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Research Article

Multidimensional quadratic-phase Fourier transform and its uncertainty principles

LUÍS PINHEIRO CASTRO* AND RITA CORREIA GUERRA

ABSTRACT. The main aim of this article is to propose a multidimensional quadratic-phase Fourier transform (MQFT) that generalises the well-known and recently introduced quadratic-phase Fourier transform (as well as, of course, the Fourier transform itself) to higher dimensions. In addition to the definition itself, some crucial properties of this new integral transform will be deduced. These include a Riemann-Lebesgue lemma for the MQFT, a Plancherel lemma for the MQFT and a Hausdorff-Young inequality for the MQFT. A second central objective consists of obtaining different uncertainty principles for this MQFT. To this end, using techniques that include obtaining various auxiliary inequalities, the study culminates in the deduction of L^p-type Heisenberg-Pauli-Weyl uncertainty principles and L^p-type Donoho-Stark uncertainty principles for the MQFT.

Keywords: Multidimensional quadratic-phase Fourier transform, Donoho-Stark uncertainty principle, Heisenberg-Pauli-Weyl uncertainty principle, Riemann-Lebesgue lemma, Hausdorff-Young inequality.

2020 Mathematics Subject Classification: 42B10, 42A38, 26D15, 44A15, 47A05, 47B38.

1. INTRODUCTION

The main theme of this work is the "multidimensional quadratic-phase Fourier transform", which is introduced here for the first time, generalising the well-known (one-dimensional) quadratic-phase Fourier transform [2, 3]. This last quadratic-phase Fourier transform has proved to be an integral operator with substantial virtues in the field of applications, showing great potential in terms of the flexibility of the possibilities for choosing its five free parameters. This can be seen in several recent publications, such as [1, 7, 11, 13, 14, 15, 16, 17, 18, 19] (among many other papers). Now, with the current introduction of the multidimensional quadratic-phase Fourier transform, where the roles of these parameters are now various matrices, it is expected that this new operator will also be well received and used, especially in the field of applications (even outside the discipline of Mathematics).

To better understand the structure of the proposed multidimensional quadratic-phase Fourier transform, we will deduce some of its fundamental properties, exhibit some of its relationships with other existing transforms and operators, and then derive some uncertainty principles associated with such new multidimensional quadratic-phase Fourier transform.

On this last point, it should be noted that in the scientific community in general, of all scientific disciplines, the most famous notion of uncertainty principles is related to Quantum Mechanics and directly associated to the fact that Heisenberg concluded that "the position and the momentum of an electron in an atom cannot be both determined explicitly, but only probabilistically under a certain uncertainty". Already in the Harmonic Analysis and Signal Processing

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*Corresponding author: Luís Pinheiro Castro; castro@ua.pt
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community, the classic Heisenberg uncertainty principle for the Fourier transform is that the product of the duration and bandwidth of a signal $f(x)$ has a lower bound (which depends on the square of the L^2-norm of f). This inequality has been explored in various contexts and for various integral transforms other than the Fourier transform, becoming commonly known as the Heisenberg-Pauli-Weyl [9, 10, 20, 21] uncertainty principle. Another uncertainty principle that we will consider here is called Donoho-Stark and involves different concepts and quantities, based in particular on the so-called ϵ-concentration and on the measures of certain subsets.

This article is organised as follows. Section 2 is devoted to the introduction of the multidimensional quadratic-phase Fourier transform and the deduction of its fundamental properties (such as a Riemann-Lebesgue lemma, a Plancherel type theorem, an inversion formula and a Hausdorff-Young inequality), which are also useful tools in the following sections. In section 3, we obtain sufficient conditions to guarantee an uncertainty principle of the Heisenberg-Pauli-Weyl type, in a framework of $L^p(\mathbb{R}^n)$ spaces (with $1 \leq p \leq 2$), for the multidimensional quadratic-phase Fourier transform. In the last section, we will study various structural inequalities related to the multidimensional quadratic-phase Fourier transform, which will culminate in obtaining $L^p(\mathbb{R}^n)$ type Donoho-Stark uncertainty principles (in a first subsection for $p = 2$ and then, in a second subsection, for any integrability exponent p between 1 and 2).

2. The multidimensional quadratic-phase Fourier transform

In this section, we will introduce the multidimensional quadratic-phase Fourier transform and deduce some of its fundamental properties.

As briefly mentioned in the previous section, our main motivation in this work has to do with the introduction of a new integral transform that conveniently generalises several well-known integral transforms. In this sense, our goal was to be able to generalise the Fourier transform, the fractional Fourier transform, the linear canonical transform, the offset linear canonical transform and the quadratic-phase Fourier transform to a multidimensional context, and to make this generalisation as global as possible using as few restrictions as possible. These restrictions are essentially related to the concern that the new transform continues to have good elementary and useful properties so that it has great potential for applicability (particularly in the fields of engineering and applied physics). So, in addition to the purely mathematical aspect of obtaining a new "object" that generalises various other existing mathematical concepts, care was also taken to frame the new definition with elements that would allow us to verify the existence of interesting and crucial properties that would enhance the use of this new mathematical tool in various contexts of applicability.

