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AN ANALYSIS ON THE SHAPE-PRESERVING CHARACTERISTICS OF $\lambda-$SCHURER OPERATORS

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Abstract. This study investigates the shape-preserving characteristics of λ-Schurer operators, a class of operators derived from a modified version of the classical Schurer bases by incorporating a shape parameter λ. The primary focus is on understanding how these operators maintain the geometric features of the functions they approximate, which is crucial in fields like computer graphics and geometric modelling. By examining the fundamental properties and the divided differences associated with λ-Schurer bases, we derive vital results that confirm the operators' capability to preserve essential shape attributes under various conditions. The findings have significant implications for the application of these operators in computational analysis and other related areas, providing a solid foundation for future research.

1. Introduction

In recent years, the study of shape-preserving approximation methods has gained significant attention due to their critical role in applications such as computer graphics, CAD modelling, and numerical analysis. Shape-preserving operators ensure that the essential geometric features of functions, such as monotonicity and convexity, are maintained during approximation [1]. Bézier bases have become particularly popular among these methods due to their ability to offer smooth and continuous approximations with limited control points [6, 11].

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In 2010, Ye et al. [12] established a new class of bases, so-called Bézier bases, based on shape parameters λ chosen from the interval $[-1,1]$. Bézier bases are fundamental in approximation methods that aim to preserve shapes, playing a crucial role in computer graphics and geometric modelling. The recent works related to some shape parameters including λ are given as: In their exploration of the modified λ -Bernstein-polynomial, Ayman-Mursaleen et al. [7] thoroughly analyzed its approximation properties, providing valuable insights into its behavior and potential applications. Su et al. [21] conducted a rigorous analysis of the shape-preserving properties of λ -Bernstein operators, demonstrating their ability to maintain crucial geometric characteristics such as monotonicity and convexity during the approximation process. Ansari et al. [2] delved into the approximation properties of bivariate Bernstein-Kantorovich operators, extending their application by incorporating a summability method and establishing connections with related GBS operators. Kajla et al. [14] introduced the innovative Bézier-Baskakov-Beta type operators, a novel class designed to enhance shape-preserving approximation and offer improved flexibility in controlling the geometric features of the approximated function. Rao et al. [18] investigated the approximation capabilities of modified Baskakov-Durrmeyer operators, focusing on the influence of a shape parameter α on their ability to represent complex functions while preserving their fundamental geometric properties accurately. Özger et al. [17] examined the convergence behaviour of generalized blending-type Bernstein-Kantorovich operators, establishing the rate of weighted statistical convergence and providing a deeper understanding of their approximation characteristics.

Bézier bases provide a mathematical framework that ensures smoothness and continuity, making them ideal for accurately approximating complex shapes like fonts, logos, and CAD models. Bézier bases allow for precise control over curve shapes with a limited number of control points, giving designers and engineers the flexibility to fine-tune approximations while maintaining the integrity of the original shape. This ability to preserve essential features during the approximation process highlights the importance of Bézier bases in achieving visually and geometrically accurate representations. Due to all these facts these bases have become prevalent among researchers, and there have been many variations of Bézier bases inaugurated to the literature (see [4, 8, 13]).

Schurer [19] introduced a remarkable variation of the classical Bernstein operators by incorporating a nonnegative parameter ϑ, which is both linear and positive. Most recently, Özger [16] constructed a modified version of these bases, namely $\lambda-$Schurer bases, as follows: For shape parameter $\lambda \in [-1,1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer bases are

$$\begin{aligned}
\widehat{s}_{r,0}(\lambda;\tau) &= s_{r,0}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,1}(\tau), \\
\widehat{s}_{r,p}(\lambda;\tau) &= s_{r,p}(\tau) + \lambda \left\{ \frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1} s_{r+1,p}(\tau) \right. \\
&\qquad \left. - \frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1} s_{r+1,p+1}(\tau) \right\}, \qquad p = 1,2,\ldots,r+\vartheta-1, \\
\widehat{s}_{r,r+\vartheta}(\lambda;\tau) &= s_{r,r+\vartheta}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,r+\vartheta}(\tau),
\end{aligned} \tag{1}$$

where $s_{r,p}(\tau)$ are the fundamental Schurer bases of degree $r+\vartheta$ defined as

$$s_{r,p}(\tau) = \binom{r+\vartheta}{p} \tau^p (1-\tau)^{r+\vartheta-p}, \qquad p = 0,1,\ldots,r+\vartheta. \tag{2}$$

Then using the $\lambda-$Schurer bases given in (1), Özger established the $\lambda-$Schurer operators $S^{\lambda}_{r,\vartheta}(g;\tau) : C[0,1+\vartheta] \to C[0,1]$

$$S^{\lambda}_{r,\vartheta}(g;\tau) = \sum_{p=0}^{r+\vartheta} \widehat{s}_{r,p}(\lambda;\tau)\, g\left(\frac{p}{r}\right), \quad \tau \in [0,1], \ r \in \mathbb{N}, \tag{3}$$

for any g in $C[0,1+\vartheta]$. In [16], the statistical convergence properties of operators in (3) is examined, and an estimation for the rate of weighted A-statistical convergence is provided. Furthermore, two Voronovskaja-type theorems are established, one of which employs weighted A-statistical convergence.

