Journal of New Results in Science 13 (3) (2024) 175-185

Journal of New Results in Science
https://dergipark.org.tr/en/pub/jnrs
Research Article
Open Access

E-ISSN: 1304-7981 https://doi.org/10.54187/jnrs.1537253

Coerciveness and isomorphism of discontinuous Sturm-Liouville problems with transmission conditions

Mustafa Kandemir[1], Murat Küçük[2]

Keywords
BVP, Functional-transmission conditions, Isomorphism, Coerciveness, Solvability

Abstract — This study investigates a discontinuous Sturm-Liouville boundary value problem (BVP) on two intervals with functionals and transmission conditions in the direct sum of Sobolev spaces. Moreover, it presents the differential operator generated by the problem under investigation. The definition space of this operator is the direct sum of Sobolev spaces, and the value space of the operator is the space obtained by adding the complex spaces where the boundary conditions are evaluated about the direct sum of Sobolev spaces. This paper establishes the solvability of the problem and some important spectral properties of the operator, such as isomorphism, Fredholmness, and coerciveness concerning spectral parameters. In addition, the conclusion section discusses how different original problems can be produced.

Subject Classification (2020): 34A36, 34B08

1. Introduction

The discontinuous differential operator problems have recently drawn the attention of theoretical researchers due to their potential applications in physics. For example, discontinuous problems and additional transmission conditions are commonly seen in various disciplines, including solid mechanics, magnetostatics, and electrostatics [1–3]. Many researchers have investigated the solvability and certain spectral properties of nonlocal Sturm-Liouville problems [4–7]. In recent years, there has been a surge in interest in generalizing classical boundary value problems for ordinary linear differential equations because of its potential applications in physical sciences and applied mathematics.

The so-called functional boundary value problem is a significant specific case of the generalized boundary value problems. Numerous authors have addressed these issues [3, 5, 8, 9]. Some boundary-value transmission problems that arise while analyzing nonclassical problems cannot be resolved using typical methods for solving classical boundary-value problems. Boundary-value problems for ordinary differential equations are often studied in classical theory for equations with continuous coefficients and boundary conditions containing only the endpoints of the interval under consideration. This study, however, discusses one nonclassical boundary-value problem for a second-order ordinary differential equation with discontinuous coefficients and boundary conditions containing not only endpoints of the considered interval but also a point of discontinuity and linear functionals.

[1]mkandemir5@yahoo.com (Corresponding Author); [2]murat_kucuk_@hotmail.com
[1,2]Department of Mathematics, Faculty of Education, Amasya University, Amasya, Türkiye
Article History: Received: 22 Aug 2024 - Accepted: 10 Oct 2024 - Published: 31 Dec 2024

Namely, we consider a Sturm-Liouville diffraction defined on $[-1,0)\cup(0,1]$ given by

$$L(\lambda)u := \tau(x)u''(x) + (\sigma(x)-\lambda^2)u(x) = f(x) \tag{1.1}$$

with functional boundary-transmission conditions given by

$$L_k u := \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \delta_k u^{(m_k)}(1) + \mathcal{F}_k = f_k \tag{1.2}$$

for $k \in \{1,2,3,4\}$. Here, $\tau(x)$ is piecewise constant function such that for $x \in [-1,0)$, $\tau(x) = \tau_1$ where $\tau_1 \neq 0$ and for $x \in (0,1]$, $\tau(x) = \tau_2$ where $\tau_2 \neq 0$, λ complex parameter, for $i \in \{1,2\}$ and $k \in \{1,2,3,4\}$, τ_i, α_k, β_k, η_k, δ_k, and f_k are complex numbers, $f(x)$ is complex-valued function, for $k \in \{1,2,3,4\}$, m_k are nonnegative integers, and $\sigma(x)$ is integrable function on $[-1,0)\cup(0,1]$. We assume that $|\alpha_k|+|\beta_k|+|\eta_k|+|\delta_k| \neq 0$, and $\mathcal{F}_k$ is a linear functional in the space $L_q(-1,1)$. After applying the method of separation of variables to a variety of physical problems, such as heat and mass transfer problems [1, 3, 6, 10], diffraction problems [11], vibrating string problems (when the string loaded additional with point masses) [8, 12], and some special cases of the considered boundary value problem (1.1)-(1.2) arise. Yakubov [3] and Mukhtarov [6] investigated discontinuity problems with transmission conditions in mechanics. Triebel [13], Yakubov and Yakubov [14], Imanbaev and Sadybekov [15], Shakhmurov [16], Aliyev [17], and Rasulov [18] studied the various spectral properties of some nonlocal boundary-value problems for differential-operator equations. It should be noted that [4, 19–23] explored some novel problems with boundary values with nonlocal boundary conditions.

2. Preliminaries

This section presents some properties that are needed in the following sections.

Theorem 2.1. [24] Let $T \in (X,Y)$ be semi-Fredholm. If A is a T-compact operator from X to Y, then $S = T + A \in (X,Y)$ is also semi-Fredholm with $indS = indT$.

Note 2.2. [14] Consider an ordinary differential equation with constant coefficients and with weight 1 on the whole axis

$$L_0(\lambda)u := \lambda^m u(x) + \lambda^{m-1}u^{'}(x) + \cdots + a_m u^{(m)}(x) = f(x) \tag{2.1}$$

where a_k are complex numbers. Enumerate the roots of the equation

$$a_m\omega^m + a_{m-1}\omega^{m-1} + \cdots + 1 = 0 \tag{2.2}$$

by $\omega_j, j \in \{1,2,3,...,m\}$. Let numbers ω_j be $p-$separated. Denote

$$\underline{\omega} := \min\{\arg\omega_1, ..., \arg\omega_p, \arg\omega_{p+1}+\pi, ..., \arg\omega_m+\pi\}$$

$$\overline{\omega} := \max\{\arg\omega_1, ..., \arg\omega_p, \arg\omega_{p+1}+\pi, ..., \arg\omega_m+\pi\}$$

and the value $\arg\omega_j$ is chosen up to a multiple of 2π, so that $\overline{\omega} - \underline{\omega} < \pi$.

