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GEOMETRY OF TIMELIKE NORMAL SURFACES IN DE SITTER 3-SPACE

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ABSTRACT

In this paper, we characterize the normal surfaces of regular curves in de Sitter 3-space, which is a pseudosphere in 4-dimensional Minkowski space. The results obtained in terms of curvatures, which are the basic parameters of curves in differential geometry, have always been meaningful. Here we examine with the help of the given normal surface curvatures associated with a regular curve. First of all, we obtain the geometric components of the surface, such as fundamental forms and curvatures, in terms of the curvatures of the curve. Then, we examine the special conditions of this surface, such as minimalness and flatness. Finally, we obtain some new characterizations, theorems, and results with the help of the obtained equations.

Keywords: Normal surface, Weingarten surface, Minimal surface, Flat surface, de Sitter space

1 INTRODUCTION

The study of the geometry of surfaces associated with a curve by differential operations is a very convenient field of study for mathematics and some applied sciences. The surface associated with the curve results from the curvilinear or linear motion of a curve. In this study, the relationship between the Frenet elements and curvatures of the curve and mean, principal, and Gaussian curvatures of surface is established. As a result, surface characterizations obtained in terms of the curvatures of the curve yield meaningful results [1-11].

Some examples of surfaces associated with curves can be given as T (tangent), N (normal), and B (binormal) surfaces: Surfaces formed by the parallel movement of the curve to vector fields T, N, and B are named with these vector fields. The characteristics of these surfaces in different spaces have been widely studied. For some, see [5-7]. The normal surfaces that we will refer to in this study are formed by the transport of a curve in the vector field N direction. Let Υ be the regular curve with unit speed parameter (s) and N be the (unit) normal vector field of Υ. In this case, the normal surface associated with the Υ is expressed by

$$\Psi(s,v) = \Upsilon + vN,$$

where v is the motion parameter [5].

Hyperbolic geometry in Minkowski space-time corresponds to the geometry of spherical space in Euclidean geometry. The place in differential geometry and mathematics of Euclidean and Minkowski spaces is undeniable [2], [4], [6-12], [14-17]. In addition, these spaces provide important geometrical explanations in other applied sciences, especially in physics. For centuries, spherical space served as the geometric foundation for Newtonian mechanics, but it fell short in explaining Einstein's theory of special relativity. At this point, Minkowski space-time and hyperbolic geometry come into play. Hyperbolic space geometry offers the geometric framework necessary for the explanation of special relativity. In our study, we characterize the surfaces in the de Sitter space

$$S_1^3 = \{X \in R_1^4;\ X \circ X = 1\},$$

from the hyperbolic space forms in *4D* Minkowski space R_1^4. Here, "$\circ$" denotes the Lorentzian inner product [2].

In this article, we examine the normal surfaces of unit speed regular curves in S_1^3. After giving the necessary definitions, theorems, and formulas, we obtain the geometric components of the surface, such as fundamental forms, mean curvature, principal curvatures, and Gaussian curvature. With the help of these components, which we obtained in terms of the curvatures of the curve, we examine the characteristic conditions of the surface, such as being minimal, flat, and Weingarten surfaces. We obtain some theorems and corollaries from the equations we have obtained from here.

2 SURFACES AND TIMELIKE CURVES IN DE SITTER 3-SPACE

In this part, we introduce *4D* Minkowski space and its pseudo-sphere, the de Sitter space S_1^3. In addition, we give the basic theorems, definitions, and equations for the surface characterizations that we will examine in this space.