In particular, let us recall that the well-known linear canonical transform of a given function f is defined by

$$\mathcal{L}_{\{a,b,c,d\}} f(x) = \frac{1}{\sqrt{2\pi i b}} \int_{\mathbb{R}} e^{\frac{i}{2b}(ay^2 - 2yx + dx^2)} f(y) dy$$

for $b \neq 0$, and by $\sqrt{d}\, e^{\frac{i}{2} cdx^2} f(d\,x)$, if $b = 0$. The four real parameters a, b, c and d are restricted to $ad - bc = 1$ and so only three parameters are free, thus transforming the linear canonical transform into a three-parameter integral transform. Initially, this was proposed independently for reasons deeply associated with the canonical transforms of paraxial optics [5] and quantum mechanics [12]. In fact, as is now well-known, the discovery and development of the theory of linear canonical transforms in the early 1970s was motivated by independent work on two quite different physical models: paraxial optics and nuclear physics. In the first case, the integral kernel of the linear canonical transform was written as a descriptor of the propagation of light in the paraxial regime by Stuart A. Collins Jr. [5] and, in the second case, the linear canonical transform was identified by Marcos Moshinsky and Christiane Quesne [12] as a powerful tool while they were working on certain problems on the alpha clustering and decay of radioactive nuclei.

In addition, there is also a very natural generalisation of the linear canonical transform itself, called the offset linear canonical transform (OLCT) (or "special affine Fourier transform"), which has additional flexibility by additionally presenting a time-shifted and frequency-modulated. Indeed, having in mind a set of six real parameters $a, b, c, d, \tau, \eta \in \mathbb{R}$, such that $ad-bc=1$, it is usual to denote $A=(a,b,c,d,\tau,\eta)$, and for a function f (e.g. in $L^2(\mathbb{R})$), the OLCT of f is defined by

$$\mathcal{O}_A f(x) = \int_{\mathbb{R}} f(y)\, K_A(y,x)\, dy,$$

with

$$K_A(y,x) = \frac{1}{\sqrt{i2\pi|b|}} e^{i\frac{d\tau^2}{2b}}\, e^{i\left[\frac{a}{2b}y^2+\frac{1}{b}y(\tau-x)-\frac{1}{b}x(d\tau-b\eta)+\frac{d}{2b}x^2\right]},$$

if $b \neq 0$, and by $\sqrt{d}e^{i\frac{cd}{2}(x-\tau)^2+i\eta x} f\left[d(x-\tau)\right]$ if $b=0$ (i.e., in the case of $b=0$, the OLCT is simply a chirp multiplication operator). This generalisation has revealed a wide range of important applications, particularly in the area of signal processing and the modelling of optical systems. Naturally, this wide applicability is closely linked to the flexibility of the OLCT and its wide range of generalisations of other integral transforms, such as the Fourier transform and the fractional Fourier transform, the Fresnel transform, the shifted fractional Fourier transform and the linear canonical transform itself.

Moreover, for parameters $a, b, c, d, e \in \mathbb{R}$ (with $b \neq 0$), and the quadratic-phase function

$$Q_{(a,b,c,d,e)}(x,y) := ax^2 + bxy + cy^2 + dx + ey, \tag{2.1}$$

in [2] it was introduced the so-called quadratic-phase Fourier transform $\mathbb{Q}$ given by

$$(\mathbb{Q}f)(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} f(y)\, e^{iQ_{(a,b,c,d,e)}(x,y)}\, dy, \tag{2.2}$$

where $f \in L^1(\mathbb{R})$ or $f \in L^2(\mathbb{R})$. Thus, we may observe that when $a=c=d=e=0$ and $b=\pm 1$, $\mathbb{Q}$ is simply the Fourier and inverse Fourier integral transforms, respectively. Moreover, when $d=e=0$, the kernel generated by (2.1) includes the kernel of the linear canonical transform as well as of the one of the fractional Fourier transform (up to the choice of some constant factors that do not change the properties of corresponding integral operators). Given the above definitions, it is also clear that the quadratic-phase Fourier transform encompasses the OLCT as a particular case.

It is in this framework that we propose to introduce a generalisation of the quadratic-phase Fourier transform (2.2) to the n-dimensional setting, thus performing several generations of the aforementioned integral transforms at once. To this end, the central idea of the proposed definition was to consider the most appropriate possible replacement of the real parameters that appear in the quadratic-phase function (cf. (2.1)) of the kernel of the quadratic-phase Fourier transform by matrices (with real entries) and to take sufficient care to ensure that these matrices were arranged appropriately (given the non-commutativity of their multiplication) and that, as a result, fundamental properties of this new integral operator could be demonstrated.

It is therefore in this context and expectation that we propose the following definition of the multidimensional quadratic-phase Fourier transform.

Definition 2.1. *Let A, B, C, D and E be $n \times n$ matrices with B being symmetric and $\det(B) \neq 0$. The multidimensional quadratic-phase Fourier transform (MQFT) of $f \in L^1(\mathbb{R}^n)$ is defined by*

$$[\mathcal{Q}_M(f)](x) := \int_{\mathbb{R}^n} f(y)\, \mathcal{K}_M^{\mathcal{Q}}(x,y)\, dy,$$

where

$$\mathcal{K}_M^{\mathcal{Q}}(x,y) := \Omega(B,n) e^{iQ_{(A-E)}(x,y)}$$

with $\Omega(B,n) := \left(\frac{i}{2\pi}\right)^{n/2} (\det(B))^{1/2}$, $Q_{(A-E)}(x,y) := x^T Ax + x^T By + y^T Cy + \vec{1} Dx + \vec{1} Ey$, and $\vec{1} := (1,1,\dots,1)$, and where the symbol ^{T} is denoting the transpose operator.