Theorem 2.3. [14] Let $m \geq 1, a_m \neq 0$ and the roots of (2.2) be p-separated. Then, for any $\varepsilon > 0$ and for all complex numbers λ satisfying $\frac{\pi}{2} - \underline{\omega} + \varepsilon < \arg\lambda < \frac{3\pi}{2} - \varpi - \varepsilon$, the operator $L_0(\lambda) : u \to L_0(\lambda)u$ from $W^l_{q,\gamma}(\mathbb{R})$ onto $W^{l-m}_{q,\gamma}(\mathbb{R})$,where an integer $l \geq m$, $q \in (1,\infty)$, $-\frac{1}{q} < \gamma < \frac{1}{q}$, is an isomorphism, and for these λ, the following estimates hold for a solution of (2.1):

$$\sum_{k=0}^{l} |\lambda|^{l-k} \|u\|_{W^k_{q,\gamma}(\mathbb{R})} \leq C(\varepsilon)\left(\|f\|_{W^{l-m}_{q,\gamma}(\mathbb{R})} + |\lambda|^{l-m}\|f\|_{L_{q,\gamma}(\mathbb{R})}\right), \upsilon = 1,2$$

$$\sum_{k=0}^{l} |\lambda|^{m-k} \left\|u^{(k+p)}\right\|_{L_{q,\gamma}(\mathbb{R})} \leq C(\varepsilon)\left\|f^{(p)}\right\|_{L_{q,\gamma}(\mathbb{R})}, 0 \leq p \leq l-m$$

Theorem 2.4. [14] Let the following conditions be satisfied:

i. $\{E_0, E_1\}$ is an interpolation couple

ii. γ_0, γ_1 are real numbers, $l \in \{1, 2, 3, ...\}$ and $1 \leq p_0,\ p_1 \leq \infty$

iii. s is integer number $0 \leq s \leq l-1$, $\gamma_0 + s + \frac{1}{p_0} > 0$, and $\gamma_1 + s + \frac{1}{p_1} < l$

iv. $\theta = \frac{\gamma_0+s+\frac{1}{p_0}}{l+\gamma_0-\gamma_1+\frac{1}{p_0}-\frac{1}{p_1}}$ and $\frac{1}{p} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$

Then, for $u \in W^l(p_0, \gamma_0, E_0; p_1, \gamma_1, E_1)$, the following inequality holds:

$$\left\| u^{(s)}(0) \right\|_{(E_0,E_1)_{\theta,p}} \leq C \left(\|u\|_{L_{p_0,\gamma_0}((0,1);E_0)} + \|u\|^{(l)}_{L_{p_1,\gamma_1}((0,1);E_1)} \right)$$

Theorem 2.5. [14] Under the conditions of Theorem 2.4, for $u \in W^l(p_0, \gamma_0, E_0; p_1, \gamma_1, E_1)$, $\mathbb{C}$, and $|\lambda| \to \infty$, the following inequalities hold:

i. $|\lambda|^{l-s} \left\| u^{(s)}(0) \right\|_{(E_0,E_1)_{\theta,p}} \leq C \left(|\lambda|^{l+\gamma_0+\frac{1}{p_0}} \|u\|_{Lp_0,\gamma_0}((0,1), E_0) + |\lambda|^{\gamma_1+\frac{1}{p_1}} \|u\|_{Lp_1,\gamma_1}((0,1), E_1) \right)$

ii. $|\lambda|^{l-s+\gamma+\frac{1}{p}} \|u^{(s)}\|_{L_{p,\gamma}((0,1);(E_0,E_1)_{\theta,p})} \leq C \left(|\lambda|^{l+\gamma_0+\frac{1}{p_0}} \|u\|_{Lp_0,\gamma_0}((0,1), E_0) + |\lambda|^{\gamma_1+\frac{1}{p_1}} \|u^l\|_{Lp_1,\gamma_1}((0,1), E_1) \right)$

where $\gamma > -\frac{1}{p}$.

3. Solvability and Coerciveness of Problems for Homogeneous Equation with Nonhomogeneous Boundary Transmission Conditions

This section first considers the following boundary value problem for the homogeneous differential equation

$$L_0(\lambda)u := \tau(x)u''(x) - \lambda^2 u(x) = 0 \tag{3.1}$$

together with the nonlocal and nonhomogeneous boundary conditions, for $k \in \{1, 2, 3, 4\}$,

$$L_{k0}u := \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \delta_k u^{(m_k)}(1) = f_k \tag{3.2}$$

For convenience, we use the following notations:

$$\omega_1 := -\tau_1^{-\frac{1}{2}}, \quad \omega_2 := \tau_1^{-\frac{1}{2}}, \quad \omega_3 := -\tau_2^{-\frac{1}{2}}, \quad \omega_4 = \tau_2^{-\frac{1}{2}}$$

$$\underline{\omega} := \min\{\arg\tau_1, \arg\tau_2\}, \quad \bar{\omega} := \max\{\arg\tau_1, \arg\tau_2\}$$

$$\theta := \begin{vmatrix} \alpha_1\omega_1^{m_1} & \beta_1\omega_2^{m_1} & \eta_1\omega_3^{m_1} & \delta_1\omega_4^{m_1} \\ \alpha_2\omega_1^{m_2} & \beta_2\omega_2^{m_2} & \eta_2\omega_3^{m_2} & \delta_2\omega_4^{m_2} \\ \alpha_3\omega_1^{m_3} & \beta_3\omega_2^{m_3} & \eta_3\omega_3^{m_3} & \delta_3\omega_4^{m_3} \\ \alpha_4\omega_1^{m_4} & \beta_4\omega_2^{m_4} & \eta_4\omega_3^{m_4} & \delta_4\omega_4^{m_4} \end{vmatrix}$$

and

$$B_\varepsilon(\underline{\omega}, \bar{\omega}) := \{\lambda \in \mathbb{C}\ :\ \pi + \bar{\omega} + \varepsilon < \arg\lambda < 3\pi + \underline{\omega} - \varepsilon\}$$

for real $\varepsilon > 0$ small enough.