Building on the foundational work of Ye et al. [12] on Bézier bases, this paper explores the $\lambda-$Schurer operators in (3), a variation introduced by Özger given in (3), which extends the classical Schurer operators by incorporating a shape parameter λ. These operators are designed to provide more flexibility in controlling the shape of the approximated function, making them a powerful tool for shape-preserving approximation. The primary objective of this study is to analyze the shape-preserving properties of these operators and to establish their effectiveness through rigorous mathematical proofs and computational analysis. The manuscript is organized as follows: Section 2 covers the fundamental concepts of fundamental Schurer bases, divided differences, as well as the notions of $0-$convex, $1-$convex, and $2-$convex functions, including the relevant relationships and results. Section 3 presents the primary theoretical, computational, and numerical results and discussions regarding the shape-preserving properties of $\lambda-$Schurer operators. In the last section, we provide an elaborate conclusion.

2. Auxiliary Results

In this section, we give the fundamental properties of the $\lambda-$Schurer bases and some essentials on the divided differences. We commence our work by providing the binomial coefficient formula as

$$\binom{r}{p} = \begin{cases} \frac{r!}{p!(r-p)!}, & 0 \le p \le r, \\ 0, & \text{otherwise.} \end{cases}$$

In the next lemma, we give some basic properties of $s_{r,p}(\tau)$, such as, recursive relation, degree raising, derivative formula and endpoint interpolating properties.

Lemma 1. *For integer $\vartheta \ge 0$, the fundamental Schurer bases $s_{r,p}(\tau)$ in (2) satisfy the following identities:*

$$s_{r,p}(\tau) = 0 \quad \textit{if} \quad p > r+\vartheta \quad \textit{or} \quad p < 0, \tag{4}$$

$$s_{r,p}(\tau) = (1-\tau)\, s_{r-1,p}(\tau) + \tau s_{r-1,p-1}(\tau), \tag{5}$$

$$s_{r,p}(\tau) = \left(1 - \tfrac{p}{r+\vartheta+1}\right) s_{r+1,p}(\tau) + \left(\tfrac{p+1}{r+\vartheta+1}\right) s_{r+1,p+1}(\tau), \tag{6}$$

$$\frac{d}{d\tau}\left[s_{r,p}(\tau)\right] = (r+\vartheta)\left[s_{r-1,p-1}(\tau) - s_{r-1,p}(\tau)\right], \tag{7}$$

and

$$s_{r,p}(0) = \begin{cases} 0 & \textit{if} \;\; p \ne 0, \\ 1 & \textit{if} \;\; p = 0, \end{cases} \qquad s_{r,p}(1) = \begin{cases} 0 & \textit{if} \;\; p \ne r+\vartheta, \\ 1 & \textit{if} \;\; p = r+\vartheta. \end{cases} \tag{8}$$

Proof. The proof of (4) and (8) are a direct consequence of definitions of the binomial coefficient and $s_{r,p}(\tau)$ in (2), so they are omitted. To prove (5), we only apply basic algebra to the definition (2) of Schurer polynomials, which yields

$$\begin{aligned}(1-\tau)\, s_{r-1,p}(\tau) + \tau s_{r-1,p-1}(\tau) &= (1-\tau)\tbinom{r+\vartheta-1}{p}\tau^p (1-\tau)^{r+\vartheta-1-p} \\ &\quad + \tau\tbinom{r+\vartheta-1}{p-1}\tau^{p-1}(1-\tau)^{(r+\vartheta-1)-(p-1)} \\ &= \left[\tbinom{r+\vartheta-1}{p} + \tbinom{r+\vartheta-1}{p-1}\right]\tau^p (1-\tau)^{r+\vartheta-p}.\end{aligned}$$

Since $\binom{r+\vartheta-1}{p} + \binom{r+\vartheta-1}{p-1} = \binom{r+\vartheta}{p}$, we then have the desired result. Next to prove (6), we first note that

$$\begin{aligned}\tau s_{r,p}(\tau) &= \tbinom{r+\vartheta}{p}\tau^{p+1}(1-\tau)^{r+\vartheta-p} \\ &= \frac{\binom{r+\vartheta}{p}}{\binom{r+\vartheta+1}{p+1}}\tbinom{r+\vartheta+1}{p+1}\tau^{p+1}(1-\tau)^{(r+\vartheta+1)-(p+1)} \\ &= \left(\tfrac{p+1}{r+\vartheta+1}\right) s_{r+1,p+1}(\tau),\end{aligned} \tag{9}$$

and also

$$\begin{aligned}(1-\tau)\, s_{r,p}(\tau) &= \tbinom{r+\vartheta}{p}\tau^p(1-\tau)^{r+\vartheta+1-p} \\ &= \frac{\binom{r+\vartheta}{p}}{\binom{r+\vartheta+1}{p}}\tbinom{r+\vartheta+1}{p}\tau^p(1-\tau)^{(r+\vartheta+1)-p} \\ &= \left(1 - \tfrac{p}{r+\vartheta+1}\right) s_{r+1,p}(\tau).\end{aligned} \tag{10}$$