Remark 2.1. *As previously announced this is a generalisation, for the multidimensional case, of several other operators (or integral transforms), as it is the case of the "Quadratic-Phase Fourier Transform" introduced in* [2] *(and also related with the framework of* [3]*).*

Remark 2.2. *The just introduced multidimensional quadratic-phase Fourier transform is also a generalisation of several other multidimensional integral transforms. Namely:*

(i) for $A = C = D = E = 0$ and $B = I$, we recover the multidimensional Fourier transform;

(ii) for $D = E = 0$,

$$A = C = \frac{1}{2}\mathrm{diag}(\cot(\alpha_1), \cot(\alpha_2), \dots, \cot(\alpha_n))$$

and

$$B = -\mathrm{diag}(\csc(\alpha_1), \csc(\alpha_2), \dots, \csc(\alpha_n)),$$

with $\alpha_p \neq k\pi$, for all $k \in \mathbb{N}_0$ and $p = 1, \dots, n$, we obtain the multidimensional fractional Fourier transform;

(iii) considering the multidimensional LCT (MLCT) defined in [4] *and the corresponding matrix*

$$M = \begin{bmatrix} G & H \\ I & J \end{bmatrix},$$

we obtain this transform, through the MQFT, considering $D = E = 0$ and

$$\begin{aligned} A &= \frac{JH^{-1}}{2}, \\ B &= -\, H^{-T}, \\ C &= \frac{H^{-1}G}{2}, \end{aligned}$$

with A, C being symmetric matrices. In this way, the matrix M (that characterises the MLCT), in terms of the matrices that appear in the kernel of the MQFT, is given by

$$M = \begin{bmatrix} -2B^{-T}C & -B^{-T} \\ B - 4AB^{-T}C^T & -2AB^{-T} \end{bmatrix},$$

being this M a symplectic matrix (under the present conditions).

Moreover, note that we can rewrite the MQFT in terms of the Fourier transform $\mathcal{F}$, some variable transformations and also certain chirp functions, in the form

$$[\mathcal{Q}_M(f)](x) = i^{n/2} (\det(B))^{1/2} e^{i(x^T Ax + \vec{1} Dx)} \left[\mathcal{F}(f(y) e^{i(y^T Cy + \vec{1} Ey)}) \right] (B^T x), \tag{2.3}$$

where

$$(\mathcal{F}f)(x) = \left(\frac{1}{\sqrt{2\pi}}\right)^n \int_{\mathbb{R}^n} f(y) e^{ix^T y}\, dy.$$

In this work, we will often use the usual $L^p(\mathbb{R}^n)$ norms (for $p \in [1,\infty]$) and denote them by $\|\cdot\|_{L^p(\mathbb{R}^n)}$.

Lemma 2.1 (Riemann-Lebesgue Lemma for the MQFT). *$\mathcal{Q}_M$ is a bounded linear operator from $L^1(\mathbb{R}^n)$ into $C_0(\mathbb{R}^n)$. Namely, if $f \in L^1(\mathbb{R}^n)$, then $\mathcal{Q}_M(f) \in C_0(\mathbb{R}^n)$ and*

$$\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}\|f\|_{L^1(\mathbb{R}^n)}.$$

Proof. Using the identity (2.3), and the Riemann-Lebesgue Lemma for the Fourier transform, we see that $\mathcal{Q}_M(f) \in C_0(\mathbb{R}^n)$, provided $f \in L^1(\mathbb{R}^n)$. Moreover, from the definition of $\mathcal{Q}_M$, we have

$$\begin{aligned}\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} &= \sup_{x\in\mathbb{R}^n}\left|\frac{i^{n/2}(\det(B))^{1/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n} e^{iQ_{(A-E)}(x,y)}f(y)\,dy\right| \\ &\leq \sup_{x\in\mathbb{R}^n}\frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}\int_{\mathbb{R}^n}\left|e^{iQ_{(A-E)}(x,y)}\right||f(y)|\,dy \\ &= \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}\|f\|_{L^1(\mathbb{R}^n)}.\end{aligned}$$

□

We will continue with a result that shows the invertibility of the MQFT and presents a formula for its inverse.

Theorem 2.1. *If $f \in L^1(\mathbb{R}^n)$ and $\mathcal{Q}_M(f) \in L^1(\mathbb{R}^n) \cap C_0(\mathbb{R}^n)$, then*

$$f(x) = \int_{\mathbb{R}^n}\overline{\mathcal{K}^{\mathcal{Q}}_M(y,x)}[\mathcal{Q}_M(f)](y)\,dy \tag{2.4}$$

for almost every $x \in \mathbb{R}^n$, where

$$\overline{\mathcal{K}^{\mathcal{Q}}_M(y,x)} := \overline{\Omega(B,n)}e^{-iQ_{(A-E)}(y,x)}. \tag{2.5}$$

Proof. Using a substitution of variable in (2.3) allows us to rewrite the $\mathcal{Q}_M$ in the form

$$[\mathcal{Q}_M(f)](x) = i^{n/2}(\det(B))^{-1/2}e^{i(x^TAx+\vec{1}Dx)}\left[\mathcal{F}(f(B^{-1}y)e^{i((B^{-1}y)^TC(B^{-1}y)+\vec{1}E(B^{-1}y))})\right](x). \tag{2.6}$$

We shall make use of the operators τ_B and M_g, given by

$$(\tau_B f)(x) := f(Bx)$$

and

$$(M_g f)(x) := g(x)f(x)$$

for the matrix B (and its inverse), and any function g, respectively.

