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Characterizations of Curves According to Elasticity in Finsler Manifold

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ABSTRACT: Physically, inelastic curve flow is qualified by the nonexistence of any strain energy taken from the motion. We have found out the changing equations for an inelastic curve whose length is preserved over all time. In this study, we give some characterizations for curves in terms of elasticity.

Keywords: Finsler manifold, flow of a curve, frenet equations

Finsler Manifoldunda Esnekliğine Göre Eğrilerin Karakterizasyonları

ÖZET: Fiziksel olarak, elastik olmayan bir eğri akışı, hareket kaynaklı bir enerji geriliminin bulunmaması olarak karakterize edilir. Biz bir elastik olmayan düzlem eğrisinin, yani yay uzunluğu her zaman korunan bir eğri için değişim denklemlerini ortaya çıkardık. Bu çalışmada, elastiklik açısından eğrilerin bazı karakterizasyonları verildi.

Anahtar Kelimeler: Bir eğrinin akışı, finsler manifold, frenet denklemleri

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INTRODUCTION

If protected arc length of a curve the flow of this curve is referred to as inelastic. Physically, inelastic of the flow of curve brings out motions here no tension energy is excited. In (Kwon and Park, 1999; Kwon et al., 2005) consider inelastic curve flows and improvable surface in R^3. Moreover in (Latifi and Ravazi, 2008; Gürbüz, 2009; Öğrenmiş et al., 2011; Yoon, 2011; Öztekin and Bozok, 2013; Yıldız and Okuyucu, 2014) inelastic flows of curves in different ambient space are studied.

Finsler geometry is known as some generalizations of Riemannian geometry. P. Finsler 1918 began to study this issue. Firstly, he studied curves and surfaces in his thesis. Finsler geometry as in differential geometry plays an important role in other branches of science such as physics, biology, computer science and engineering. (Bejancu and Farran, 2000; Solange and Portugal, 2001; Brandt, 2005; Yılmaz and Bektaş, 2011)

In this study, we examine inelastic flows of curves defined on the 3-dimensional Finsler manifolds F^3. Conditions for a curve flow to be inelastic are phrased as a PDE with the inclusion of curvature and torsion of the curve.

MATERIAL AND METHOD

In this section of the paper there is written on the basic concepts and descriptions of 3-dimensional Finsler manifolds. Finsler manifold is a smooth manifold *M* together with a function *F* defined on the tangent bundle of *M* so that for all tangent vectors.

Let M be a m-dimensional differentiable manifold and TM the bundle of M. Indicate by Π the canonic projection of TM on M. Let M' be open submanifold of TM which is non-emtpy such that $\Pi(M')=M$ and $\theta(M)\cap M'$ is empty, where θ is section of TM where the section is zero.

Considering a differentiable function $F: M'\to(0,\infty)$ and take $F^* = F^2$. then $\{(U',\Phi'); x^i, y^i\}$ in M' satisfy the following conditions,

(i) As F is positive homogeneus of degree one according to $(y^1,...,y^m)$, we have

$$F(x^1,...x^m, ky^1,...,ky^m) = kF(x^1,...,x^m, y^1,...,y^m) \quad (1)$$

for whatever $(x,y)\in\Phi'(U')$ and any $k>0$.

(ii) At point (x,y)

$$g_{ij}(X,Y) = \frac{1}{2}\frac{\partial^2 F^*}{\partial y^i \partial y^j}(X,Y), i,j\in\{1,...,m\} \quad (2)$$

are the components of definite quadratic form on R^m where the component is positive. (Bejancu and Farran, 2000).

$F^m = (M,\ M',F)$ satisfying (i) and (ii) is called a Finsler manifold such that F is the finsler function of Finsler manifold.

$\mathsf{F}^{m+1} = (M,M',F)$ be given as Finsler manifold and let $F' = (C,C',F)$ is a one-dimensional submanifold of F^{m+1} where C is a differentiable curve in M given by the equations in locally

$$x^i = x^i(s), i\in\{1,...,m+n\},\ s\in(a,b), \quad (3)$$

s parameter is handle the on C such that the parameter is the arclength. (s,v) is indicated the coordinates on C'. Later we write

$$y^i(s,v) = v\frac{xd^i}{sd},\ i\in\{0,...,m\}. \quad (4)$$

Besides $\{\frac{\partial}{\partial s},\frac{\partial}{\partial v}\}$ is a natural field of frames on C where $\frac{\partial}{\partial v}$ is a unit Finsler vector field (Bejancu and Farran, 2000).

Let $\mathsf{F}^3 = (M,M',F)$ be a 3D Finsler manifold and C a differentiable curve in M given locally by the parametric equations

$$x^i = x^i(s)\ ;\ (x^1(s), x^2(s), x^3(s)) \neq (0,0,0), \quad (5)$$

where s is the arclength parameter on C.