Subsequently, summation of (9) and (10) yields property (6). Lastly, by taking the derivative of $s_{r,p}(\tau)$ with respect to τ by means of basic algebra rules, we obtain the property (7) as

$$\begin{aligned}\frac{d}{d\tau}\left[s_{r,p}(\tau)\right] &= \tbinom{r+\vartheta}{p}p\tau^{p-1}(1-\tau)^{r+\vartheta-p} - \tbinom{r+\vartheta}{p}(r+\vartheta-p)\tau^{p}(1-\tau)^{r+\vartheta-1-p} \\ &= (r+\vartheta)\Big[\tfrac{(r+\vartheta-1)!}{(p-1)!(r+\vartheta-p)!}\tau^{p-1}(1-\tau)^{(r+\vartheta-1)-(p-1)} \\ &\qquad -\tfrac{(r+\vartheta-1)!}{p!(r+\vartheta-1-p)!}\tau^{p}(1-\tau)^{(r+\vartheta-1)-p}\Big] \\ &= (r+\vartheta)\left[s_{r-1,p-1}(\tau) - s_{r-1,p}(\tau)\right].\end{aligned}$$

□

The following lemma will present some auxiliary results that are essential for our main outcomes.

Lemma 2. *For $\lambda \in [-1,1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer bases in (1) satisfy the following properties:*

$$\widehat{s}_{r,p}(\lambda;\tau) \geq 0, \tag{11}$$

$$\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}(\lambda;\tau) = 1, \tag{12}$$

$$\widehat{s}_{r,p}(\lambda;\tau) = \tilde{s}_{r,r-p}(\lambda;1-\tau). \tag{13}$$

Proof. In order to prove property (11), we first note that $s_{r,p}(\tau) \geq 0$ for all $r \in \mathbb{N}$ and $\tau \in [0,1]$ where $\vartheta \geq 0$ is integer by definition of the binomial coefficient formula. Next, we rewrite $\lambda-$Schurer bases given in (1) as

$$\begin{aligned}\widehat{s}_{r,p}(\lambda;\tau) = \frac{1}{r+\vartheta+1}\Bigg\{&\left(p+1-\lambda\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)s_{r+1,p+1}(\tau) \\ &+\left(r+\vartheta+1-p+\lambda\frac{r+\vartheta-2p+1}{r+\vartheta-1}\right)s_{r+1,p}(\tau)\Bigg\},\end{aligned}$$

by employing degree raising property (6). Since $1 \leq p \leq r+\vartheta-1$, one can easily find that $0 \leq \frac{p-1}{r+\vartheta-1} \leq 1 - \frac{1}{r+\vartheta-1} \leq 1$. Then utilizing the fact $-1 \leq \lambda \leq 1$ yields $-1 \leq \lambda\left(1-\frac{2(p-1)}{r+\vartheta-1}\right) \leq 1$. Subsequently, we get

$$0 \leq r+\vartheta-p \leq r+\vartheta+1-p+\lambda\frac{r+\vartheta-2p+1}{r+\vartheta-1}. \tag{14}$$

Analogously, one can derive $-1 \leq \lambda\left(1-\frac{2p}{r+\vartheta-1}\right) \leq 1$ which implies

$$0 \leq (p+1)-1 \leq p+1-\lambda\frac{r+\vartheta-2p-1}{r+\vartheta-1}. \tag{15}$$

Hence, we have $\widehat{s}_{r,p}(\lambda;\tau) \geq 0$ by (14) and (15). The proof of partition of unity property (12) is given in [16], and symmetry property is a direct consequence of definitions (1)-(2), so they are omitted. □

The following divided differences definition and subsequent results are presented on the grounds of the pioneering work by Asher and Greif [3].

Definition 1 ([3])**.** *Given points $\tau_0, \tau_1, \ldots, \tau_r$ with arbitrary indices $0 \leq q < p \leq r$, the divided difference of a function g with order r is defined by*

$$[\tau_0, \tau_1, \ldots, \tau_r; g] = \sum_p g(\tau_p) \prod_{q \neq p} \frac{1}{(\tau_p - \tau_q)}.$$

The divided differences of g are linear and symmetric and satisfy the recursive formula

$$\begin{aligned} [\tau_0; g] &= g(\tau_0) \\ [\tau_0, \ldots \tau_r; g] &= \frac{[\tau_1, \ldots \tau_r; g] - [\tau_0, \ldots \tau_{r-1}; g]}{\tau_r - \tau_0}. \end{aligned}$$

By recursive formula, for $0 \leq q \leq r$, we have the following identities:

$$\begin{aligned} [\tau_q; g] &= g(\tau_q), \\ [\tau_q, \tau_{q+1}; g] &= \frac{g(\tau_{q+1}) - g(\tau_q)}{\tau_{q+1} - \tau_q}, \\ [\tau_q, \tau_{q+1}, \tau_{q+2}; g] &= \frac{[\tau_{q+1}, \tau_{q+2}; g] - [\tau_q, \tau_{q+1}; g]}{\tau_{q+2} - \tau_q}. \end{aligned}$$

Lemma 3 ([15])**.** *For a fixed $r \in \mathbb{N}$, the function g is called $r-$convex if $[\tau_0, \tau_1, \ldots, \tau_r; g] \geq 0$. In particular, if function g is*

- **i:** *nonnegative, then it is $0-$convex,*
- **ii:** *nondecreasing, then it is $1-$convex,*
- **iii:** *convex in the usual sense, then it is $2-$convex.*

3. Primary Results on the Shape-Preserving Characteristics of $\lambda-$Schurer Operators

This part is dedicated to the main results of the manuscript. We will present our findings on the positivity, linearity, endpoint preservation, monotonicity and convexity of $\lambda-$Schurer operators $S_{r,\vartheta}^{\lambda}(g;\tau)$. We commence our work by representing $S_{r,\vartheta}^{\lambda}(g;\tau)$ in terms of fundamental Schurer bases $s_{r,p}(\tau)$ in (2) and divided differences.