So, from (2.6), we can write

$$[\mathcal{Q}_M(f)](x) = [M_{ce^{w_1}}\,\mathcal{F}\,\tau_{B^{-1}}\,M_{e^{w_2}}(f)](x), \tag{2.7}$$

with

$$\begin{aligned}c :=& i^{n/2}(\det(B))^{-1/2};\\ w_1(x) :=& i(x^TAx+\vec{1}Dx);\\ w_2(x) :=& i(x^TCx+\vec{1}Ex).\end{aligned}$$

It is clear that all the operators used in the right-hand side of (2.7) are invertible in the present framework, and therefore, from (2.7), we have

$$\left[\mathcal{Q}_M^{-1}(f)\right](x) = \left[M_{e^{-w_2}}\, \tau_B\, \mathcal{F}^{-1}\, M_{c^{-1}e^{-w_1}}(f)\right](x),$$

and so (2.4) is obtained. □

Lemma 2.2 (Plancherel type Lemma for the MQFT). *If $f \in L^2(\mathbb{R}^n)$, then*

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)}. \tag{2.8}$$

Proof. Using (2.3) and having in mind the Plancherel theorem for the Fourier transform, we have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} &= \left\| i^{n/2}(\det(B))^{1/2} e^{i(x^TAx+\vec{1}Dx)} \left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](B^Tx)\right\|_{L^2(\mathbb{R}^n)} \\
&= |\det(B)|^{1/2} |\det(B)|^{-1/2} \frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)} \\
&= \frac{1}{(2\pi)^{n/2}}\|f\|_{L^2(\mathbb{R}^n)}.
\end{aligned}$$

□

Remark 2.3. It is clear from the identity (2.8) that, although the MQFT defined here is not unitary (in $L^2(\mathbb{R}^n)$), a small modification of the definition, taking into account a different constant, can compensate for the constant now obtained in the identity (2.8), transforming it into the constant one. From this perspective, it is easy to redefine the MQFT (using a different constant) to make it a unitary operator.

We recall that for $1 < p < 2$, we have

$$L^p(\mathbb{R}^n) \subset L^1(\mathbb{R}^n) + L^2(\mathbb{R}^n) = \{f_1 + f_2 : f_1 \in L^1(\mathbb{R}^n), f_2 \in L^2(\mathbb{R}^n)\}.$$

Thus, a possible way to interpret the definition of $\mathcal{Q}_M$ in $L^p(\mathbb{R}^n)$, for $1 < p < 2$, is to consider $f \in L^p(\mathbb{R}^n)$ such that $f = f_1 + f_2$, with $f_1 \in L^1(\mathbb{R}^n)$, $f_2 \in L^2(\mathbb{R}^n)$, and then read off the MQFT of f in the form $\mathcal{Q}_M(f) = \mathcal{Q}_M(f_1) + \mathcal{Q}_M(f_2)$.

For the reader's benefit, let us now briefly recall the statement of Riesz-Thorin Interpolation Theorem that we will use in the next proof.

Theorem 2.2 (Riesz-Thorin Interpolation Theorem; cf., e.g., [8]). *Let (X, μ) and (Y, ν) be measure spaces and $1 \le p_0, p_1, q_0, q_1 \le \infty$ (and the measure ν on Y is also required to be semifinite when $q_0 = q_1 = \infty$).*

If $T : (L^{p_0}(X, \mu) + L^{p_1}(X, \mu)) \to (L^{q_0}(Y, \nu) + L^{q_1}(Y, \nu))$ is a linear operator such that

$$\|Tf\|_{L^{q_0}(Y,\nu)} \le M_0\|f\|_{L^{p_0}(X,\mu)}, \quad \|Tg\|_{L^{q_1}(Y,\nu)} \le M_1\|g\|_{L^{p_1}(X,\mu)}$$

for all $f \in L^{p_0}(X, \mu)$ and $g \in L^{p_1}(X, \mu)$, and we consider the interpolated exponents

$$\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}, \quad \frac{1}{q_\theta} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$$

for some $\theta \in [0, 1]$, then $T : L^{p_\theta}(X, \mu) \to L^{q_\theta}(Y, \nu)$ is bounded and

$$\|Tg\|_{L^{q_\theta}(Y,\nu)} \le M_0^{1-\theta} M_1^{\theta}\|g\|_{L^{p_\theta}(X,\mu)}$$

for all $f \in L^{p_\theta}(X, \mu)$.