∇^* is the the Levi-Civita connection of F^3.

Then we have

$$\nabla^*_{\frac{\partial}{\partial s}} \frac{\partial}{\partial v} = \kappa n,$$
$$\nabla^*_{\frac{\partial}{\partial s}} n = -\kappa \frac{\partial}{\partial v} + \tau b$$
$$\nabla^*_{\frac{\partial}{\partial s}} b = \tau n. \tag{6}$$

where n and b are respectively the principal normal and the binormal vector field on C. Here $\left\{\frac{\partial}{\partial v}, n, b\right\}$ be the Frenet frame of C in F^3. κ and τ are respectively the curvature and the torsion of C (Bejancu and Farran, 2000).

Throughout this paper, we assume that $M : [0,l]\times[0,t_\infty] \to F^3$ is a family of smooth curves in F^3, where the arclength l is parameter of the first curve.

The arclength of M is given by

$$s(u) = \int_0^u \left|\nabla^*_{\frac{\partial}{\partial u}} M\right| du. \tag{7}$$

Putting $v = \left|\nabla^*_{\frac{\partial}{\partial u}} M\right|$, the operator $\nabla^*_{\frac{\partial}{\partial u}}$ is defined by

$$\nabla^*_{\frac{\partial}{\partial s}} = \frac{1}{v} \nabla^*_{\frac{\partial}{\partial u}}, \tag{8}$$

while $sd = vdu$ is the arclength parameter. Any flow of M family written as

$$\nabla^*_{\frac{\partial}{\partial t}} M = f \frac{\partial}{\partial v} + gn + hb, \tag{9}$$

where f, g, h are smooth functions on M.

Because exposure to the expansion or compression, the arclength variation in Euclidean space is given by

$$\nabla^*_{\frac{\partial}{\partial t}} s(u,t) = \int_0^u \nabla^*_{\frac{\partial}{\partial t}} v = 0 \tag{10}$$

for all $u \in [0,l]$

Evolution of a curve $M(u,t)$ and its flow $\nabla^*_{\frac{\partial}{\partial t}} M$ in 3-dimensional Finsler manifold F^3 are said to be inelastic if $\nabla^*_{\frac{\partial}{\partial t}} \left|\nabla^*_{\frac{\partial}{\partial u}} M\right| = 0.$

RESULTS AND DISCUSSION

Conditions for inelastic of the flow in 3-dimensional Finsler manifold F^3 are given by the Theorem 3.1.

Theorem 3.1. Let $\nabla^*_{\frac{\partial}{\partial t}} M = f \frac{\partial}{\partial v} + gn + hb$ be a differentiable flow of the curve M in F^3. A flow of M in F^3 is inelastic if and only if $\nabla^*_{\frac{\partial}{\partial s}} f = g\kappa$

Proof. According to definition of M, we have

$$v^2 = \langle \nabla^*_{\frac{\partial}{\partial u}} M, \nabla^*_{\frac{\partial}{\partial u}} M \rangle, \tag{11}$$

where $\langle\, , \,\rangle$ is inner product in F^3. $\nabla^*_{\frac{\partial}{\partial u}}$ and $\nabla^*_{\frac{\partial}{\partial t}}$ commute since u and t are independent coordinates. So we get

$$\begin{aligned} 2v\nabla^*_{\frac{\partial}{\partial t}} v &= \nabla^*_{\frac{\partial}{\partial t}} \langle \nabla^*_{\frac{\partial}{\partial u}} M, \nabla^*_{\frac{\partial}{\partial u}} M \rangle \\ &= 2\langle \nabla^*_{\frac{\partial}{\partial u}} M, \nabla^*_{\frac{\partial}{\partial u}} (\nabla^*_{\frac{\partial}{\partial t}} M) \rangle \\ &= 2\langle \nabla^*_{\frac{\partial}{\partial u}} M, \nabla^*_{\frac{\partial}{\partial u}} (f \frac{\partial}{\partial v} + gn + hb) \rangle \\ &= 2v\langle \frac{\partial}{\partial v}, (\nabla^*_{\frac{\partial}{\partial u}} f) \frac{\partial}{\partial v} + fv\kappa n + (\nabla^*_{\frac{\partial}{\partial u}} g) n - vg\kappa \frac{\partial}{\partial v} + v\tau b + (\nabla^*_{\frac{\partial}{\partial u}} h) b + hv\tau n \rangle \\ &= 2v(\nabla^*_{\frac{\partial}{\partial u}} f - vgk). \end{aligned} \tag{12}$$

In this way, we derive

$$\nabla^*_{\frac{\partial}{\partial t}} v = \nabla^*_{\frac{\partial}{\partial u}} f - vg\kappa \tag{13}$$

Now assume that $\nabla^*_{\frac{\partial}{\partial s}} M$ is inelastic. From "Eq. 13." we have

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial t}} s(u,t) &= \int_0^u (\nabla^*_{\frac{\partial}{\partial t}} t)\,du \\ &= \int_0^u (\nabla^*_{\frac{\partial}{\partial u}} f - vg\kappa)\,du \\ &= 0\end{aligned} \tag{14}$$

for all $u \in [0,l]$. This implies that $\nabla^*_{\frac{\partial}{\partial u}} f = vq$ *(kappa),* or $\nabla^*_{\frac{\partial}{\partial s}} f = g\kappa$ If we think the reverse of the above calculation, the proof is completed.