Lemma 4. *For any $\lambda \in [-1,1]$ and integer $\vartheta \geq 0$, the $\lambda-$Schurer operators in (3) can be rewritten as*

$$S_{r,\vartheta}^{\lambda}(g;\tau) = B_{r,\vartheta}(g;\tau) + \frac{\lambda}{r} \sum_{p=0}^{r+\vartheta-1} \left(\frac{r+\vartheta-2p-1}{(r+\vartheta)^2 - 1} \right) s_{r+1,p+1}(\tau) \left[\tfrac{p}{r}, \tfrac{p+1}{r}; g\right], \quad (16)$$

where $s_{r,p}(\tau)$ are as in (2) and

$$B_{r,\vartheta}(g;\tau) = \sum_{p=0}^{r+\vartheta} s_{r,p}(\tau)\left[\tfrac{p}{r};g\right],$$

are the Bernstein-Schurer operators constructed in [19].

Proof. Substitution of (1) to the expression (3) of $\lambda-$Schurer operators yields

$$\begin{aligned} S_{r,\vartheta}^{\lambda}(g;\tau) = & \left[s_{r,0}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,1}(\tau)\right] g(0) \\ & + \sum_{p=1}^{r+\vartheta-1}\left[s_{r,p}(\tau) + \lambda\left(\frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1} s_{r+1,p}(\tau)\right.\right. \\ & \left.\left. - \frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1} s_{r+1,p+1}(\tau)\right)\right] g\left(\frac{p}{r}\right) \\ & + \left[s_{r,r+\vartheta}(\tau) - \frac{\lambda}{r+\vartheta+1} s_{r+1,r+\vartheta}(\tau)\right] g\left(\frac{r+\vartheta}{r}\right), \end{aligned}$$

which can also be written as

$$\begin{aligned} S_{r,\vartheta}^{\lambda}(g;\tau) = & \sum_{p=0}^{r+\vartheta} s_{r,p}(\tau)\, g\left(\frac{p}{r}\right) - \lambda \sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{(r+\vartheta)^2-1}\right) s_{r+1,p+1}(\tau)\, g\left(\frac{p}{r}\right) \\ & + \lambda \sum_{p=1}^{r+\vartheta}\left(\frac{r+\vartheta-2p+1}{(r+\vartheta)^2-1}\right) s_{r+1,p}(\tau)\, g\left(\frac{p}{r}\right), \end{aligned}$$

after simplifying similar terms. Reindexing the last summation in the above equation and then utilizing the notation of divided differences given in Definition 1, we obtain the desired result in (16). $\square$

Remark 1. *In the special case $\vartheta = 0$ and $p \to p-1$ in (16), we get equation (6) in [21].*

Now, we are ready to present our principal conclusions on the shape-preserving properties of the $\lambda-$Schurer operators. The following theorem is on the geometric properties of $S_{r,\vartheta}^{\lambda}(g;\tau)$, such as nonnegativity, linearity and endpoint interpolation.

Theorem 1. *Let $\lambda \in [-1,1]$, $r \in \mathbb{N}$, and $\vartheta \geq 0$ integer. The $\lambda-$Schurer operators in (3) satisfy the following properties:*

i: *Nonnegativity: For $g \in C[0,1+\vartheta]$, $S_{r,\vartheta}^{\lambda}(g;\tau) \geq 0$ whenever $g(\tau) \geq 0$.*

ii: *Linearity: For $g_1, g_2 \in C[0,1+\vartheta]$ and $\beta_1, \beta_2 \in \mathbb{R}$,*

$$S_{r,\vartheta}^{\lambda}(\beta_1 g_1 + \beta_2 g_2;\tau) = \beta_1 S_{r,\vartheta}^{\lambda}(g_1;\tau) + \beta_2 S_{r,\vartheta}^{\lambda}(g_2;\tau).$$

iii: *Endpoint interpolation: $S_{r,\vartheta}^{\lambda}(g;0) = [0;g]$.*

Proof. We begin our work by writing $\lambda-$Schurer operators in (3) in terms of divided differences as

$$S_{r,\vartheta}^{\lambda}\left(g;\tau\right)=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}\left(\lambda;\tau\right)\left[\tfrac{p}{r};g\right].$$

For the proof of part (**i**), assume that $g\left(\tau\right)\geq 0$. Consequently, we have $S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\geq 0$ by (11) and Lemma 3. Next, by the linearity of the divided differences and summation operator, we obtain

$$\begin{aligned}S_{r,\vartheta}^{\lambda}\left(\beta_1 g_1+\beta_2 g_2;\tau\right)&=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}\left(\lambda;\tau\right)\left[\tfrac{p}{r};\beta_1 g_1+\beta_2 g_2\right]\\&=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}\left(\lambda;\tau\right)\left(\beta_1\left[\tfrac{p}{r};g_1\right]+\beta_2\left[\tfrac{p}{r};g_2\right]\right)\\&=\beta_1 S_{r,\vartheta}^{\lambda}\left(g_1;\tau\right)+\beta_2 S_{r,\vartheta}^{\lambda}\left(g_2;\tau\right),\end{aligned}$$

