {n_{1i}}^1)}, (x_1, {n_{11}}^2, \ldots, {n_{1i}}^2), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2, n_{21}, \ldots, n_{2i})\alpha(x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), (x_1, {n_{11}}^2, \ldots, {n_{1i}}^2), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2, n_{21}, \ldots, n_{2i})(\alpha_2(x_1), \alpha_1(n_{11})^1, \ldots, \alpha_1(n_{1i})^1), (x_1, {n_{11}}^2, \ldots, {n_{1i}}^2), \ldots, (x_1, \ldots, {n_{1i}}^j))$$

ii) ${d_m}^{ij^h} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{ij-1}$ $\quad i \geq 0$ and $j \geq 1$

$${d_m}^{ij^h}((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j)) = ((x_2, n_{21}, \ldots, n_{2i}),$$
$$(x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^m, \ldots, {n_{1i}}^m)(x_1, {n_{11}}^{m+1}, \ldots, {n_{1i}}^{m+1}), \ldots, (x_1, \ldots, {n_{1i}}^j))$$

iii) ${d_j}^{ij^h} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{ij-1}$ $\quad i \geq 0$ and $j \geq 1$

$${d_j}^{ij^h}((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^{j-1}, \ldots, {n_{1i}}^{j-1}))$$

iv) ${s_m}^{ij^h} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{ij+1}$ $\quad i, j \geq 0$

$${s_m}^{ij^h}((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^m, \ldots, {n_{1i}}^m), (1, 1, \ldots, 1),$$
$$(x_1, {n_{11}}^{m+1}, \ldots, {n_{1i}}^{m+1}) \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$

Similarly we can define the vertical face and degeneracy maps:

i) ${d_0}^{ij^v} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{i-1j}$ $\quad i \geq 1$ and $j \geq 0$

$${d_0}^{ij^v}((x_2, n_{21}, \ldots, n_{2i}), (x_1, {n_{11}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1, {n_{11}}^j, \ldots, {n_{1i}}^j))$$
$$= (({x_2}^{n_{21}}, n_{22} \ldots, n_{2i}), ({x_1}^{{n_{11}}^1}, {n_{12}}^1, \ldots, {n_{1i}}^1), \ldots, ({x_1}^{{n_{11}}^j}, {n_{12}}^j, \ldots, {n_{1i}}^j))$$
$$= ((x_2 f(n_{21}), n_{22} \ldots, n_{2i}), (x_1 \partial({n_{11}}^1), {n_{12}}^1, \ldots, {n_{1i}}^1), \ldots, (x_1 \partial({n_{11}}^j), {n_{12}}^j, \ldots, {n_{1i}}^j))$$

ii) $d_m{}^{ij^v} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{i-1j} \qquad i \geq 1$ and $j \geq 0$
$d_m{}^{ij^v}((x_2, n_{21}, \ldots, n_{2i}), (x_1, n_{11}{}^1, \ldots, n_{1i}{}^1), \ldots, (x_1, n_{11}{}^j, \ldots, n_{1i}{}^j))$
$= ((x_2, n_{21}, \ldots, (n_{2m})(n_{2m+1}), \ldots, n_{2i}), (x_1, n_{11}{}^1, \ldots, (n_{1m})(n_{1m+1}), \ldots n_{1i}{}^1), \ldots,$
$(x_1, n_{11}{}^j, \ldots, (n_{1m})(n_{1m+1}), \ldots, n_{1i}{}^j))$

iii) $d_i{}^{ij^v} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{i-1j} \qquad i \geq 1$ and $j \geq 0$
$d_i{}^{ij^v}((x_2, n_{21}, \ldots, n_{2i}), (x_1, n_{11}{}^1, \ldots, n_{1i}{}^1), \ldots, (x_1, n_{11}{}^j, \ldots, n_{1i}{}^j))$
$= ((x_2, n_{21}, \ldots, n_{2i-1}), (x_1, n_{11}{}^1, \ldots, n_{1i-1}{}^1), \ldots, (x_1, n_{11}{}^j, \ldots, n_{1i-1}{}^j))$

iv) $s_m{}^{ij^v} : \mathscr{B}_{ij} \longrightarrow \mathscr{B}_{i+1j} \qquad i, j \geq 0$
$s_m{}^{ij^v}((x_2, n_{21}, \ldots, n_{2i}), (x_1, n_{11}{}^1, \ldots, n_{1i}{}^1), \ldots, (x_1, n_{11}{}^j, \ldots, n_{1i}{}^j))$
$= ((x_2, n_{21}, \ldots, (n_{2m}), 1, (n_{2m+1}), \ldots, n_{2i}), (x_1, n_{11}{}^1, \ldots, (n_{1m}), 1, (n_{1m+1}), \ldots n_{1i}{}^1),$
$\ldots, (x_1, n_{11}{}^j, \ldots, (n_{1m}), 1, (n_{1m+1}), \ldots, n_{1i}{}^j))$