Now we take curve as arclength parametrized. The coordinate u according to the curve arclength s. Now we can give the following Lemma.

Lemma 3.1. Let $\nabla^*_{\frac{\partial}{\partial t}} M = f\frac{\partial}{\partial v} + gn + hb$ be a differentiable flow of the curve M in F^3. Then, the differentiations of $\{\frac{\partial}{\partial v}, n, b\}$ with respect to t is

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial t}} \frac{\partial}{\partial v} &= (f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)n + (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)b, \\ \nabla^*_{\frac{\partial}{\partial t}} n &= -(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)\frac{\partial}{\partial v} + \psi b, \\ \nabla^*_{\frac{\partial}{\partial t}} b &= -(g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)\frac{\partial}{\partial v} - \psi n,\end{aligned} \tag{15}$$

where $\psi = \langle \nabla^*_{\frac{\partial}{\partial t}} n, b\rangle$.

Proof. Using "Eq. 6." and Theorem 3.1., we calculate

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial t}} \frac{\partial}{\partial v} &= \nabla^*_{\frac{\partial}{\partial t}} (\nabla^*_{\frac{\partial}{\partial s}} M) = \nabla^*_{\frac{\partial}{\partial u}} (f\frac{\partial}{\partial v} + gn + hb) \\ &= (\nabla^*_{\frac{\partial}{\partial s}} f)\frac{\partial}{\partial v} + f\kappa n + (\nabla^*_{\frac{\partial}{\partial s}} g)n - g\kappa\frac{\partial}{\partial v} + g\tau b + (\nabla^*_{\frac{\partial}{\partial s}} h)b + h\tau n \\ &= (f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)n + (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)b\end{aligned} \tag{16}$$

Differentiating the Frenet frame with respect to t gives

$$\begin{aligned}0 = \nabla^*_{\frac{\partial}{\partial t}} \langle \frac{\partial}{\partial v}, n\rangle &= \langle \nabla^*_{\frac{\partial}{\partial t}} \frac{\partial}{\partial v}, n\rangle + \langle \frac{\partial}{\partial v}, \nabla^*_{\frac{\partial}{\partial t}} n\rangle \\ &= (f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau) + \langle \frac{\partial}{\partial v}, \nabla^*_{\frac{\partial}{\partial t}} n\rangle, \\ 0 = \nabla^*_{\frac{\partial}{\partial t}} \langle \frac{\partial}{\partial v}, b\rangle &= \langle \nabla^*_{\frac{\partial}{\partial t}} \frac{\partial}{\partial v}, b\rangle + \langle \frac{\partial}{\partial v}, \nabla^*_{\frac{\partial}{\partial t}} b\rangle \\ &= (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h) + \langle \frac{\partial}{\partial v}, \nabla^*_{\frac{\partial}{\partial t}} b\rangle, \\ 0 = \nabla^*_{\frac{\partial}{\partial t}} \langle n, b\rangle &= \langle \nabla^*_{\frac{\partial}{\partial t}} n, b\rangle + \langle n, \nabla^*_{\frac{\partial}{\partial t}} b\rangle \\ &= \psi + \langle n, \nabla^*_{\frac{\partial}{\partial t}} b\rangle.\end{aligned} \tag{17}$$

Thus, since $\langle \nabla^*_{\frac{\partial}{\partial t}} n, n\rangle = \langle \nabla^*_{\frac{\partial}{\partial t}} b, b\rangle = 0$, we obtain

$$\nabla^*_{\frac{\partial}{\partial t}} n = -(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)\frac{\partial}{\partial v} + \psi b \tag{18}$$

and

$$\nabla^*_{\frac{\partial}{\partial t}} b = -(g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)\frac{\partial}{\partial v} - \psi n, \tag{19}$$

which completes the proof.