which completes the proof of part (**ii**). Lastly, for part (**iii**), substitution of (8) in (1) yields

$$\widehat{s}_{r,p}\left(\lambda;0\right)=\begin{cases}0 & \text{if } p\neq 0\\ 1 & \text{if } p=0\end{cases},$$

which consequently implies

$$S_{r,\vartheta}^{\lambda}\left(g;0\right)=\sum_{p=0}^{r+\vartheta}\widehat{s}_{r,p}\left(\lambda;0\right)\left[\tfrac{p}{r};g\right]=\widehat{s}_{r,0}\left(\lambda;0\right)\left[0;g\right]+\sum_{p=1}^{r+\vartheta}\widehat{s}_{r,p}\left(\lambda;0\right)\left[\tfrac{p}{r};g\right]=\left[0;g\right].$$

□

Prior to the presentation of our primary findings on the monotonicity preservation of $\lambda-$Schurer operators, we will present the first derivative of these operators in the following lemma.

Lemma 5. *For any* $\lambda\in[-1,1]$ *and* $g:[0,1+\vartheta]\to\mathbb{R}$, $\vartheta\geq 0$ *integer, the* $\lambda-$*Schurer operators in (3) satisfy the following identity*

$$\begin{aligned}\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]=\frac{1}{r}\Bigg\{&\sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r,p}\left(\tau\right)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\\&+\sum_{p=0}^{r+\vartheta-1}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r,p+1}\left(\tau\right)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\Bigg\}.\end{aligned}\tag{17}$$

Proof. One can differentiate equation (16)

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]=\left(r+\vartheta\right)\left\{\sum_{p=1}^{r+\vartheta}s_{r-1,p-1}\left(\tau\right)\left[\tfrac{p}{r};g\right]-\sum_{p=0}^{r+\vartheta-1}s_{r-1,p}\left(\tau\right)\left[\tfrac{p}{r};g\right]\right\}$$

$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right],$$

by utilizing (7) and (4), respectively. Next, reindexing the summation with $s_{r-1,p-1}(\tau)$ term and then applying divided differences identity of first order yield

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=\frac{(r+\vartheta)}{r}\sum_{p=0}^{r+\vartheta-1}s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]$$
$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right].$$

Using property (6) implies

$$\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]=\frac{(r+\vartheta)}{r}\sum_{p=0}^{r+\vartheta-1}\left(\left(1-\tfrac{p}{r+\vartheta}\right)s_{r,p}(\tau)+\left(\tfrac{p+1}{r+\vartheta}\right)s_{r,p+1}(\tau)\right)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]$$
$$+\frac{\lambda}{r}\sum_{p=0}^{r+\vartheta-1}\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\left[s_{r,p}(\tau)-s_{r,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right],$$

and subsequently, combining the summations with similar terms produces the first derivative given in (17). □

Remark 2. *In the special case $\vartheta=0$ in (17), we obtain equation (7) in [21].*

Theorem 2 (**Monotonicity**). *If g is increasing (or decreasing) on the interval $[0,1+\vartheta]$, then so are all the corresponding $\lambda-$Schurer operators for all $\lambda\in[-1,1]$ and $r\in\mathbb{N}$.*

Proof. In order to prove that $S_{r,\vartheta}^{\lambda}(g;\tau)$ is increasing whenever g is also increasing on $[0,1+\vartheta]$, it is sufficient to show that the first derivative given in Lemma 5 is nonnegative. Firstly, for an increasing function g; i.e., $1-$convex, we have

$$\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\geq 0 \tag{18}$$

by Lemma 3. Moreover, for $0\leq p\leq r+\vartheta-1$, we have $-1\leq 1-\frac{2p}{r+\vartheta-1}\leq 1$. Since $-1\leq\lambda\leq 1$, we get $-1\leq\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)\leq 1$ which leads to

$$0\leq r+\vartheta-p-1\leq r+\vartheta-p+\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right), \tag{19}$$

and

$$0\leq(p+1)-1\leq p+1-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right). \tag{20}$$

Subsequently, we obtain $\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]\geq 0$ due to inequalities (18)-(20). Analogously, for a decreasing function g on $\left[0,1+\vartheta\right]$, we have

$$\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\leq 0. \tag{21}$$

Then by (19)-(21), we have $\frac{d}{d\tau}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]\leq 0$ which implies $S_{r,\vartheta}^{\lambda}\left(g;\tau\right)$ is also decreasing on $\left[0,1+\vartheta\right]$. Hence the proof is complete. □

Remark 3. *The $\vartheta=0$ case is presented as Theorem 3.1 in [21].*

Lemma 6. *For any $\lambda\in[-1,1]$ and $g:[0,1+\vartheta]\to\mathbb{R}$, $\vartheta\geq 0$ integer, the $\lambda-$Schurer operators in (3) satisfy the following identity*

$$\begin{aligned}\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]=\lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}&\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)\right.\\&\left.+s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\}\\+\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}&\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]\\&\times s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\\+\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}&\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]\\&\times s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right].\end{aligned}\tag{22}$$