Lorentzian inner product "$\circ$" defined in *4D* Minkowski space R_1^4 is given by

$$X \circ Y = -x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4,$$

for any $X, Y \in R_1^4$. Vector product for any $X, Y, Z \in R_1^4$ is defined by

$$X \times Y \times Z = \begin{vmatrix} -e_1 & e_2 & e_3 & e_4 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{vmatrix},$$

where $\{e_1, e_2, e_3, e_4\}$ is the canonical basis. The de-Sitter space S_1^3, which is a pseudo-sphere in R_1^4, is defined by [2]

$$S_1^3 = \{X \in R_1^4;\ X \circ X = 1\}.$$

Let Υ be a unit speed reguler curve in S_1^3. If Υ is a timelike curve, then the Serret-Frenet frame formulas and curvatures for the curve are given by [2]

$$\nabla_T T = \eta_1 N + \Upsilon$$

$$\nabla_T N = \eta_1 T + \eta_2 B$$

$$\nabla_T B = -\eta_2 N$$

and

$$\eta_1 = \|\nabla_T T - \Upsilon\|,$$

$$\eta_2 = -\frac{det(\Upsilon, \nabla_T \Upsilon, \nabla_T^2 \Upsilon, \nabla_T^3 \Upsilon)}{{\eta_1}^2}.$$

And also, let's remember the basic components for the characterization of surfaces. For any surface $\Psi(s, v)$ field of normal vectors and the first and second fundamental forms are respectively defined by the equations

$$n = \frac{\Psi_{\{s\}} \times \Psi_{\{v\}}}{\|\Psi_{\{s\}} \times \Psi_{\{v\}}\|}$$

where $\Psi_{\{v\}} = \frac{\partial \Psi}{\partial v}, \Psi_{\{s\}} = \frac{\partial \Psi}{\partial s}$, v parameterizes movement of curve Υ, and

$$I = Ads^2 + 2Bdsdv + Cdv^2,$$

$$II = ads^2 + 2bdsdv + cdv^2,$$

where

$$\begin{aligned} A &= \Psi_{\{s\}} \circ \Psi_{\{s\}}, B = \Psi_{\{s\}} \circ \Psi_{\{v\}}, C = \Psi_{\{v\}} \circ \Psi_{\{v\}}, \\ a &= \Psi_{\{ss\}} \circ n, \quad b = \Psi_{\{sv\}} \circ n, \quad c = \Psi_{\{vv\}} \circ n. \end{aligned} \tag{1}$$

Also, Gaussian, mean, and principal curvatures K, H, and κ_1, κ_2 are defined as [3-7]:

$$K = \frac{ac - b^2}{AC - B^2}, \qquad H = \frac{Ac - 2Bb + Ca}{2\left(AC - B^2\right)}, \tag{2}$$

and

$$\kappa_1 = H + \sqrt{H^2 - K}, \quad \kappa_2 = H - \sqrt{H^2 - K}. \tag{3}$$

After giving these basic equations, we remind the following theorems to examine the cases where the normal surface is a minimal, flat, or Weingarten surface:

Definition 1. If the mean and Gaussian curvatures of a surface Ψ satisfy the equation

$$\left(\frac{\partial K}{\partial v}\right)\left(\frac{\partial H}{\partial s}\right) - \left(\frac{\partial H}{\partial v}\right)\left(\frac{\partial K}{\partial s}\right) = 0,$$

the surface Ψ is called Weingarten surface [11].

Theorem 2. A surface Ψ is a minimal surface if and only if its mean curvature is zero for all points of the surface [1].

Theorem 3. A surface Ψ is developable (flat) if and only if its Gaussian curvature is zero for all points of the surface [1].

3 TIMELIKE NORMAL SURFACES IN S_1^3

In this section, we characterize normal surfaces formed by timelike curves in S_1^3 with the help of reminders from previous sections.

Theorem 4. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. Then, the first and second fundamental forms of surface Ψ are

$$I = ((v\eta_1)^2 + 2v\eta_1 + v^2{\eta_2}^2 + 1)ds^2 + dv^2,$$

$$II=\left(\frac{-\eta_1{}'\eta_2v^2-\eta_2{}'v-\eta_1\eta_2{}'v^2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}\right)ds^2+\left(\frac{-4\eta_2\eta_1v-2\eta_2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}\right)dsdv.$$