Now, we will show that these maps satisfy the usual Bisimplicial identities;

1) a) for $k+1 = m$
$d_k{}^{ij^h} d_m{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^j)$
$= d_k{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^m (x_1, n_{1i})^{m+1}, \ldots, (x_1, n_{1i})^j)$
$= ((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^m (x_1, n_{1i})^{m+1}, \ldots, (x_1, n_{1i})^j) \ldots\ldots (1)$
$d_{m-1}{}^{ij^h} d_k{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^j)$
$= d_{m-1}{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^{k+1}, \ldots, (x_1, n_{1i})^j)$
$= ((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^{k+1} (x_1, n_{1i})^{k+2}, \ldots, (x_1, n_{1i})^j) \ldots\ldots (2)$,
from $(1) = (2)$, we obtain that;

$$d_k{}^{ij^h} d_m{}^{ij^h} = d_{m-1}{}^{ij^h} d_k{}^{ij^h}.$$

for $k+1 < m$
$d_k{}^{ij^h} d_m{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^j)$
$= d_k{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^m (x_1, n_{1i})^{m+1}, \ldots, (x_1, n_{1i})^j)$
$= ((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^{k+1}, \ldots, (x_1, n_{1i})^m (x_1, n_{1i})^{m+1}, \ldots, (x_1, n_{1i})^j) \ldots (1)$
$d_{m-1}{}^{ij^h} d_k{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^j)$
$= d_{m-1}{}^{ij^h}((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^{k+1}, \ldots, (x_1, n_{1i})^j)$
$= ((x_2, n_{2i}), (x_1, n_{1i})^1, \ldots, (x_1, n_{1i})^k (x_1, n_{1i})^{k+1}, \ldots, (x_1, n_{1i})^m (x_1, n_{1i})^{m+1}, \ldots, (x_1, n_{1i})^j) \ldots (2)$,
from $(1) = (2)$ for $k < m$, we obtain that;

$$d_k{}^{ij^h} d_m{}^{ij^h} = d_{m-1}{}^{ij^h} d_k{}^{ij^h}.$$

b) for $k+1 = m$
$d_k{}^{ij^v} d_m{}^{ij^v}((x_2, n_{2i}), (x_1, n_{1i})^j) =$
$d_k{}^{ij^v}((x_2, \ldots, n_{2m} n_{2m+1}, \ldots, n_{2i}), (x_1, \ldots, n_{1m} n_{1m+1}, \ldots, n_{1i})^1, \ldots, (x_1, \ldots, n_{1m} n_{1m+1}, \ldots, n_{1i})^j) =$
$((x_2, \ldots, n_{2k} n_{2m} n_{2m+1}, \ldots, n_{2i}), (x_1, \ldots, n_{1k} n_{1m} n_{1m+1}, \ldots, n_{1i})^1, \ldots, (x_1, \ldots, n_{1k} n_{1m} n_{1m+1}, \ldots, n_{1i})^j) \ldots (1)$
$d_{m-1}{}^{ij^v} d_k{}^{ij^v}((x_2, n_{2i}), (x_1, n_{1i})^j) =$
$d_{m-1}{}^{ij^v}((x_2, \ldots, n_{2k} n_{2k+1}, \ldots, n_{2i}), (x_1, \ldots, n_{1k} n_{1k+1}, \ldots, n_{1i})^1, \ldots, (x_1, \ldots, n_{1k} n_{1k+1}, \ldots, n_{1i})^j) =$
$((x_2, \ldots, n_{2k} n_{2k+1} n_{2k+2}, \ldots, n_{2i}), (x_1, \ldots, n_{1k} n_{1k+1} n_{1k+2}, \ldots, n_{1i})^1, \ldots, (x_1, \ldots, n_{1k} n_{1k+1} n_{1k+2}, \ldots, n_{1i})^j)$
$\ldots (2)$,

from **(1)** = **(2)** , we obtain that;

$$d_k{}^{ij^v} d_m{}^{ij^v} = d_{m-1}{}^{ij^v} d_k{}^{ij^v}.$$

for $k+1<m$

$d_k{}^{ij^v} d_m{}^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k{}^{ij^v}((x_2,\ldots,n_{2m}n_{2m+1},\ldots,n_{2i}),(x_1,\ldots,n_{1m}n_{1m+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1m}n_{1m+1},\ldots,n_{1i})^j) =$