Proof. Differentiation of the first derivative in (17) by using property (7) results in

$$\begin{aligned}\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]=\frac{1}{r}&\left\{\sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p+\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]\right.\\&\times(r+\vartheta)\left[s_{r-1,p-1}(\tau)-s_{r-1,p}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\\&+\sum_{p=0}^{r+\vartheta-1}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]\\&\left.\times(r+\vartheta)\left[s_{r-1,p}(\tau)-s_{r-1,p+1}(\tau)\right]\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\right\},\end{aligned}$$

which can also be rewritten as

$$\begin{aligned}\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]=\frac{(r+\vartheta)}{r}&\left\{\sum_{p=0}^{r+\vartheta-2}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]s_{r-1,p}(\tau)\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\right.\\&-\sum_{p=0}^{r+\vartheta-1}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\end{aligned}$$

$$+\sum_{p=-1}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r-1,p+1}(\tau)\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]$$
$$\left.-\sum_{p=0}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]\right\},$$

after use of property (4) and reindexing of summations. Finally, employing the fact that

$$\left[\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]-\left[\tfrac{p}{r},\tfrac{p+1}{r};g\right]=\frac{2}{r}\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right],$$

by Definition 1 yields the desired second derivative given in (22). □

Remark 4. *In the special case $\vartheta=0$ in (22), we obtain the second derivative presented in Lemma 3.3 in [21].*

Remark 5. *To demonstrate the convexity preservation property of $\lambda-$Schurer operators $S_{r,\vartheta}^{\lambda}(g;\tau)$, it must be shown that the second derivative, as presented in Lemma 6, is nonnegative whenever the associated function g is convex. Firstly, in view of Lemma 3, we have*

$$\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\geq 0, \tag{23}$$

for any convex function g. Secondly, for $0\leq p\leq r+\vartheta-2$, we have $0\leq\frac{2(p+1)}{r+\vartheta-1}\leq 2$ which implies $-1\leq 1-\frac{2(p+1)}{r+\vartheta-1}\leq 1$ Since $-1\leq\lambda\leq 1$, it is clear to see that $-1\leq\lambda\left(1-\frac{2(p+1)}{r+\vartheta-1}\right)\leq 1$ which leads to

$$0\leq r+\vartheta-p-2\leq r+\vartheta-p-1+\lambda\left(1-\frac{2(p+1)}{r+\vartheta-1}\right)=r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right). \tag{24}$$

In a similar fashion, for $0\leq p\leq r+\vartheta-2\leq r+\vartheta-1$ and $-1\leq\lambda\leq 1$, one can write $-1\leq-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)\leq 1$ which implies

$$0\leq(p+1)-1\leq p+1-\lambda\left(1-\frac{2p}{r+\vartheta-1}\right)=p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right). \tag{25}$$

Consequently, we affirm that

$$\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[r+\vartheta-p-1+\lambda\left(\frac{r+\vartheta-2p-3}{r+\vartheta-1}\right)\right]$$
$$\times s_{r-1,p}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]$$
$$+\frac{2(r+\vartheta)}{r^2}\sum_{p=0}^{r+\vartheta-2}\left[p+1-\lambda\left(\frac{r+\vartheta-2p-1}{r+\vartheta-1}\right)\right]$$
$$\times s_{r-1,p+1}(\tau)\left[\tfrac{p}{r},\tfrac{p+1}{r},\tfrac{p+2}{r};g\right]\geq 0,$$

due to (23)-(25). In opposition, the term

$$\lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)+s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\}$$

may produce negative or positive values depending on the choice of shape parameter $\lambda\in[-1,1]$. *Furthermore, the monotonic behavior of function g will also have an effect on the determination of the sign of second derivative given in (22) since*

$$-\left[0,\tfrac{1}{r};g\right]\leq 0 \quad \textit{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0, \tag{26}$$

for monotone increasing g and

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \textit{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\leq 0, \tag{27}$$

for monotone decreasing g by Lemma 3. On the grounds of this discussion, one can expect that $S_{r,\vartheta}^{\lambda}(g;\tau)$ *is not necessarily convex for all* $\lambda\in[-1,1]$ *and g on* $[0,1]$. *We verify this line of reasoning by demonstrating the following numerical examples.*

Example 1. *In this first example, we consider the monotone increasing and convex function* $g(\tau)=e^{\tau}-\log_{10}[(\tau+1)^2]$ *on* $[0,1]$, *and form Table 1 in which the intervals are given where* $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]\geq 0$ *for different values of* λ, r, *and* ϑ.

To begin with, we have inequalities in (26) hold true since g is monotone increasing on $[0,1]$. *By inspecting the intervals from Table 1, one can say that* $\lambda-$*Schurer operators successfully preserve the convexity of the associated g function for* $\lambda>-\frac{1}{2}$ *for all* $\vartheta\geq 0$ *without loss of generality. Contrarily, it requires to utilize larger r values to maintain the convexity for* $-1\leq\lambda<-\frac{1}{2}$. *For instance,* $S_{r,1}^{-1}(g;\tau)$ *and* $S_{r,1}^{-7/8}(g;\tau)$ *are convex on* $[0,1]$ *for* $r\geq 14$ *and* $r\geq 9$, *respectively, when* $\vartheta=1$. *Moreover, performing calculations by taking bigger* ϑ *values definitely improves the results. For example,* $S_{r,3}^{-1}(g;\tau)$ *and* $S_{r,3}^{-7/8}(g;\tau)$ *are convex on* $[0,1]$ *for* $r\geq 6$ *and* $r\geq 2$, *respectively, when* $\vartheta=3$, *and* $S_{r,4}^{-1}(g;\tau)$ *is convex on* $[0,1]$ *for* $r\geq 2$ *when* $\vartheta=4$.