Theorem 2.3 (Hausdorff-Young Inequality for $\mathcal{Q}_M$)**.** *Let $1 \leq p \leq 2$ and take p' as the conjugate exponent of p (meaning that $p' \geq 2$ and $\frac{1}{p} + \frac{1}{p'} = 1$). If $f \in L^p(\mathbb{R}^n)$ then $\mathcal{Q}_M(f) \in L^{p'}(\mathbb{R}^n)$ and*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We recall that from Lemma 2.2 we already know that for $p = 2$ it holds

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \frac{1}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)}, \quad f \in L^2(\mathbb{R}^n), \tag{2.9}$$

and from Lemma 2.1, for $p = 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)}, \quad f \in L^1(\mathbb{R}^n). \tag{2.10}$$

Thus, using the Riesz-Thorin Interpolation Theorem, we obtain that $\mathcal{Q}_M(f) : L^p(\mathbb{R}^n) \to L^{p'}(\mathbb{R}^n)$ is a bounded operator for $p \in [1, 2]$ (with p' being the conjugate exponent of p). In addition, the interpolation exponent θ must satisfy

$$\frac{\theta}{1} + \frac{1-\theta}{2} = \frac{1}{p}.$$

Thus, $\theta = \frac{2}{p} - 1$ and so, again from (2.9) and (2.10), it follows

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{\theta/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}, \quad f \in L^p(\mathbb{R}^n).$$

□

3. Heisenberg-Pauli-Weyl uncertainty principle

In this section, we present a L^p-type Heisenberg-Pauli-Weyl uncertainty principle associated with the MQFT.

Theorem 3.4. *If $1 \leq p \leq 2$, $f \in L^2(\mathbb{R}^n)$, $yf \in L^p(\mathbb{R}^n)$, $x\mathcal{Q}_M(f) \in L^p(\mathbb{R}^n)$, then*

$$\|yf\|_{L^p(\mathbb{R}^n)} \|x\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)} \geq \frac{|\det(B)|^{1/2-1/p}}{\sigma_{max}(B)} \frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}, \tag{3.11}$$

where $\sigma_{max}(B)$ is the maximum singular value of the matrix B. Moreover, the equality holds if and only if $p = 2$, $\lambda_{max}(BB^T) = \lambda_{min}(BB^T)$ (where $\lambda(BB^T)$ represents an eigenvalue of the matrix BB^T) and $f(y)e^{i(y^T Cy + \vec{1} Ey)}$ is a Gaussian function.

Proof. From (2.3), we know that

$$[\mathcal{Q}_M(f)](x) = i^{n/2} (\det(B))^{1/2} e^{i(x^T Ax + \vec{1} Dx)} \left[\mathcal{F}(f(y)e^{i(y^T Cy + \vec{1} Ey)})\right](B^T x).$$

Moreover, $\|yf\|_{L^p(\mathbb{R}^n)} = \|yf(y)e^{i(y^T Cy + \vec{1} Ey)}\|_{L^p(\mathbb{R}^n)}$ and

$$\begin{aligned}
&\|(B^T x)\,[\mathcal{Q}_M(f)](x)\|_{L^p(\mathbb{R}^n)} \\
&= \left\|(B^T x) i^{n/2} (\det(B))^{1/2} e^{i(x^T Ax + \vec{1} Dx)} \left[\mathcal{F}(f(y)e^{i(y^T Cy + \vec{1} Ey)})\right](B^T x)\right\|_{L^p(\mathbb{R}^n)} \\
&= |\det(B)|^{1/2} \left\|(B^T x) \left[\mathcal{F}(f(y)e^{i(y^T Cy + \vec{1} Ey)})\right](B^T x)\right\|_{L^p(\mathbb{R}^n)} \\
&= |\det(B)|^{1/2-1/p} \left\|x \left[\mathcal{F}(f(y)e^{i(y^T Cy + \vec{1} Ey)})\right](x)\right\|_{L^p(\mathbb{R}^n)}.
\end{aligned}$$

If $f \in L^2(\mathbb{R}^n)$, then $f(y)e^{i(y^TCy+\vec{1}Ey)} \in L^2(\mathbb{R}^n)$. Using the Heisenberg-Pauli-Weyl uncertainty principle for the multidimensional Fourier transform (cf. Lemma 5 of [4]), we have

$$\begin{aligned} &\|yf(y)\|_{L^p(\mathbb{R}^n)}\|(B^Tx)\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\ &\quad = |\det(B)|^{1/2-1/p}\|yf(y)e^{i(y^TCy+\vec{1}Ey)}\|_{L^p(\mathbb{R}^n)}\left\|x\left[\mathcal{F}(f(y)e^{i(y^TCy+\vec{1}Ey)})\right](x)\right\|_{L^p(\mathbb{R}^n)} \\ &\quad \geq |\det(B)|^{1/2-1/p}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \end{aligned} \tag{3.12}$$

Additionally, $|B^Tx|^2 = x^TBB^Tx$. We note that the matrix BB^T is a real and symmetric matrix, so there exists an orthogonal matrix U such that

$$U^T(BB^T)U = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n],$$

where $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of BB^T.