$((x_2,\ldots,n_{2k}n_{2k+1},n_{2m}n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,\ldots,n_{1k}n_{1k+1},n_{1m}n_{1m+1},\ldots,n_{1i})^j)\ldots(1)$

$d_{m-1}{}^{ij^v} d_k{}^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_{m-1}{}^{ij^v}((x_2,\ldots,n_{2k}n_{2k+1},\ldots,n_{2i}),(x_1,\ldots,n_{1k}n_{1k+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1k}n_{1k+1},\ldots,n_{1i})^j) =$

$((x_2,\ldots,n_{2k}n_{2k+1},n_{2m}n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,\ldots,n_{1k}n_{1k+1},n_{1m}n_{1m+1},\ldots,n_{1i})^j)\ldots(2)$,

from **(1)** = **(2)**, we obtain that;

$$d_k{}^{ij^v} d_m{}^{ij^v} = d_{m-1}{}^{ij^v} d_k{}^{ij^v}.$$

2) a) for $k+1=m$

$d_k{}^{ij^h} s_m{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)\ldots(1)$

$s_{m-1}{}^{ij^h} d_k{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$s_{m-1}{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},(1,1,\ldots,1),(x_1,n_{1i})^{k+2},\ldots,(x_1,n_{1i})^j)\ldots(2)$,

from **(1)** = **(2)**, we obtain that;

$$d_k{}^{ij^h} s_m{}^{ij^h} = s_{m-1}{}^{ij^h} d_k{}^{ij^h}.$$

for $k+1<m$

$d_k{}^{ij^h} s_m{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)$
$\ldots(1)$

$s_{m-1}{}^{ij^h} d_k{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$s_{m-1}{}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)$
$\ldots(2)$,

from **(1)** = **(2)**, we obtain that;

$$d_k{}^{ij^h} s_m{}^{ij^h} = s_{m-1}{}^{ij^h} d_k{}^{ij^h}.$$

b) for $k+1=m$

$d_k{}^{ij^v} s_m{}^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k{}^{ij^v}((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j) =$

$((x_2,n_{21},\ldots,n_{2k}n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1k}n_{1m},1,n_{1m+1},\ldots,n_{1i})^j)\ldots(1)$

$s_{m-1}{}^{ij^v} d_k{}^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$s_{m-1}{}^{ij^v}((x_2,n_{21},\ldots,n_{2k}n_{2k+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1k}n_{1k+1},\ldots,n_{1i})^j) =$

$((x_2,n_{21},\ldots,n_{2k}n_{2k+1},1,n_{2k+2},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1k}n_{1k+1},1,n_{1k+2},\ldots,n_{1i})^j)\ldots(2)$

from **(1)** = **(2)**, we obtain that;

$$d_k^{ij^v} s_m^{ij^v} = s_{m-1}^{ij^v} d_k^{ij^v}.$$

for $k+1<m$

$d_k^{ij^v} s_m^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k^{ij^v}((x_2,n_{21},\dots,n_{2m},1,n_{2m+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1m},1,n_{1m+1},\dots,n_{1i})^j) =$

$((x_2,n_{21},\dots,n_{2k}n_{2k+1},n_{2m},1,n_{2m+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k}n_{1k+1},n_{1m},1,n_{1m+1},\dots,n_{1i})^j)$

$\dots(1)$

$s_{m-1}^{ij^v} d_k^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$s_{m-1}^{ij^v}((x_2,n_{21},\dots,n_{2k}n_{2k+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k}n_{1k+1},\dots,n_{1i})^j) =$

$((x_2,n_{21},\dots,n_{2k}n_{2k+1},n_{2m},1,n_{2m+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k}n_{1k+1},n_{1m},1,n_{1m+1},\dots,n_{1i})^j)$

$\dots(2)$,

from **(1)** = **(2)**, we obtain that;

$$d_k^{ij^v} s_m^{ij^v} = s_{m-1}^{ij^v} d_k^{ij^v}.$$

3) a) It can be seen that;

$d_k^{ij^h} s_k^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^k,(1,1,\dots,1),(x_1,n_{1i})^{k+1},\dots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^k,(x_1,n_{1i})^{k+1},\dots,(x_1,n_{1i})^j) = id.$

At the same time it can be seen that ;

$d_{k+1}^{ij^h} s_k^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_{k+1}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^k,(1,1,\dots,1),(x_1,n_{1i})^{k+1},\dots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^k,(x_1,n_{1i})^{k+1},\dots,(x_1,n_{1i})^j) = id.$

b) Similarly it can be seen that;