Example 2. *In this scheme, we consider* $g(\tau)=e^{-\tau}$, *which is monotone decreasing and convex on* $[0,1]$. *Similar to Example 1, we calculate the intervals when* $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}(g;\tau)\right]\geq 0$ *as listed in Table 2.*

Since g is monotone decreasing, the inequalities in (27) are satisfied. Without loss of generality, one can conclude that $S_{r,\vartheta}^{\lambda}(g;\tau)$ *preserve the convexity of this particular function g for* $|\lambda|<\frac{1}{2}$. *On the other hand, the efficiency of convexity preservation decreases for* $-1\leq\lambda<-\frac{1}{2}$ *and* $\frac{1}{2}<\lambda\leq 1$. *For example,* $S_{r,1}^{-7/8}(g;\tau)$, $S_{r,1}^{-11/20}(g;\tau)$ *and* $S_{r,1}^{13/14}(g;\tau)$ *are convex on* $[0,1]$, *for* $r\geq 25$, $r\geq 5$ *and* $r\geq 9$, *respectively. Furthermore,* $S_{r,1}^{-1}(g;\tau)$ *and* $S_{r,1}^{-1}(g;\tau)$ *do not preserve the convexity on* $[0,1]$ *for* $r\leq 260$. *The results are improved if we consider bigger* ϑ *values. For example,* $S_{r,3}^{\lambda}(g;\tau)$ *is convex on* $[0,1]$, *when* $r\geq 2$, *for all* $\lambda\geq-\frac{7}{8}$ *as listed in Table 2, even though, we observe that* $S_{r,1}^{-1}(g;\tau)$ *still do not preserve the convexity*

TABLE 1. List of intervals where $S_{r,\vartheta}^{\lambda}\left(e^{\tau} - \log_{10}[(\tau+1)^2];\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=-1$	$\lambda=-7/8$	$\lambda=-11/20$	$\lambda=-2/7$	$\lambda=3/10$	$\lambda=2/5$	$\lambda=11/16$	$\lambda=1$
				$\vartheta=1$				
2	$[0,0.772742]$	$[0,0.805152]$	$[0,0.946657]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,0.860669]$	$[0,0.891815]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,0.911864]$	$[0,0.939819]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,0.942694]$	$[0,0.967221]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,0.961928]$	$[0,0.983432]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
7	$[0,0.974392]$	$[0,0.993373]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
8	$[0,0.982752]$	$[0,0.999655]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
9	$[0,0.988529]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
10	$[0,0.992622]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
11	$[0,0.995582]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
12	$[0,0.997761]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
13	$[0,0.999387]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
14	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
				$\vartheta=3$				
2	$[0,0.989931]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,0.992123]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,0.996158]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,0.999500]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
				$\vartheta=4$				
2	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$

TABLE 2. List of intervals where $S_{r,\vartheta}^{\lambda}\left(e^{-\tau};\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=-1$	$\lambda=-7/8$	$\lambda=-11/20$	$\lambda=-2/7$	$\lambda=3/10$	$\lambda=1/2$	$\lambda=13/14$	$\lambda=1$
				$\vartheta=1$				
2	[0.286817, 1]	[0.025858, 1]	[0.130190, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.792078]	[0, 0.766560]
3	[0.195798, 1]	[0.166034, 1]	[0.041414, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.929621]	[0, 0.900579]
4	[0.135709, 1]	[0.108515, 1]	[0.000300, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.973939]	[0, 0.955284]
5	[0.097872, 1]	[0.073892, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.988329]	[0, 0.974152]
6	[0.073320, 1]	[0.052226, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.994656]	[0, 0.983036]
7	[0.056714, 1]	[0.038046, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.997889]	[0, 0.987978]
8	[0.045049, 1]	[0.028389, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999692]	[0, 0.991023]
9	[0.036581, 1]	[0.021588, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.993036]
10	[0.030259, 1]	[0.016657, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.994438]
24	[0.005779, 1]	[0.000053, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999096]
25	[0.005338, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999168]
30	[0.003741, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999426]
50	[0.001369, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999796]
100	[0.000346, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999950]
200	[0.000087, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999987]
300	[0.000052, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 0.999994]
				$\vartheta=3$				
2	[0.0196549, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0.0216047, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0.0193631, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0.0165661, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
6	[0.0140429, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
7	[0.0119356, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
8	[0.0102133, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
9	[0.0088096, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
10	[0.0302594, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
24	[0.0019536, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
25	[0.0018209, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
30	[0.0013232, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
50	[0.0005220, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
100	[0.0001399, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
200	[0.0000362, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
260	[0.0000216, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
				$\vartheta=5$				
2	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
3	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
4	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]
5	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]	[0, 1]

on $[0,1]$ *for* $r \leq 260$. *Lastly,* $S_{r,5}^{\lambda}(g;\tau)$ *is convex on* $[0,1]$, *when* $r \geq 2$, *for all* λ *values listed in Table 2.*

From the analysis presented in Remark 5 and the numerical demonstrations in Examples 1 and 2, it follows that the $\lambda-$Schurer operators may fail to maintain the convexity of associated functions with a monotonic nature for certain values of $\lambda \in [-1,1]$. To address this issue, we propose a revised result for the convexity preservation of $S_{r,\vartheta}^{\lambda}(g;\tau)$ by introducing additional conditions on the function g within the interval $[0,1+\vartheta]$.