We also have that

$$|B^Tx|^2 = x^TBB^Tx \leq \lambda_{max}(BB^T)x^TUIU^Tx = \lambda_{max}(BB^T)|x|^2. \tag{3.13}$$

Therefore,

$$|B^Tx|^p \leq \left[\lambda_{max}(BB^T)\right]^{p/2}|x|^p.$$

So, considering also now (3.12), it follows

$$\begin{aligned} &\left[\lambda_{\max}(BB^T)\right]^{1/2}\|yf(y)\|_{L^p(\mathbb{R}^n)}\|x\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\ &\quad \geq \|yf(y)\|_{L^p(\mathbb{R}^n)}\|(B^Tx)\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \\ &\quad \geq |\det(B)|^{1/2-1/p}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \end{aligned} \tag{3.14}$$

As $\lambda_{max}(BB^T) = \sigma^2_{max}(B)$, then the inequality can be rewritten as

$$\|yf(y)\|_{L^p(\mathbb{R}^n)}\|x\left[\mathcal{Q}_M(f)\right](x)\|_{L^p(\mathbb{R}^n)} \geq \frac{|\det(B)|^{1/2-1/p}}{\sigma_{max}(B)}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}. \tag{3.15}$$

From (3.13), we have that $|B^Tx|^2 = \lambda_{max}(BB^T)|x|^2$ if and only if

$$\lambda_{max}(BB^T) = \lambda_{min}(BB^T) = \sigma^2_{max}(B) = \sigma^2_{min}(B) = \sigma^2(B). \tag{3.16}$$

According to Lemma 5 of [4] (and also [6], for the unidimensional case), the equality in (3.14) is attained if and only if $p = 2$ and $f(y)e^{i(y^TCy+\vec{1}Ey)}$ is a Gaussian function, that is,

$$f(y)e^{i(y^TCy+\vec{1}Ey)} = ce^{k|y|^2},$$

where c is a constant and $k < 0$. So, we have

$$\|yf\|_{L^p(\mathbb{R}^n)}\|x\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)} = \frac{1}{\sigma_{max}(B)}\frac{n\|f\|_{L^2(\mathbb{R}^n)}}{2}$$

if and only if B satisfies (3.16) and $f(y)e^{i(y^TCy+\vec{1}Ey)} = ce^{k|y|^2}$. □

4. Donoho-Stark uncertainty principles

In this section, we study the Donoho-Stark uncertainty principles of type L^p. In a first subsection, we will do so in the most standard framework of $p = 2$, and then, in a second subsection, we will consider the case of p between 1 and 2.

4.1. L^2**-type Donoho-Stark uncertainty principles.** We start by defining two operators on $L^2(\mathbb{R}^n)$:

$$P_\Lambda f = \chi_\Lambda f$$

and

$$Q_\Gamma f = \mathcal{Q}_M^{-1}\left[\chi_\Gamma \mathcal{Q}_M(f)\right],$$

where Λ and Γ are measurable sets on $\mathbb{R}^n$, and χ_Γ denotes the characteristic function on Γ.

Definition 4.2. *(i) Let Λ be a measurable set on $\mathbb{R}^n$, $0 < \varepsilon_\Lambda < 1$ and $f \in L^2(\mathbb{R}^n)$. f is called ε_Λ-concentrated on Λ if*

$$\|P_{\Lambda^c} f\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_\Lambda \|f\|_{L^2(\mathbb{R}^n)}.$$

(ii) Let Γ be a measurable set on $\mathbb{R}^n$, $0 < \varepsilon_\Gamma < 1$ and $f \in L^2(\mathbb{R}^n)$. $\mathcal{Q}_M(f)$ is said to be ε_Γ-concentrated on Γ if

$$\|Q_{\Gamma^c} f\|_{L^2(\mathbb{R}^n)} \leq \varepsilon_\Gamma \|f\|_{L^2(\mathbb{R}^n)}.$$

We will make use of the usual operator norms of $P_\Lambda, Q_\Gamma : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ defined by

$$\|P_\Lambda\| := \sup_{f \in L^2(\mathbb{R}^n)} \frac{\|P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}}$$

and

$$\|Q_\Gamma\| := \sup_{f \in L^2(\mathbb{R}^n)} \frac{\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}},$$

respectively.

In addition, we will also use the Hilbert-Schmidt norm of operators $\mathcal{L} : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ of the form $(\mathcal{L}f)(x) = \int_{\mathbb{R}^n} f(y) K(x,y)\, dy$, where $f \in L^2(\mathbb{R}^n)$ and $K(x,y) \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$. We recall that the Hilbert-Schmidt norm of $\mathcal{L}$ is given by

$$\|\mathcal{L}\|_{HS} := \left(\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |K(x,y)|^2 dy dx\right)^{1/2}.$$

Lemma 4.3. *Let Λ and Γ be two measurable sets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$. Then,*

$$\|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2} |\Omega(B,n)| |\Lambda|^{1/2} |\Gamma|^{1/2}.$$

Proof. From the definitions of P_Λ and Q_Γ, we have

$$\begin{aligned}
[Q_\Gamma P_\Lambda f](t) =& \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(\chi_\Lambda f)](t) \\
=& \int_\Gamma \int_{\mathbb{R}^n} (\chi_\Lambda f)(y) \mathcal{K}_M^{\mathcal{Q}}(x,y) \overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)} dy dx \\
=& \int_{\mathbb{R}^n} (\chi_\Lambda f)(y) \int_\Gamma \mathcal{K}_M^{\mathcal{Q}}(x,y) \overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)} dy dx \\
=& \int_{\mathbb{R}^n} f(y) \chi_\Lambda(y) K(t,y) dy,
\end{aligned}$$

with $h_t(x) := K(t,y) = \int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx$. Let us now compute

$$\begin{aligned}
[\mathcal{Q}_M(\chi_\Lambda h_t)](x_1) &= \int_{\mathbb{R}^n} \chi_\Lambda(y)\mathcal{K}_M^{\mathcal{Q}}(x_1,y)\left(\int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right)dy \\
&= \int_{\mathbb{R}^n} \chi_\Lambda(y)\left(\int_{\mathbb{R}^n} \chi_\Gamma(t)\mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right)\mathcal{K}_M^{\mathcal{Q}}(x_1,y)\,dy \\
&= \mathcal{Q}_M[\chi_\Lambda(\mathcal{Q}_M^{-1}(\chi_\Gamma\mathcal{K}_M^{\mathcal{Q}}))(t)](x_1) \\
&= \chi_\Lambda(t)\chi_\Gamma(x_1)\mathcal{K}_M^{\mathcal{Q}}(x_1,t).
\end{aligned}$$