$d_k^{ij^v} s_k^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k^{ij^v}((x_2,n_{21},\dots,n_{2k},1,n_{2k+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k},1,n_{1k+1},\dots,n_{1i})^j) =$

$((x_2,n_{21},\dots,n_{2k},n_{2k+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k},n_{1k+1},\dots,n_{1i})^j) = id.$

and also it can be shown that;

$d_{k+1}^{ij^v} s_k^{ij^v}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_{k+1}^{ij^v}((x_2,n_{21},\dots,n_{2k},1,n_{2k+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k},1,n_{1k+1},\dots,n_{1i})^j) =$

$((x_2,n_{21},\dots,n_{2k},n_{2k+1},\dots,n_{2i}),\dots,(x_1,n_{11},\dots,n_{1k},n_{1k+1},\dots,n_{1i})^j) = id.$

4) a) For $k=m+2$

$d_k^{ij^h} s_m^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^m,(1,1,\dots,1),(x_1,n_{1i})^{m+1},\dots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^m,(1,1,\dots,1),(x_1,n_{1i})^{m+1}(x_1,n_{1i})^{m+2},\dots,(x_1,n_{1i})^j)\dots(1)$

$s_m^{ij^h} d_{k-1}^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$s_m^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^{k-1}(x_1,n_{1i})^k,\dots,(x_1,n_{1i})^j) =$

$((x_2,n_{2i}),(x_1,n_{1i})^1,\dots,(x_1,n_{1i})^m,(1,1,\dots,1),(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},\dots,(x_1,n_{1i})^j)\dots(2)$

from **(1)** = **(2)** we obtain that;

$$d_k^{ij^h} s_m^{ij^h} = s_m^{ij^h} d_{k-1}^{ij^h}.$$

For $k>m+2$

$d_k^{ij^h} s_m^{ij^h}((x_2,n_{2i}),(x_1,n_{1i})^j) =$

$d_k{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j) =$
$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},(x_1,n_{1i})^k(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j)$
$\ldots(1)$
$s_m{}^{ijh}d_{k-1}{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_m{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^{k-1}(x_1,n_{1i})^k,\ldots,(x_1,n_{1i})^j) =$
$((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},(x_1,n_{1i})^{k-1}(x_1,n_{1i})^k,\ldots,(x_1,n_{1i})^j)$
$\ldots(2)$
from **(1)** = **(2)**, we obtain that;

$$d_k{}^{ijh}s_m{}^{ijh} = s_m{}^{ijh}d_{k-1}{}^{ijh}.$$

b) For $k = m+2$
$d_k{}^{ijv}s_m{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$d_k{}^{ijv}((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j) =$
$((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1}n_{2k},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1}n_{1k},\ldots,n_{1i})^j)\ldots(1)$
$s_m{}^{ijv}d_{k-1}{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_m{}^{ijv}((x_2,n_{21},\ldots,n_{2k-1}n_{2k},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1k-1}n_{1k},\ldots,n_{1i})^j) =$
$((x_2,n_{21},\ldots,n_{2m},1,n_{2k-1}n_{2k},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1k-1}n_{1k},\ldots,n_{1i})^j)\ldots(2)$
from **(1)** = **(2)**. we obtain that;

$$d_k{}^{ijv}s_m{}^{ijv} = s_m{}^{ijv}d_{k-1}{}^{ijv}.$$

For $k > m+2$
$d_k{}^{ijv}s_m{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$d_k{}^{ijv}((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j) =$
$((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1},n_{2k}n_{2k+1},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1},n_{1k}n_{1k+1},\ldots,n_{1i})^j)$
$\ldots(1)$
$s_m{}^{ijv}d_{k-1}{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_m{}^{ijv}((x_2,n_{21},\ldots,n_{2k-1}n_{2k},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1k-1}n_{1k},\ldots,n_{1i})^j) =$
$((x_2,n_{21},\ldots,n_{2m},1,n_{2m+1},n_{2k-1}n_{2k},\ldots,n_{2i}),\ldots,(x_1,n_{11},\ldots,n_{1m},1,n_{1m+1},n_{1k-1}n_{1k},\ldots,n_{1i})^j)$
$\ldots(2)$
from **(1)** = **(2)**, we obtain that;

$$d_k{}^{ijv}s_m{}^{ijv} = s_m{}^{ijv}d_{k-1}{}^{ijv}.$$

5) a) For $k = m$
$s_k{}^{ijh}s_m{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_k{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)$
$= ((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)\ldots(1)$
$s_{m+1}{}^{ijh}s_k{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_{m+1}{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k,(1,1,\ldots,1),(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j)$
$= ((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k,(1,1,\ldots,1),(1,1,\ldots,1),(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j)\ldots(2)$
from **(1)** = **(2)**, we obtain that;

$$s_k{}^{ijh}s_m{}^{ijh} = s_{m+1}{}^{ijh}s_k{}^{ijh}.$$

For $k < m$
$s_k{}^{ijh}s_m{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_k{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^m,(1,1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)$
$= ((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k,(1,\ldots,1),(x_1,n_{1i})^m,(1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)$