Theorem 3 (**Convexity**). *Let* g *be a function that is nonincreasing on* $(0,\tau_0)$ *and nondecreasing* $(\tau_0, 1+\vartheta)$ *for any point* $\tau_0 \in (0,1+\vartheta)$ *for* $\vartheta \geq 0$ *integer. If* g *is convex on* $[0,1]$, *then so are all the corresponding* $\lambda-$*Schurer operators for all* $\lambda \in [0,1]$ *and* $r > r_0(\tau_0)$.

Proof. Due to Remark 5, it is sufficient to establish that

$$\lambda\frac{(r+\vartheta)(r+\vartheta+1)}{r(r+\vartheta-1)}\left\{s_{r-1,0}(\tau)\left(-\left[0,\tfrac{1}{r};g\right]\right)+s_{r-1,r+\vartheta-1}(\tau)\left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\right\}\geq 0, \tag{28}$$

holds. To begin with, let $\lambda \in [0,1]$ and $\vartheta \geq 0$ be integer. Now, depending on the choice of point $\tau_0 \in (0,1+\vartheta)$, we will encounter the following cases :

Case 1: When $\tau_0 < \frac{1}{2}$, one can choose r suitably so that $\frac{1}{r} < \tau_0 < \frac{1}{2}$. Therefore, g is nonincreasing on $\left(0,\frac{1}{r}\right)$ and nondecreasing on $\left(\frac{r-1}{r},1+\vartheta\right)$, which implies

$$-\left[0,\tfrac{1}{r};g\right]=g(0)-g\left(\tfrac{1}{r}\right)\geq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]=g\left(\tfrac{r+\vartheta}{r}\right)-g\left(\tfrac{r+\vartheta-1}{r}\right)\geq 0,$$

So inequality (28) is accurate.

Case 2: Next, we consider $\frac{1}{2} < \tau_0$ and accordingly choose r such that $\frac{1}{2} < \tau_0 < \frac{r-1}{r}$. Hence, g is nonincreasing on $\left(0,\frac{1}{r}\right)$ and nondecreasing on $\left(\frac{r-1}{r},1+\vartheta\right)$, which implies

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \text{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0.$$

The inequality (28) remains valid.

Case 3: In this last scheme, we pick $\tau_0 = \frac{1}{2}$. Subsequently, it is straightforward to see that $\frac{1}{r} < \frac{1}{2} = \tau_0 < \frac{r-1}{r}$ which insinuates inequality (28) is true for all $r \geq 2$. □

Remark 6. *The* $\vartheta = 0$ *case is presented as Theorem 3.2 in [21].*

We establish the following numerical example as an implementation of the Theorem 3.

Example 3. *For this scheme, we consider the convex function* $g(\tau) = \left(\tau - \frac{1}{3}\right)^4$, *which is nonincreasing on* $\left(0,\frac{1}{3}\right)$ *and nondecreasing* $\left(\frac{1}{3},1+\vartheta\right)$ *for nonnegative integer* ϑ. *Hence, we have*

$$-\left[0,\tfrac{1}{r};g\right]\geq 0 \quad \textit{and} \quad \left[\tfrac{r+\vartheta-1}{r},\tfrac{r+\vartheta}{r};g\right]\geq 0.$$

Next, we obtain Table 3 in which the intervals are given where $\frac{d^2}{d\tau^2}\left[S_{r,\vartheta}^{\lambda}\left(g;\tau\right)\right]\geq 0.$

TABLE 3. List of intervals where $S_{r,\vartheta}^{\lambda}\left(\left(\tau-\frac{1}{3}\right)^{4};\tau\right)$ is convex for the associated values of λ, ϑ and r.

r	$\lambda=2/15$	$\lambda=3/10$	$\lambda=4/7$	$\lambda=11/16$	$\lambda=17/20$	$\lambda=1$
			$\vartheta=1$			
2	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
			$\vartheta=3$			
2	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
3	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
4	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
5	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$
6	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$	$[0,1]$

The numeric values from Table 3 confirm that $S_{r,\vartheta}^{\lambda}\left(g;\tau\right)$ *preserves the convexity of the affiliated function* $g\left(\tau\right)$ *on* $[0,1]$ *for all* $\lambda\in[0,1]$ *and integer* $\vartheta\geq 0$ *when* $r\geq 2$*. Thus, we can conclude that if the function* $g\left(\tau\right)$ *is selected according to the conditions outlined in Theorem 3, we achieve enhanced results regarding the preservation of convexity for the corresponding* $\lambda-$*Schurer operators.*

4. Conclusions and Future Work

This paper has provided a comprehensive analysis of the shape-preserving characteristics of λ-Schurer operators, highlighting their potential as a robust tool in approximation theory. The results demonstrate that these operators not only preserve the essential geometric features of the approximated functions but also offer enhanced control through the adjustable shape parameter λ. The theoretical insights and auxiliary results presented in this study contribute to a deeper understanding of shape-preserving approximation techniques and pave the way for further research into their applications in diverse fields, such as computer-aided geometric design and numerical analysis. Future studies could explore the extension of these operators to higher dimensions and their integration into practical computational tools. Moreover, we intend to further our research on the shape-preserving characteristics of the operators constructed in [5, 9, 10, 20, 22], respectively.

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