Proof. The normal surface of Υ is given as

$$\Psi(s,v)=\Upsilon+vN.$$

Then, by partial derivatives of Ψ, we get

$$\Psi_{\{s\}}\ =\ (v\eta^1+1)T+\eta^2vB,$$

$$\Psi_{\{ss\}}\ =\ (v\eta_1+1)\Upsilon+\eta_1{}'vT+(\eta_1+v(\eta_1{}^2-\eta_2{}^2))N+v\eta_2{}'B,$$

$$\Psi_{\{v\}}\ =\ N,\quad \Psi_{\{vv\}}=0,\quad \Psi_{\{sv\}}=\eta_1T+\eta_2B,$$

where $\eta_1{}'=\frac{\partial\eta_1}{\partial s}$ and $\eta_2{}'=\frac{\partial\eta_2}{\partial s}$. From these equalities, field of unit normal vectors of Ψ is got as

$$n=\frac{\Psi_{\{s\}}\times\Psi_{\{v\}}}{\left\|\Psi_{\{s\}}\times\Psi_{\{v\}}\right\|}=\frac{-\eta_2vT-(1-v\eta_1)B}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}$$

Also, from equalities (1), we obtain

$$\begin{aligned}&A=(v\eta^1+1)^2+v^2\eta^{22},\quad B=\ C=c=0,\\&a=\frac{-\eta_1{}'\eta_2v^2-\eta_2{}'v-\eta_1\eta_2{}'v^2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}},\quad b=\frac{-2\eta_2\eta_1v-\eta_2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}\end{aligned}\tag{4}$$

Therefore, I and II for Ψ surface are calculated as

$$I=((v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1)ds^2+dv^2,$$

$$II=\left(\frac{-\eta_1{}'\eta_2v^2-\eta_2{}'v-\eta_1\eta_2{}'v^2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}\right)ds^2+\left(\frac{-4\eta_2\eta_1v-2\eta_2}{\sqrt{(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1}}\right)dsdv.$$

Corollary 5. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. Then, the mean H and Gaussian curvatures K of the surface Ψ are given by the following equations:

$$H=\frac{-\eta_1{}'\eta_2v^2-\eta_2{}'v-\eta_1\eta_2{}'v^2}{2[(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1]^{\frac{3}{2}}},\tag{5}$$

$$K=\frac{-(2\eta_2\eta_1v+\eta_2)^2}{[(v\eta_1)^2+2v\eta_1+v^2\eta_2{}^2+1]^3}.\tag{6}$$

Proof. From equations (2) and (4), it's obtained

$$H = \frac{Ac - 2Bb + Ca}{2(AC - B^2)} = \frac{-\eta_1'\eta_2 v^2 - \eta_2' v - \eta_1\eta_2' v^2}{2[(v\eta_1)^2 + 2v\eta_1 + v^2{\eta_2}^2 + 1]^{3/2}},$$

$$K = \frac{ac - b^2}{AC - B^2} = \frac{-(2\eta_2\eta_1 v + \eta_2)^2}{[(v\eta_1)^2 + 2v\eta_1 + v^2{\eta_2}^2 + 1]^3}.$$

Theorem 6. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. Then, the surface Ψ is minimal if and only if

$$\eta_1'\eta_2 v^2 = -\eta_2' v - \eta_1\eta_2' v^2.$$

Proof. Let Ψ be minimal surface. Then, from Theorem 2, H =0. Hence, it's obtained $-\eta_1'\eta_2 v^2 - \eta_2' v - \eta_1\eta_2' v^2 = 0$. Conversely, let be $\eta_1'\eta_2 v^2 = -\eta_2' v - \eta_1\eta_2' v^2$. Then, $H = 0$. Therefore, Ψ is a minimal surface.

Theorem 7. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. Then, the surface Ψ is flat if and only if

$$\eta_2 = 0 \ \ (\eta_1 \neq ((-1)/v))$$

or

$$\eta_1 = ((-1)/(2v)).$$

Proof. From Theorem 3 and Eq. (6), the proof is clear.