Note that $\chi_\Lambda(\lambda)h_t(\lambda) \in L^2(\mathbb{R}^n)$. Using the last identity and the Plancherel Theorem, we have

$$\begin{aligned}
\|Q_\Gamma P_\Lambda\|_{HS}^2 &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(y)K(t,y)|^2\,dydt \\
&= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(y)h_t(y)|^2\,dydt \\
&= (2\pi)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |[\mathcal{Q}_M(\chi_\Lambda h_t)](x)|^2\,dxdt \\
&= (2\pi)^n\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\chi_\Lambda(t)\chi_\Gamma(x)\mathcal{K}_M^{\mathcal{Q}}(x,t)|^2\,dxdt \\
&= (2\pi)^n|\Omega(B,n)|^2|\Lambda||\Gamma|.
\end{aligned}$$

So, $\|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. □

The next Lemma gives a relation between $\|P_\Lambda Q_\Gamma\|_{HS}$ and $\|Q_\Gamma P_\Lambda\|_{HS}$.

Lemma 4.4. *Let Λ and Γ be subsets of $\mathbb{R}^n$ with finite (nonzero) measure. Then,*

$$\|P_\Lambda Q_\Gamma\|_{HS} = \|Q_\Gamma P_\Lambda\|_{HS}.$$

Proof. Let $K(t,y) = \int_\Gamma \mathcal{K}_M^{\mathcal{Q}}(x,y)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx$. We have that $K(t,y) = \overline{K(y,t)} \in L^2(\mathbb{R}^n)$ with respect to y. Let $f \in L^2(\mathbb{R}^n)$ and $g \in C_c^\infty(\mathbb{R}^n)$. Then, we have

$$\begin{aligned}
&\left|\int_\Gamma [\mathcal{Q}_M(f)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
=&\left|\int_\Gamma [\mathcal{Q}_M(f-g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx + \int_\Gamma [\mathcal{Q}_M(g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
\leq&\left|\int_{\mathbb{R}^n} [\mathcal{Q}_M(f-g)](x)\chi_\Gamma(x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right| + \left|\int_\Gamma [\mathcal{Q}_M(g)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
=&\left|\int_{\mathbb{R}^n} [\mathcal{Q}_M(f-g)](x)\chi_\Gamma(x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\right| + \left|\int_\Gamma g(y)K(t,y)dy - \int_{\mathbb{R}^n} f(y)K(t,y)\,dy\right| \\
\leq&|\Gamma|^{1/2}|\Omega(B,n)|\|\mathcal{Q}_M(f-g)\|_{L^2(\mathbb{R}^n)} + \left|\int_\Gamma (f-g)(y)K(t,y)dy\right| \\
\leq&\frac{1}{(2\pi)^{n/2}}|\Gamma|^{1/2}|\Omega(B,n)|\|f-g\|_{L^2(\mathbb{R}^n)} + \|f-g\|_{L^2(\mathbb{R}^n)}\|K(t,y)\|_{L^2(\mathbb{R}^n)} \\
<&c\varepsilon
\end{aligned}$$

for a constant c and an arbitrarily small positive ε. So, this allows us to conclude that

$$\begin{aligned}[P_\Lambda Q_\Gamma f](t) =&\chi_\Lambda \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(f)](t)\\ =&\chi_\Lambda(t)\int_\Gamma [\mathcal{Q}_M(f)](x)\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}dx\\ =&\chi_\Lambda(t)\int_{\mathbb{R}^n} f(y)K(t,y)\,dy.\end{aligned}$$

Now, the last information together with the Plancherel Theorem give us

$$\begin{aligned}\|P_\Lambda Q_\Gamma\|_{HS}^2 =&\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|K(t,y)|^2\,dydt\\ =&\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|h_t(y)|^2\,dydt\\ =&(2\pi)^n\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|\mathcal{Q}_M(h_t)(x)|^2\,dxdt\\ =&(2\pi)^n\int_{\mathbb{R}^n}\chi_\Lambda(t)\int_{\mathbb{R}^n}|\chi_\Gamma(x)\mathcal{K}_{\mathcal{Q}}(x,t)|^2\,dxdt\\ =&(2\pi)^n|\Omega(B,n)|^2|\Lambda||\Gamma|.\end{aligned}$$

Therefore, $\|P_\Lambda Q_\Gamma\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. □

Corollary 4.1. *Suppose that f, Λ and Γ satisfy the conditions of Lemmas 4.3 and 4.4. Then,*

(i) $\|Q_\Gamma P_\Lambda\| \le \|Q_\Gamma P_\Lambda\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$,
(ii) $\|P_\Lambda Q_\Gamma\| \le \|P_\Lambda Q_\Gamma\|_{HS} = (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$.