For an integer $k \geq 0$ and real $q > 1$, the direct sum of Sobolev spaces $W_q^k(-1,0)\dot{+}W_q^k(0,1)$ is defined as Banach space of complex-valued functions $u = u(x)$ defined on $[-1,0) \cup (0,1]$ which belong to $W_q^k(-1,0)$ and $W_q^k(0,1)$ on intervals $(-1,0)$ and $(0,1)$, respectively, with the norm

$$\|u\|_{W_q^k(-1,0)\cup(0,1)} = \|u\|_{W_q^k(-1,0)} + \|u\|_{W_q^k(0,1)}$$

Here, $W_q^k(a,b)$ is a Sobolev space, i.e., the Banach space consisting of all the measurable functions $u(x)$

that have generalized derivatives up to k-th order inclusive on the interval (a,b) with the finite norm

$$\|u\|_{W_q^k(a,b)} = \sum_{i=0}^{k} \left(\int_a^b \left| u^{(i)}(x) \right|^q dx \right)^{\frac{1}{q}}$$

It can be observed that $W_q^0(a,b) = L_q(a,b)$.

Theorem 3.1. If $\theta \neq 0$, then for all $\varepsilon > 0$, there exists an $R_\varepsilon > 0$ such that for all $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$, for which $|\lambda| > R_\varepsilon$, (3.1)-(3.2) has a unique solution $u(x,\lambda)$ that belongs to $W_q^n(-1,0) \dot{+} W_q^n(0,1)$, for arbitrary $n \geq \max\{2, \max\{m_1, m_2, m_3, m_4\} + 1\}$, and for these λ's, the coercive estimate

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W_q^k(-1,0)\cup(0,1)} \leq C(\varepsilon) \sum_{j=1}^{4} |\lambda|^{n-m_j-\frac{1}{q}} |f_j| \tag{3.3}$$

is valid.

Proof. For $i \in \{1,2,3,4\}$, define four basic solutions $u_i = u_i(x,\lambda)$ of (3.1), given by

$$u_i(x,\lambda) = \begin{cases} e^{\omega_i \lambda (x-\xi_i)}, & x \in I_i \\ 0, & x \notin I_i \end{cases}$$

where $\xi_1 = -1$, $\xi_2 = \xi_3 = 0$, $\xi_4 = 1$, $I_1 = I_2 = [-1,0)$, and $I_3 = I_4 = (0,1]$. It can be observed that the general solution of (3.1) can be written in the form

$$u(x,\lambda) = \sum_{k=1}^{4} c_k u_k(x,\lambda) \tag{3.4}$$

Substituting (3.4) into (3.2) yields a system of linear homogeneous equations concerning the variables C_1, C_2, C_3, and C_4, given by

$$\begin{aligned} f_k &= C_1(\omega_1\lambda)^{m_k}(\alpha_k + \beta_k e^{\omega_1\lambda}) + C_2(\omega_2\lambda)^{m_k}(\alpha_k e^{-\omega_2\lambda} + \beta_k) + C_3(\omega_3\lambda)^{m_k}(\eta_k + \delta_k e^{\omega_3\lambda}) \\ &\quad + C_4(\omega_4\lambda)^{m_k}(\eta_k e^{-\omega_4\lambda} + \delta_k) \end{aligned} \tag{3.5}$$

such that $k \in \{1,2,3,4\}$. Since $\lambda \in B_\varepsilon(\underline{\omega}, \overline{\omega})$, it follows that

$$\frac{\pi + \varepsilon}{2} < \arg(\omega_i \lambda) < \frac{3\pi - \varepsilon}{2}, \quad i \in \{1,3\}$$

and

$$-\frac{\pi - \varepsilon}{2} < \arg(\omega_i \lambda) < \frac{\pi - \varepsilon}{2}, \quad i \in \{2,4\}$$

Consequently, for these λ's and for an arbitrary $\varepsilon > 0$ small enough,

$$(-1)^{k+1}\mathrm{Re}(\omega_k \lambda) \leq -|\lambda| |\omega_k| \sin\frac{\varepsilon}{2}, \quad k \in \{1,2,3,4\}$$

Hence, the determinant of (3.5) has the form

$$\begin{aligned} \Delta(\lambda) &= \lambda^{\sum\limits_{i=1}^{4} m_i} \left(\begin{vmatrix} \alpha_1\omega_1^{m_1} & \beta_1\omega_2^{m_1} & \eta_1\omega_3^{m_1} & \delta_1\omega_4^{m_1} \\ \alpha_2\omega_1^{m_2} & \beta_2\omega_2^{m_2} & \eta_2\omega_3^{m_2} & \delta_2\omega_4^{m_2} \\ \alpha_3\omega_1^{m_3} & \beta_3\omega_2^{m_3} & \eta_3\omega_3^{m_3} & \delta_3\omega_4^{m_3} \\ \alpha_4\omega_1^{m_4} & \beta_4\omega_2^{m_4} & \eta_4\omega_3^{m_4} & \delta_4\omega_4^{m_4} \end{vmatrix} + e^{\lambda \sum\limits_{i=1}^{4} (-1)^{i+1}\omega_i} \begin{vmatrix} \beta_1\omega_1^{m_1} & \alpha_1\omega_2^{m_1} & \delta_1\omega_3^{m_1} & \eta_1\omega_4^{m_1} \\ \beta_2\omega_1^{m_2} & \alpha_2\omega_2^{m_2} & \delta_2\omega_3^{m_2} & \eta_2\omega_4^{m_2} \\ \beta_3\omega_1^{m_3} & \alpha_3\omega_2^{m_3} & \delta_3\omega_3^{m_3} & \eta_3\omega_4^{m_3} \\ \beta_4\omega_1^{m_4} & \alpha_4\omega_2^{m_4} & \delta_4\omega_3^{m_4} & \eta_4\omega_4^{m_4} \end{vmatrix} \right) \\ &= \lambda^m (\theta + r(\lambda)) \end{aligned}$$