As a result of Theorem 6 and Theorem 7:

Corollary 8. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. Then, the following results are obtained:

The surface Ψ is both minimal and flat if and only if the torsion of Υ is zero for all points of the curve: $\eta_2 = 0 \ (\eta_1 \neq ((-1)/v))$,

Let curvature η_1 of Υ be a constant. Then, the surface Ψ is a minimal surface if and only if the torsion η_2 is constant or $\eta_1 = -1/v \ (\eta_2 \neq 0)$.

Corollary 9. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. In this case, principal curvatures for the surface Ψ are obtained as

$$\kappa_1 = [\sqrt{(\eta_1'\eta_2 v^2 + \eta_2' v + \eta_2'\eta_1 v^2)^2 + 4(2\eta_2\eta_1 v + \eta_2)^2} - \eta_2' v \\ -\eta_1'\eta_2 v^2 - \eta_2'\eta_1 v^2]/2[(v\eta_1)^2 + 2v\eta_1 + v^2{\eta_2}^2 + 1]^{3/2},$$

$$\kappa_2 = [-\sqrt{(\eta_1{}'\eta_2 v^2 + \eta_2{}'v + \eta_2{}'\eta_1 v^2)^2 + 4(2\eta_2\eta_1 v + \eta_2)^2} - \eta_2{}'v$$
$$-\eta_1{}'\eta_2 v^2 - \eta_2{}'\eta_1 v^2]/2[(v\eta_1)^2 + 2v\eta_1 + v^2\eta_2{}^2 + 1]^{3/2}.$$

Proof. From Eq. (3) and Corollary 5, the proof is clear.

As a result of Corollary 9:

Corollary 10. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. If the curve Υ has constant curvatures, principal curvatures for the surface Ψ are expressed by

$$\kappa_1 = \frac{2\eta_2\eta_1 v + \eta_2}{[(v\eta_1)^2 + 2v\eta_1 + v^2\eta_2{}^2 + 1]^{3/2}}$$

and

$$\kappa_2 = \frac{-2\eta_2\eta_1 v - \eta_2}{[(v\eta_1)^2 + 2v\eta_1 + v^2\eta_2{}^2 + 1]^{3/2}}.$$

Theorem 11. Let Ψ be normal surface of timelike curve Υ with unit speed in S_1^3. If the equation

$$\begin{aligned}
&[2(-2v\eta_2\eta_1{}' - \eta_2{}'\eta_1 v - \eta_2{}' - \eta_1 v\eta_2{}')((v\eta_1)^2 + 2v\eta_1 + 1 \\
&+v^2\eta_2{}^2) - (6v\eta_2{}^2 + 6\eta_1{}^2 v + 6\eta_1)(-(\eta_1 v^2 + v)\eta_2{}' \\
&-\eta_2 v^2\eta_1{}')][(6\eta_2 v\eta_1 + 3\eta_2)(2(\eta_1 v^2 + v)\eta_1{}' + 2v^2\eta_2\eta_2{}') \\
&-2((v\eta_1)^2 + 2v\eta_1 + v^2\eta_2{}^2 + 1)(2v\eta_2\eta_1{}' - 2\eta_2{}'v\eta_1 - \eta_2{}')] \\
&= [\eta_2(\eta_1 + v(4\eta_1{}^3 v + 5\eta_1{}^2 + 4\eta_1\eta_2{}^2 v + 3\eta_2{}^2))][4(v^2\eta_2{}^2 \\
&+(v\eta_1 + 1)^2)(v^2\eta_2\eta_1{}'' + 2v^2\eta_1{}'\eta_2{}' + \eta_2{}''v^2\eta_1 + \eta_2{}''v) \\
&+6v(\eta_1{}' + v\eta_1\eta_1{}' + v\eta_2\eta_2{}')(v\eta_2\eta_1{}' + \eta_2{}'v\eta_1 + \eta_2{}')]
\end{aligned}$$

is satisfied, then surface Ψ is called Weingarten surface.