Theorem 3.2. Let $\nabla^*_{\frac{\partial}{\partial t}} M = f\frac{\partial}{\partial v} + gn + hb$ be a smooth flow of the curve M in F^3. A flow of M in F^3 is inelastic if and only if the system of PDE holds:

$$\begin{aligned}
\nabla^*_{\frac{\partial}{\partial t}}\kappa &= \nabla^*_{\frac{\partial}{\partial s}}(f\kappa) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} g + \nabla^*_{\frac{\partial}{\partial s}}(h\tau) + g\tau^2 + \tau\nabla^*_{\frac{\partial}{\partial s}} h\\
\nabla^*_{\frac{\partial}{\partial t}}\tau &= -\kappa(g\tau + \nabla^*_{\frac{\partial}{\partial s}} h) + \nabla^*_{\frac{\partial}{\partial s}}\psi,\\
\kappa\psi &= \tau(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau) + \nabla^*_{\frac{\partial}{\partial s}}(g\tau) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} h
\end{aligned} \tag{20}$$

Proof. We know that $\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial t}}(\frac{\partial}{\partial v}) = \nabla^*_{\frac{\partial}{\partial t}}\nabla^*_{\frac{\partial}{\partial s}}(\frac{\partial}{\partial v})$. This expression is used, we get

$$\begin{aligned}
\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial t}}(\frac{\partial}{\partial v}) &= \nabla^*_{\frac{\partial}{\partial s}}[(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)n + (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)b]\\
&= [\nabla^*_{\frac{\partial}{\partial s}}(f\kappa) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} g + \nabla^*_{\frac{\partial}{\partial s}}(h\tau)]n + (f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)(-k\frac{\partial}{\partial v} + \tau b)\\
&\quad + [\nabla^*_{\frac{\partial}{\partial s}}(g\tau) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} h]b + (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)(\tau n),
\end{aligned} \tag{21}$$

while

$$\begin{aligned}
\nabla^*_{\frac{\partial}{\partial t}}\nabla^*_{\frac{\partial}{\partial s}}(\frac{\partial}{\partial v}) &= \nabla^*_{\frac{\partial}{\partial t}}(\kappa n)\\
&= n\nabla^*_{\frac{\partial}{\partial t}}\kappa + \kappa\nabla^*_{\frac{\partial}{\partial t}} n\\
&= n\nabla^*_{\frac{\partial}{\partial t}}\kappa + \kappa[-(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)\frac{\partial}{\partial v} + \psi b].
\end{aligned} \tag{22}$$

Thus we see that

$$\nabla^*_{\frac{\partial}{\partial t}}\kappa = \nabla^*_{\frac{\partial}{\partial s}}(f\kappa) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} g + \nabla^*_{\frac{\partial}{\partial s}}(h\tau) + g\tau^2 + \tau\nabla^*_{\frac{\partial}{\partial s}} h \tag{23}$$

and

$$\kappa\psi = \tau(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau) + \nabla^*_{\frac{\partial}{\partial s}}(g\tau) + \nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} h \tag{24}$$

Since $\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial t}} b = \nabla^*_{\frac{\partial}{\partial t}}\nabla^*_{\frac{\partial}{\partial s}} b,$ we have

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial t}} b &= \nabla^*_{\frac{\partial}{\partial s}}[-(g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)\frac{\partial}{\partial v} - \psi n] \\ &= -[\nabla^*_{\frac{\partial}{\partial s}}\nabla^*_{\frac{\partial}{\partial s}} h + \nabla^*_{\frac{\partial}{\partial s}}(g\tau)]\frac{\partial}{\partial v} - (g\tau + \nabla^*_{\frac{\partial}{\partial s}} h)\kappa n - (\nabla^*_{\frac{\partial}{\partial s}}\psi)n - \psi(-k\frac{\partial}{\partial v} + \tau b)\end{aligned} \tag{25}$$

while

$$\begin{aligned}\nabla^*_{\frac{\partial}{\partial t}}\nabla^*_{\frac{\partial}{\partial s}} b &= \nabla^*_{\frac{\partial}{\partial t}}(\tau n) \\ &= (\nabla^*_{\frac{\partial}{\partial t}}\tau)n + \tau[-(f\kappa + \nabla^*_{\frac{\partial}{\partial s}} g + h\tau)\frac{\partial}{\partial v} + \psi b]\end{aligned} \tag{26}$$

Thus we get

$$\nabla^*_{\frac{\partial}{\partial t}}\tau = -\kappa(g\tau + \nabla^*_{\frac{\partial}{\partial s}} h) + \nabla^*_{\frac{\partial}{\partial s}}\psi, \tag{27}$$

which completes the proof.

CONCLUSION

Inelastic time evolutions of curves and surfaces have an important role in computer vision, robotics and physical science. The aim of this study is to derive and study the inelastic flows of curves some spaces. We propose to study various situation for an inelastic flow of curve are given as a PDE inclusion of curvature and torsion of the curve. In this manuscript, we define inelastic flow of curves in a Finsler Manifold, this study can be shed light on these areas.

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