where $m = \sum\limits_{i=1}^{4} m_i$. It can be observed that $r(\lambda) \to 0$ if $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ and $|\lambda| \to \infty$. Since $\theta \neq 0$, there exists an $\iota_\varepsilon > 0$ such that for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$ and $|\lambda| > \iota_\varepsilon$, we have

$\Delta(\lambda) \neq 0$. Therefore, for these λ's, (3.5) has a unique solution

$$C_i(\lambda) = \frac{1}{\Delta(\lambda)} \sum_{k=1}^{4} \Delta_{ik}(\lambda) f_k, \quad i \in \{1,2,3,4\}$$

where $\Delta_{ik}(\lambda)$ denotes the algebraic complement of (i,k)-th element of the determinant $\Delta(\lambda)$. The determinant has the representation

$$\Delta_{ik}(\lambda) = (\theta_{ik} + r_{ik}(\lambda))\, \lambda^{\sum\limits_{j\neq k}(m-m_j)}$$

where θ_{ik} are complex numbers and $r_{ik} \to 0$ as $|\lambda| \to \infty$ in the angle $B_\varepsilon(\underline{\omega}, \overline{\omega})$. Then,

$$C_i(\lambda) = \sum_{k=1}^{4} \lambda^{-m_k} \frac{\theta_{ik} + r_{ik}(\lambda)}{\theta + r(\lambda)} f_k, i \in \{1,2,3,4\}$$

Thus, the solution of (3.1)-(3.2) has the form

$$u(x,\lambda) = \sum_{i=1}^{4}\sum_{k=1}^{4} \lambda^{-m_k} \frac{\theta_{ik} + r_{ik}(\lambda)}{\theta + r(\lambda)} f_k u_i(x,\lambda)$$

From this, it follows that for each integer $n \geq 0$

$$\left\| u^{(n)} \right\|_{L_q(-1,1)} \leq C \sum_{k=1}^{4} \left(|\lambda|^{(n-m_k)} |f_k| \sum_{i=1}^{4} \|u_i(.,\lambda)\|_{L_q(I_i)} \right) \tag{3.6}$$

Further, by (3.4),

$$\begin{aligned} \|u_1(x,\lambda)\|^q_{L_q(-1,0)} &= \int_{-1}^{0} e^{q\mathrm{Re}(\omega_1\lambda)(x+1)} dx \\ &\leq \int_{-1}^{0} e^{-q|\lambda||\omega_1|\sin(\varepsilon/2)(x+1)} dx \\ &= (-q\,|\lambda|\,|\omega_1| \sin(\varepsilon/2))^{-1} \left(e^{-q|\lambda||\omega_1|\sin(\varepsilon/2)} - 1 \right) \\ &\leq C(\varepsilon)\, |\lambda|^{-1} \end{aligned}$$

as $|\lambda| \to \infty$ in the angle $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. In a similar way,

$$\|u_i(x,\lambda)\|_{L_q(I_i)} \leq C(\varepsilon)\, |\lambda|^{-1}, \quad i \in \{2,3,4\}$$

as $|\lambda| \to \infty$ in the angle $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. Substituting these inequalities in (3.6) yields

$$\left\| u^{(n)} \right\|_{L_q(-1,1)} \leq C(\varepsilon) \sum_{k=1}^{4} |\lambda|^{(n-m_k-1/q)} |f_k|$$

which, in turn, provides us the needed estimation (3.3). □

4. Fredholmness of the Problem with General Functional-Transmission Conditions

This section investigates the property of the differential operator of the problem, a Fredholm operator. Let E and F be Banach spaces and F^* be the adjoint of F. The linear operator $T : E \to F$ is called a Fredholm operator if the following conditions are satisfied:

i. The range $R(T) = \{Tu : u \in D(T)\}$ is closed in F.

ii. $\ker T = \{u \in D(T) : Tu = 0\}$ and $\operatorname{coker} T = \{u^* \in F^* : u^*(Tu) = 0,$ for all thinspace $u \in D(T)\}$ are finite dimensional subspaces in E and F^*, respectively.

iii. $\dim\ker T = \dim\operatorname{coker} T$

Suppose that $n \geq \max\{2, \max\{m_1, m_2, m_3, m_4\} + 1\}$ and define a linear operator $\mathcal{L}$ from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ into $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1) + \mathbb{C}^4$ by action low

$$\mathcal{L} : u \longrightarrow \mathcal{L}u := (L(\lambda)u, L_1u, L_2u, L_3u, L_4u)$$

Theorem 4.1. Assume that the following conditions are satisfied:

i. $\tau(x) = \tau_1$ at $x \in [-1,0)$, $\tau(x) = \tau_2$ at $x \in (0,1]$, $\tau_1 \neq 0$, $\tau_2 \neq 0$, $m_k \geq 0$, and $\theta \neq 0$.

ii. $\sigma(x)$ is measurable function on $[-1,0) \cup (0,1]$.

iii. For $k \in \{1,2,3,4\}$, the functionals $\mathcal{F}_k$, in $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ are continuous.