Proof. From Definition 1, if surface Ψ satisfy the equation

$$\left(\frac{\partial K}{\partial v}\right)\left(\frac{\partial H}{\partial s}\right) - \left(\frac{\partial H}{\partial v}\right)\left(\frac{\partial K}{\partial s}\right) = 0,$$

surface Ψ is a Weingarten surface. Partial derivatives of mean and Gaussian curvatures of surface Ψ are obtained as

$$\left(\frac{\partial H}{\partial v}\right) = \left(\frac{-3(2\eta_1(\eta_1 v + 1) + 2v\eta_2{}^2)(-\eta_2 v^2\eta_1{}' - v(\eta_1 v + 1)\eta_2{}')}{4((\eta_1 v + 1)^2 + v^2\eta_2{}^2)^{5/2}}\right)$$
$$+\left(\frac{-2v\eta_2\eta_1{}' - \eta_1 v\eta_2{}' - (\eta_1 v + 1)\eta_2{}'}{2((\eta_1 v + 1)^2 + v^2\eta_2{}^2)^{3/2}}\right),$$

$$\left(\frac{\partial H}{\partial s}\right)=\left(\frac{-3(2v^2\eta_2\eta_2'+2v\eta_1'(v\eta_1+1))(-v^2\eta_2\eta_1'-v\eta_2'(v\eta_1+1)}{4((\eta_1 v+1)^2+v^2\eta_2^2)^{5/2}}\right)$$
$$+\left(\frac{-v^2\eta_2\eta_1''-v^2\eta_1'\eta_2'-v^2\eta_1'\eta_2'-v\eta_2''(v\eta_1+1}{2((\eta_1 v+1)^2+v^2\eta_2^2)^{3/2}}\right),$$

and

$$\left(\frac{\partial K}{\partial v}\right)=\left(\frac{2\eta_2^2(2\eta_1 v+1)(4\eta_1^3v^2+5\eta_1^2v+4\eta_1\eta_2^2v^2+\eta_1+3\eta_2^2v)}{(\eta_1^2v^2+2\eta_1 v+\eta_2^2v^2+1)^4}\right),$$

$$\left(\frac{\partial K}{\partial s}\right)=\left(\frac{3\eta_2^2(2v\eta_1+1)^2(2v(v\eta_1+1)\eta_1'+2v^2\eta_2\eta_2')}{(\eta_1^2v^2+2\eta_1 v+\eta_2^2v^2+1)^4}\right)$$
$$+\left(\frac{2\eta_2(2v\eta_1+1)(2v\eta_2\eta_1'-(2v\eta_1+1)\eta_2')}{((v\eta_1+1)^2+v^2\eta_2^2)^3}\right),$$

Hence, it's obtained

$$\begin{gathered}[2(-2v\eta_2\eta_1'-\eta_2'\eta_1 v-\eta_2'-\eta_1 v\eta_2')((v\eta_1)^2+2v\eta_1+1\\ +v^2\eta_2^2)-(6v\eta_2^2+6\eta_1^2v+6\eta_1)(-(\eta_1 v^2+v)\eta_2'\\ -\eta_2v^2\eta_1')][(6\eta_2 v\eta_1+3\eta_2)(2(\eta_1v^2+v)\eta_1'+2v^2\eta_2\eta_2')\\ -2((v\eta_1)^2+2v\eta_1+v^2\eta_2^2+1)(2v\eta_2\eta_1'-2\eta_2'v\eta_1-\eta_2')]\\ =[\eta_2(\eta_1+v(4\eta_1^3v+5\eta_1^2+4\eta_1\eta_2^2v+3\eta_2^2))][4(v^2\eta_2^2\\ +(v\eta_1+1)^2)(v^2\eta_2\eta_1''+2v^2\eta_1'\eta_2'+\eta_2''v^2\eta_1+\eta_2''v)\\ +6v(\eta_1'+v\eta_1\eta_1'+v\eta_2\eta_2')(v\eta_2\eta_1'+\eta_2'v\eta_1+\eta_2')]\end{gathered}$$

as a result of Theorem 6 and Theorem 11:

Corollary 12. Let Ψ be the normal surface of a timelike curve Υ with unit speed in S_1^3. If the curve Υ has constant curvatures, then the surface Ψ is both minimal and Weingarten surface.