$\ldots(1)$
$s_{m+1}{}^{ijh}s_k{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_{m+1}{}^{ijh}((x_2,n_{2i}),(x_1,n_{1i})^1,\ldots,(x_1,n_{1i})^k,(1,1,\ldots,1),(x_1,n_{1i})^{k+1},\ldots,(x_1,n_{1i})^j)$
$= ((x_2,n_{2i}),\ldots,(x_1,n_{1i})^k,(1,1,\ldots,1),(x_1,n_{1i})^{k+1},(1,\ldots,1),(x_1,n_{1i})^{m+1},\ldots,(x_1,n_{1i})^j)\ldots(2)$
from $(1)=(2)$, we obtain that; for $k<m$

$$s_k{}^{ijh}s_m{}^{ijh} = s_{m+1}{}^{ijh}s_k{}^{ijh}.$$

b) For $k+1=m$
$s_k{}^{ijv}s_m{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_k{}^{ijv}((x_2,\ldots,n_{2m},1,n_{2m+1},\ldots,n_{2i}),(x_1,\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j) =$
$((x_2,\ldots,n_{2k},1,n_{2m},1,n_{2m+1},\ldots,n_{2i}),(x_1,\ldots,n_{1k},1,n_{1m},1,n_{1m+1},\ldots,n_{1i})^1,\ldots,$
$(x_1,\ldots,n_{1k},1,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j)\ldots(1)$
$s_{m+1}{}^{ijv}s_k{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_{m+1}{}^{ijv}((x_2,\ldots,n_{2k},1,n_{2k+1},\ldots,n_{2i}),(x_1,\ldots,n_{1k},1,n_{1k+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1k},1,n_{1k+1},\ldots,n_{1i})^j) =$
$((x_2,\ldots,n_{2k},1,n_{2k+1},1,n_{2m+1},\ldots,n_{2i}),(x_1,\ldots,n_{1k},1,n_{1k+1},1,n_{1m+1},\ldots,n_{1i})^1,\ldots,$
$(x_1,\ldots,n_{1k},1,n_{1k+1},1,n_{1m+1},\ldots,n_{1i})^j)\ldots(2)$
from $(1)=(2)$, we obtain that;

$$s_k{}^{ijv}s_m{}^{ijv} = s_{m+1}{}^{ijv}s_k{}^{ijv}.$$

For $k+1<m$
$s_k{}^{ijv}s_m{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_k{}^{ijv}((x_2,\ldots,n_{2m},1,n_{2m+1},\ldots,n_{2i}),(x_1,\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1m},1,n_{1m+1},\ldots,n_{1i})^j) =$
$((x_2,\ldots,n_{2k},1,n_{2k+1},n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,\ldots,n_{1k},1,n_{1k+1},n_{1m},1,n_{1m+1},\ldots,n_{1i})^j)\ldots(1)$
$s_{m+1}{}^{ijv}s_k{}^{ijv}((x_2,n_{2i}),(x_1,n_{1i})^j) =$
$s_{m+1}{}^{ijv}((x_2,\ldots,n_{2k},1,n_{2k+1},\ldots,n_{2i}),(x_1,\ldots,n_{1k},1,n_{1k+1},\ldots,n_{1i})^1,\ldots,(x_1,\ldots,n_{1k},1,n_{1k+1},\ldots,n_{1i})^j) =$
$((x_2,\ldots,n_{2k},1,n_{2k+1},n_{2m},1,n_{2m+1},\ldots,n_{2i}),\ldots,(x_1,\ldots,n_{1k},1,n_{1k+1},n_{1m},1,n_{1m+1},\ldots,n_{1i})^j)\ldots(2)$
from $(1)=(2)$, we obtain that;

$$s_k{}^{ijv}s_m{}^{ijv} = s_{m+1}{}^{ijv}s_k{}^{ijv}.$$

Therefore, we obtain from the action of the crossed module ∂ on the homomorphism of the groups f given by the double map α, a bisimplicial set $(X_2//N_2)//(X_1//N_1)$.

Author Contributions

All authors contributed equally to this work. They all read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare no conflict of interest.

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