Then, the linear operator

$$\mathcal{L} : u \longrightarrow \mathcal{L}u := (\tau(x)u''(x) + \sigma(x)u, L_1u, L_2u, L_3u, L_4u\ )$$

from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ onto $W_q^{n-2}(-1,d_1)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$ is bounded and Fredholm.

Proof. The operator $\mathcal{L}$ can be rewritten in the form $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where

$$\mathcal{L}_1u = (\tau(x)u''(x), u(-1), u'(-1), u(-0) - u(+0), u'(-0) - u'(+0))$$

and

$$\mathcal{L}_2u = (\sigma(x), L_1u - u(-1), L_2u - u'(-1), L_3u - u(-0) + u(+0), L_4u - u'(-0) + u'(+0))$$

Let $f \in L_q(-1,1)$. Then, from the condition i and $\frac{1}{p} + \frac{1}{q} = 1$, it follows that $\tau^{-1}(x)f(x) \in L_1(-1,1) \cap L_q(-1,1)$. By Schwartz inequality,

$$\begin{aligned}\int_0^1 \left|\tau^{-1}(x)f(x)\right| dx &\leq \left(\int_0^1 x^{-p}(x)dx\right)^{\frac{1}{p}} \left(\int_0^1 x^q(x)\left|f(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq C\, x^{\frac{1}{p}-1}\Big|_0^1 \left(\int_0^1 x^q(x)\left|f(x)\right|^q dx\right)^{\frac{1}{q}} \\ &\leq C\left\|f\right\|_{L_q(0,1)}\end{aligned} \tag{4.1}$$

Consequently, a solution of the problem

$$\tau(x)u''(x) = f(x), \quad x \in (-1,0) \cup (0,1), \quad u(-1) = h_1$$
$$u'(-1) = h_2, \quad u(-0) - u(+0) = h_3, \quad u'(-0) - u'(+0) = h_4$$

has the form

$$u(x) = \begin{cases} \int_{-1}^{x} (x-y)\tau^{-1}(y)f(y)dy + n_1 + n_2x, & x \in (-1,0) \\ \int_0^x (x-y)\tau^{-1}(y)f(y)dy + n_3 + n_4x, & x \in (0,1) \end{cases}$$

Accordingly, we obtain the solution to the problem as follows:

$$u(x) = \begin{cases} \int_{-1}^{x} (x-y)\tau^{-1}(y)f(y)dy + h_1 + (x+1)h_2, & x \in (-1,0) \\ \int_{-1}^{0} (x-y)\tau^{-1}(y)f(y)dy + \left(\int_{-1}^{0} (x-y)\tau^{-1}(y)f(y)dy + h_2 - h_4\right)x & x \in (0,1) \\ \quad + \int_0^x (x-y)\tau^{-1}(y)f(y)dy + h_1 + h_2 - h_3, & \end{cases} \tag{4.2}$$

If $f \in W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)$, then (4.2) implies $u''(x) = \tau^{-1}(x)f(x)$ and $u^{(t+2)}(x) = \tau^{-1}(x)f^{(t)}(x)$ such that $t \in \{0,1,2,...,n-2\}$.

Thus, from the condition i, (4.1), and Theorem 3.1, we obtain that the operator $\mathcal{L}_1$ from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ onto $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$ is isomorphism. Further, it can be observed that the linear operator $\mathcal{L}_2$ acts compactly from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ onto $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$. Consequently, we can apply Theorem 2.1 to the operator $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, from which it follows that the operator $\mathcal{L}$ is Fredholm. Besides, it is obvious that the operator $\mathcal{L}$ is bounded. □

5. Isomorphism and Coerciveness of the Principal Part of the Main Problem

Consider (1.1)-(1.2) without functionals, namely, the following problem

$$L_0(\lambda)u := \tau(x)u''(x) - \lambda^2 u(x) = f(x) \tag{5.1}$$

$$L_{k0}u := \alpha_k u^{(m_k)}(-1) + \beta_k u^{(m_k)}(-0) + \eta_k u^{(m_k)}(+0) + \delta_k u^{(m_k)}(1) = f_k, \quad k \in \{1,2,3,4\} \tag{5.2}$$

The operator corresponding to this problem is

$$\widetilde{\mathcal{L}}_0 u = (L_0(\lambda)u, L_{10}u, L_{20}u, L_{30}u, L_{40}u)$$

Theorem 5.1. Let $\theta \neq 0$ and $n \geq \max\{2, \max\{m_1,m_2,m_3,m_4\}+1\}$. Then, for all $\varepsilon > 0$, there exists an $\iota_\varepsilon > 0$ such that for all complex numbers λ satisfying $\frac{\pi}{2} + \frac{\overline{\omega}}{2} + \varepsilon < \arg\lambda < \frac{\pi}{2} + \frac{\underline{\omega}}{2} - \varepsilon$, $|\lambda| > \iota_\varepsilon$, the operator $\widetilde{\mathcal{L}}_0(\lambda)$ from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ onto $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$ is an isomorphism, and the following coercive estimate holds for the solution of (3.1)-(3.2)

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W_q^k} \leq C(\varepsilon)\left(\|f\|_{W_q^{n-2}} + |\lambda|^{n-2}\|f\|_{L_q} + \sum_{j=1}^{4} |\lambda|^{n-m_j-\frac{1}{q}} |f_j|\right) \tag{5.3}$$

Proof. It is obvious that the linear operator $\widetilde{\mathcal{L}}_0(\lambda)$ is continuous from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ into $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$.