Proof. Let be the curve Υ has constant curvatures. From Eq. (5), $H=0$. Hence, Ψ is minimal surface and also, from Theorem 11, Ψ is Weingarten surface.

Example

Consider the unit speed curve

$$\Upsilon(s)=(sinh(s),cosh(s),0,0).$$

Since

$$\Upsilon(s)\circ\Upsilon(s)=-sinh^2(s)+cosh^2(s)+0+0=1$$

equality is satisfied, the curve is in S_1^3. Let's calculate the curvature and the torsion of curve Υ. The first, second, and third derivatives of the curve are obtained as

$$\Upsilon'(s) = (cosh(s), sinh(s), 0,0)\ ,$$

$$\Upsilon''(s) = (sinh(s), cosh(s),0,0)$$

and

$$\Upsilon'''(s) = (cosh(s), sinh(s), 0,0).$$

Also, unit vector fields T and N are obtained as

$$\Upsilon'(s) = T = (cosh(s), sinh(s),0,0)$$

and

$$N = \frac{\Upsilon''(s)}{\|\Upsilon''(s)\|} = (sinh(s), cosh(s),0,0)$$

Hence, the curvature is calculated as

$$\eta_1 = \nabla_T N \circ T = \sqrt{-sinh^2(s) + cosh^2(s)} = 1\ .$$

Since the first, second, and third derivatives of the curve are linear, the determinant is zero. Then the torsion of the curve is obtained as

$$\eta_2 = -\frac{det(\Upsilon, \Upsilon', \Upsilon'', \Upsilon''')}{{\eta_1}^2} = 0\ .$$

Therefore, η_1 and η_2 curvatures are constant. Let the normal surface of the curve be

$$\Psi(s,v) = \Upsilon + vN$$

$$= \big((1+v)sinh(s), (1+v)cosh(s), 0,0\big).$$

Then, it's obtained

$$\Psi_{\{s\}} = \big((1+v)cosh(s), (1+v)sinh(s), 0,0\big)\ ,$$

$$\Psi_{\{ss\}} = \big((1+v)sinh(s), (1+v)cosh(s), 0,0\big)\ ,$$

$$\Psi_{\{v\}} = (\sinh(s)\, , \cosh(s)\, , 0,0),$$

$$\Psi_{\{vv\}} = 0,\ \ \Psi_{\{sv\}} = (cosh(s), sinh(s),0,0)$$

n=0 (Because $\Psi_{\{s\}}$ and $\Psi_{\{v\}}$ are linear.)

Then, $a = b = c = 0$. Hence $H = 0$. Therefore, from Definition 1 and Theorem 2, the surface Ψ is both minimal and Weingarten surface. Corollary 12 is verified.

4 CONCLUSION

Curvatures of a curve are the most basic characterization parameters. The results obtained in terms of these parameters have always been favorable. In this study, we have revealed the relationship between the normal surfaces of timelike curves and some special surfaces in Sitter space in 3D. Here we studied minimal, flat, and Weingarten surfaces. Minimizing a surface means minimizing the surface area. Minimal surfaces allow the solution of minimum field problems in physics. Similarly, flat surfaces offer convenient solutions in plane geometry and Weingarten surfaces in terms of establishing the surface curvature relationship. By obtaining the relationship of these surfaces with the normal surface in terms of the curvature and torsion of the curve, we have produced very plain and convenient results. For some main results, see Theorem 6, Theorem 7, Corollary 8, and Corollary 12. In addition, we think that our study will make a meaningful contribution to the literature, given that the subject of curves and surfaces is rarely studied in hyperbolic spaces, especially in 3-dimensional de Sitter space.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

Artificial Intelligence (AI) Contribution Statement

This manuscript was entirely written, edited, analyzed, and prepared without the assistance of any artificial intelligence (AI) tools. All content, including text, data analysis, and figures, was solely generated by the author.

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