Let $(f(x), f_1, f_2, f_3, f_4) \in W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$ be an element. We seek the solution $u(x,\lambda)$ of (5.1)-(5.2) in the form of the sum $u(x,\lambda) = u_1(x,\lambda) + u_2(x,\lambda)$ as follows. We denote the restriction of $f(x)$ on the interval I_j by $f_j(x)$ such that $j \in \{1,2\}$ where $I_1 = (-1,0)$ and $I_2 = (0,1)$. Let $\widetilde{f}_j \in W_q^{n-2}(\mathbb{R})$ be an extension of $f_j \in W_q^{n-2}(I_j)$ such that the extension operator $S_j f_j := \tilde{f}_j$ from $W_q^{n-2}(I_v)$ into $W_q^{n-2}(\mathbb{R})$ is bounded for $j \in \{1,2\}$ where $\mathbb{R} = (-\infty,\infty)$. First, consider the equations

$$\tau_j(x)\widetilde{u}''(x) - \lambda^2\widetilde{u}(x) = \widetilde{f}_j(x), \quad x \in \mathbb{R}$$

for $j \in \{1,2\}$. By Theorem 2.3, we have that this equation has a unique solution $\tilde{u}_{1v} = \tilde{u}_{1v}(.,\lambda) \in W_q^n(\mathbb{R})$ and for $u_{1j}(x,\lambda)$, the restriction of $\tilde{u}_{1j}(x,\lambda)$ on the interval I_j, the estimate

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u_{1j}\|_{W_q^k(I_j)} \leq C(\varepsilon)\left(\|f\|_{W_q^{n-2}(I_j)} + |\lambda|^{n-2}\|f\|_{L_q(I_j)}\right), \quad j \in \{1,2\} \tag{5.4}$$

is valid for all complex numbers λ satisfying $\lambda \in B_\varepsilon(\underline{\omega}, \bar{\omega})$. Consequently, $u_1(x,\lambda) \in W_q^n(-1,0)\dot{+}W_q^n(0,1)$ defined by

$$u_1(x,\lambda) = \begin{cases} u_{11}(x,\lambda), & x \in (-1,0) \\ u_{12}(x,\lambda), & x \in (0,1) \end{cases}$$

which satisfies (5.1). By using this solution, we construct the following boundary-value problem

$$\tau(x)u''(x) - \lambda^2 u(x) = 0, \quad x \in (-1,0) \cup (0,1)$$

$$L_{k0}u = f_k - L_{k0}u_1(.,\lambda), \quad k \in \{1,2,3,4\}$$

By Theorem 3.1, for all $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$, sufficiently large in modulus, this problem has a unique solution $u_2 = u_2(x,\lambda)$ that belongs to $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ and for these λ's the estimate

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u_2\|_{W_q^k} \le C(\varepsilon)\sum_{j=1}^{4} |\lambda|^{n-m_j-\frac{1}{q}} \left(|L_{j0}u_1| + |f_j|\right) \tag{5.5}$$

is hold. By Theorem 3.1 and considering Theorem 2.5, we have the following estimates, for all $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$ and $n \ge \max\{2, \max\{m_1,m_2,m_3,m_4\}+1\}$:

$$\begin{aligned} |\lambda|^{n-m_j-\frac{1}{q}} |L_{j0}u_1| &\le C\,|\lambda|^{n-m_j-\frac{1}{q}} \sum_{j=1}^{2} \|u_{1j}\|_{C^{m_j}(I_j)} \\ &\le C\left(\sum_{j=1}^{2} |\lambda|^n \|u_{1j}\|_{L_q(I_j)} + \|u_{1j}\|_{W_q^n(I_j)}\right) \\ &\le C(\varepsilon)\left(\|f\|_{W_q^{n-2}} + |\lambda|^{n-2}\|f\|_{Lq}\right) \end{aligned} \tag{5.6}$$

Thus, from (5.5) and (5.6),

$$\sum_{k=0}^{n} |\lambda|^{n-2} \|u_2\|_{W_q^k} \le C(\varepsilon)\left(\|f\|_{W_q^{n-2}} + |\lambda|^{n-2}\|f\|_{Lq} + \sum_{j=1}^{4} |\lambda|^{l-m_j-\frac{1}{q}} |f_j|\right) \tag{5.7}$$

Moreover, the function $u(x,\lambda)$ defined as $u(x,\lambda) = u_1(x,\lambda) + u_2(x,\lambda)$ is the solution of (5.1)-(5.2). Taking into account (5.4) and (5.7), for this solution, the needed estimation (5.3) is valid. Further, from (5.3), it follows the uniqueness of the solution. Besides, by Theorem 4.1, the operator $\widetilde{\mathcal{L}}_0$ is Fredholm operator from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ into $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$. Isomorphism of this operator follows from the fact that it is a Fredholm and a one-to-one operator. □

6. Solvability and Coerciveness of the Main Problem

This section researches the main problem (1.1)-(1.2).

Theorem 6.1. Let $\theta \ne 0$, $n \ge \max\{2, \max\{m_1,m_2,m_3,m_4\}+1\}$, and the functionals $\mathcal{F}_v$ in $W_q^{m_j}(-1,0)$ $\dot{+}W_q^{m_j}(0,1)$ be continuous. Then, for all $\varepsilon > 0$, there exists an $\iota_\varepsilon > 0$ such that for all complex numbers λ satisfying $\frac{\pi}{2} + \frac{\overline{\omega}}{2} + \varepsilon < \arg\lambda < \frac{\pi}{2} + \frac{\underline{\omega}}{2} - \varepsilon$, $|\lambda| > \iota_\varepsilon$, the operator

$$\widetilde{\mathcal{L}}(\lambda)u := (L(\lambda)u, L_1u, L_2u, L_3u, L_4u)$$

is an isomorphism from $W_q^n(-1,0)\dot{+}W_q^n(0,1)$ onto $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$, and for these λ's, the following coercive estimate holds for the solution of (1.1)-(1.2)

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W_q^k} \quad\le\quad C(\varepsilon)\left(\|f\|_{W_q^{n-2}} + |\lambda|^{n-2}\|f\|_{Lq} + \sum_{j=1}^{4} |\lambda|^{l-m_j-\frac{1}{q}} |f_j|\right) \tag{6.1}$$

where $C(\varepsilon)$ is a constant which depends only on ε.

Proof. Let $(f(x), f_1, f_2, f_3, f_4)$ be an element of $W_q^{n-2}(-1,0)\dot{+}W_q^{n-2}(0,1)\dot{+}\mathbb{C}^4$. Assume that there exists a solution $u = u(x,\lambda)$ of (1.1)-(1.2) corresponding to this element. Then, this solution satisfies the equalities

$$L_0(\lambda)u = L(\lambda)u - \sigma(x)u \tag{6.2}$$

and

$$L_{k0}u = L_ku - \mathcal{F}_ku, \quad k \in \{1,2,3,4\} \tag{6.3}$$

By applying Theorem 5.1 to (6.2)-(6.3), we have that for this solution the following a priory estimate is hold

$$\sum_{k=0}^{n} |\lambda|^{n-k} \|u\|_{W_q^k} \leq C(\varepsilon)\left(\|L(\lambda)u-\sigma(x)u\|_{W_q^k}+|\lambda|^{n-2}\|L(\lambda)u-\sigma(x)u\|_{Lq}+\sum_{j=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}|L_ju-\mathcal{F}_ju|\right)$$

$$\leq C(\varepsilon)\left(\|f\|_{W_q^{n-2}}+|\lambda|^{n-2}\|f\|_{Lq}+\|\sigma(x)u\|_{W_q^{n-2}}+|\lambda|^{n-2}\|\sigma(x)u\|_{Lq}+\sum_{\upsilon=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}|f_j| + \sum_{j=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}(|\mathcal{F}_ju|)\right) \quad (6.4)$$

In view of [14], for all $\zeta > 0$,

$$\|u\|_{W_q^k} \leq \zeta \|u\|_{W_q^{k+1}} + C(\zeta)\|u\|_{Lq} \quad (6.5)$$

By [25], for $u \in W_q^n(-1,0) \dotplus W_q^n(0,1)$, the following estimate holds

$$|\lambda|^{n-m_j-\frac{1}{q}}\|u\|^{(m_j)}_{C[-1,1]} \leq C\left(\|u\|_{W_q^k}+|\lambda|^n\|u\|_{L_q}\right) \quad (6.6)$$

From the conditions of Theorem 5.1, (6.5)-(6.6), and [14], it follows that

$$\|\sigma(x)u\|_{W_q^{n-2}}+|\lambda|^{n-2}\|\sigma(x)u\|_{Lq}+\sum_{j=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}(|\mathcal{F}_\upsilon u|)$$
$$\leq C(\varepsilon)\left(\|f\|_{W_q^{n-2}}+|\lambda|^{n-2}\|f\|_{Lq}\right)+\zeta\left(\|u\|_{W_q^n}+|\lambda|^{n-2}\|u\|_{Lq}\right)+\sum_{j=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}\|u\|_{W_q^k}$$
$$\leq C(\varepsilon)\left(\|f\|_{W_q^{n-2}}+|\lambda|^{n-2}\|f\|_{q,0}\right)+|\lambda|^{-\frac{1}{q}}\sum_{k=0}^{4}|\lambda|^{n-k}\|u\|_{W_q^k}$$

Here, we use the following inequality:

$$|\lambda|^{n-2}\|u\|_{Lq} \leq C|\lambda|^{-1}|\lambda|^{n-1}\|u\|_{W_q^k}$$

Substituting (6.6) into (6.4),

$$\sum_{k=0}^{n}|\lambda|^{n-2}\|u\|_{q,k} \leq C(\varepsilon)\left(\|f\|_{\|u\|_{W_q^{n-2}}}+|\lambda|^{n-2}\|f\|_{Lq}+\sum_{j=1}^{4}|\lambda|^{n-m_j-\frac{1}{q}}|f_j|\right)+|\lambda|^{-\frac{1}{q}}\sum_{k=0}^{4}|\lambda|^{n-k}\|u\|_{\|u\|_{W_q^k}}$$

Thus, for $\lambda \in B_\varepsilon(\underline{\omega},\bar{\omega})$ sufficiently large in modulus, we obtain a priori estimate (6.1). From this estimate, it follows the uniqueness property of the solution of (1.1)-(1.2), i.e., the operator $\widetilde{\mathcal{L}}(\lambda)$ is a one-to-one operator. Moreover, by Theorem 4.1, the operator $\widetilde{\mathcal{L}}(\lambda)$ from $W_q^n(-1,0)\dotplus W_q^n(0,1)$ into $W_q^{n-2}(-1,0)\dotplus W_q^{n-2}(0,1)\dotplus\mathbb{C}^4$ is Fredholm. Consequently, the existence of a solution results in its uniqueness. □

7. Conclusion

In this paper, the Sturm-Liouville boundary value problem with discontinuous coefficient differential equations and the transition conditions of the discontinuity point in the boundary conditions and the functional are considered. The solvability of this problem and the spectral properties of the differential operator belonging to the problem, such as coerciveness, isomorphism, and being a Fredholm operator, are investigated. Then, theorems related to the spectral properties of this problem are proved. The problem in this study can be constructed in different ways, such as adding a linear operator or an elliptic operator to the differential equation, putting interior points in the boundary conditions, taking the differential equation to a higher order, taking more than one discontinuity point. Moreover, for each of the mentioned problems, the subject of finding the asymptotic distributions of the eigenvalues of the problem can be studied. Since each of the aforesaid problems will be original problems, completely new theses, and new articles can be derived from each.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] I. Titeux, Y. Yakubov, *Comletenes of root functions for the thermal condition in a strip with piecewise continuous coefficients*, Mathematical Models and Methods in Applied Sciences 7 (7) (1997) 1035-1050.

[2] M. Kandemir, O. Sh. Mukhtarov, *Many point boundary value problems for elliptic differential-operator equations with interior singularities*, Mediterranean Journal of Mathematics 17 (2020) Article Number 35 21 pages.

[3] O. Muhtarov, S. Yakubov, *Problems for ordinary differential equations with transmission conditions*, Applicable Analysis 81 (2002) 1033-1064.

[4] M. Kandemir, O. Sh. Mukhtarov, *Nonlocal Sturm-Liouville problems with integral terms in the boundary conditions*, Electronic Journal of Differential Equations 2017 (2017) Article Number 11 12 pages.

[5] M. Kandemir, Y.Yakubov, *Regular boundary value problems with a discontinuous coefficient, functional-multipoint conditions and a linear spectral parameter*, Israel Journal of Mathematics 180 (1) (2010) 255-270.

[6] O. Sh. Mukhtarov, *Discontinuous boundary value problem with spectral parameter in boundary conditions*, Turkish Journal of Mathematics 18 2 (1994) 183-192.

[7] O. Sh. Mukhtarov, K. Aydemir, *Eigenfunction expansion for Sturm-Liouville problems with transmission conditions at one interior point*, Acta Mathematica Scientia 35 (3) (2015) 639-649.

[8] A. V. Likov, Y. A. Mikhailov, The theory of heat and mass transfer, Qosenergaizdat (Russian), 1963.

[9] M. Kandemir, O. Sh. Mukhtarov, Y. Yakubov, *Irregular boundary value problems with discontinuous coefficients and the eigenvalue parameter*, Mediterranean Journal of Mathematics 6 (2009) 317-338.

[10] M. S. Alves, J. E. M. Rivera, M. Sepulveda, O. V. Villagran,*Transmission problem in thermoelasticity*, Hindawi 2011 (2011) Article Number 190548 33 pages.

[11] M. S. Agranovich, *Spectral properties of diffraction problems*, in: N. N. Voitovich, B. Z. Kazenelenbaum, A. N. Sivov (Eds.), The Generalized Method of Eigenoscillations in the Theory of Diffraction, Nauka, Moscow, 1977 (in Russian: translated into English Wiley-VCH, Berlin 1999).

[12] A. Stikonas, *The Sturm-Liouville problem with a nonlocal boundary condition* Lithuanian Mathematical Journal, 47 (3) (2007) 336-351.

[13] H. Triebel, Interpolation theory function spaces differential operators, in: M. Artin, H. Bass, J. Eells, W. Feit, P. J. Freyd, F. W. Gehring, H. Halberstam, L. V, Hormander, M. Kac, J. H. B. Kemperxuan, H. A. Lauwerier, W. A. J. Luxemburg, F, P. Peterson, I. M. Singer, and A. C. Zaanen (Eds.), Vol. 18, Nort Holland, Amsterdam, 1978.

[14] S. Yakubov, Y. Yakubov, Differential-operator equation ordinary and partial differential equation, in: A. Jeffrey, H. Brezis, R. G. Douglas (Eds.), Monographs and Surveys in Pure and Applied Mathematics, 1st Edition, Chapman and Hall/CRC, Boca Raton London, 1999.

[15] N. S. Imanbaev, M. A. Sadybekov, *Characteristic determinant of the spectral problem for the ordinary differential operator with the boundary load*, in: A. Ashyralyev, E. Malkowsky (Eds.), International Conference on Analysis and Applied Mathematics, Shymkent, 2014 - AIP Conference Proceedings 1611 (1) (2014) 261-265.

[16] V. B. Shakhmurov, *Linear and nonlinear abstract elliptic equations with VMO coefficients and applications*, Fixed Point Theory and Applications 2013 (2013) Article Number 6 21 pages.

[17] Z. S. Aliyev, *Basic properties of fourth order differential operator with spectral parameter in the boundary condition*, Central European Journal of Mathematics 8 (2) (2010) 378-388.

[18] M. L. Rasulov, Application of contour integral method (in Russian), Navka, Moskow, 1997.

[19] K. Aydemir, *Boundary value problems with eigenvalue depending boundary and transmission conditions*, Boundary Value Problems 2014 (2014) Article Number 131 11 pages.

[20] K. Aydemir, O. Sh. Mukhtarov, *Specturm and Green's function of a many interval Sturm-Liouville problem*, Z. Naturforsch 70 (5) (2015) 301-308.

[21] M. Kadakal, O. Sh. Mukhtarov, *Discontinuous Sturm-Liouville problems containing eigenparameter in the boundary conditions*, Acta Mathematica Sinica 22 (5) (2006) 1519-1528.

[22] M. Kandemir, *Irregular boundary value problems for elliptic differential-operator equations with discontinuous coefficients and transmission conditions*, Kuwait Journal of Science and Engineering 39 (1) (2012) 71-97.

[23] M. A. Sadybekov, B. Kh. Turmetov, B. T. Torebek, *Solvability of nonlocal boundary-value problems for the Laplace equation in the ball*, Electronic Journal of Differential Equations 2014 (2014) Article Number 157 14 pages.

[24] T. Kato, Perturbation theory for linear operators, Classics in Mathematics, Sipringer-Verlag, New York, 1966.

[25] O. V. Besov, V. P. Il'in, S. M. Nikol'skii, *Integral representation of functional and imbedding theorems*, V. H. Winston, the University of Michigan, New York, 1978.