This corollary follows directly from the definitions of $\|\cdot\|$ and $\|\cdot\|_{HS}$ and Lemmas 4.3 and 4.4.

Theorem 4.5. *Let Λ and Γ be two measurable sets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, $f \in L^2(\mathbb{R}^n)$ and $\varepsilon_1 + \varepsilon_2 < 1$. If f is ε_Λ-concentrated on Λ and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, then*

$$|\Lambda||\Gamma| \ge \frac{1}{(2\pi)^n}\left(\frac{1-\varepsilon_\Lambda-\varepsilon_\Gamma}{|\Omega(B,n)|}\right)^2. \tag{4.17}$$

Proof. By Lemma 2.2, we have that $\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \le (2\pi)^{n/2}\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} = \|f\|_{L^2(\mathbb{R}^n)}$ and so

$$\|Q_\Gamma\| = \sup_{f\in L^2(\mathbb{R}^n)}\frac{\|Q_\Gamma f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \le 1. \tag{4.18}$$

Now, we consider

$$\begin{aligned}\|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} =&\|f - Q_\Gamma f + Q_\Gamma f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}\\ \le&\|f - Q_\Gamma f\|_{L^2(\mathbb{R}^n)} + \|Q_\Gamma f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Since $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, we have that $\|f - Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \le \varepsilon_\Gamma\|f\|_{L^2(\mathbb{R}^n)}$. On the other hand, using (4.18), we have

$$\|Q_\Gamma f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \|Q_\Gamma\|\|f - P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \|f - P_\Lambda f\|_{L^2(\mathbb{R}^n)} \le \varepsilon_\Lambda\|f\|_{L^2(\mathbb{R}^n)},$$

since f is ε_Λ-concentrated on Λ.

In this way, we have $\|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \leq (\varepsilon_\Gamma + \varepsilon_\Lambda)\|f\|_{L^2(\mathbb{R}^n)}$, which gives that

$$\begin{aligned}\|Q_\Gamma P_\Lambda\| \geq & \frac{\|Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \\ \geq & \frac{\|f\|_{L^2(\mathbb{R}^n)} - \|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}}{\|f\|_{L^2(\mathbb{R}^n)}} \\ \geq & 1 - \varepsilon_\Gamma - \varepsilon_\Lambda\end{aligned}$$

(where we have used the inequality

$$\|f\|_{L^2(\mathbb{R}^n)} = \|f - Q_\Gamma P_\Lambda f + Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} \leq \|f - Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)} + \|Q_\Gamma P_\Lambda f\|_{L^2(\mathbb{R}^n)}).$$

By Corollary 4.1, we obtain $(2\pi)^{n/2}|\Omega(B,n)||\Gamma|^{1/2}|\Lambda|^{1/2} \geq 1 - \varepsilon_\Gamma - \varepsilon_\Lambda$, that is equivalent to (4.17). □

Theorem 4.6. *Let $\Lambda, \Gamma \subseteq \mathbb{R}^n$ be two measurable sets such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^2(\mathbb{R}^n)$. Let $\varepsilon_\Lambda, \varepsilon_\Gamma > 0$ be such that $\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2 < 1$. If f is ε_Λ-concentrated on Λ and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ, then*

$$|\Lambda||\Gamma| \geq \frac{1}{(2\pi)^n}\left(\frac{1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2}}{|\Omega(B,n)|}\right)^2.$$

Proof. We have

$$I = P_\Lambda + P_{\Lambda^c} = P_\Lambda Q_\Gamma + P_\Lambda Q_{\Gamma^c} + P_{\Lambda^c},$$

where I is the identity operator. From this identity, we obtain

$$\|f - P_\Lambda Q_\Gamma f\|^2_{L^2(\mathbb{R}^n)} = \|P_\Lambda Q_{\Gamma^c} f + P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)}.$$

From the orthogonality between P_Λ and P_{Λ^c}, we have

$$\|f - P_\Lambda Q_\Gamma f\|^2_{L^2(\mathbb{R}^n)} = \|P_\Lambda Q_{\Gamma^c} f + P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)} \leq \|Q_{\Gamma^c} f\|^2_{L^2(\mathbb{R}^n)} + \|P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)}.$$

This implies that

$$\begin{aligned}\|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} &\leq \left(\|P_{\Lambda^c} f\|^2_{L^2(\mathbb{R}^n)} + \|Q_{\Gamma^c} f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} \\ &\leq \left(\varepsilon_\Lambda^2\|f\|^2_{L^2(\mathbb{R}^n)} + \varepsilon_\Gamma^2\|f\|^2_{L^2(\mathbb{R}^n)}\right)^{1/2} \\ &\leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \geq & \|f\|_{L^2(\mathbb{R}^n)} - \|P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \\ \geq & \|f\|_{L^2(\mathbb{R}^n)} - \|P_\Lambda Q_\Gamma\|\|f\|_{L^2(\mathbb{R}^n)} \\ = & \left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)}.\end{aligned}$$

Consequently, we have

$$\left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)} \leq \|f - P_\Lambda Q_\Gamma f\|_{L^2(\mathbb{R}^n)} \leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)}.$$

Corollary 4.1 gives us that $\|P_\Lambda Q_\Gamma\| \leq (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}$. Hence,

$$\begin{aligned}\left(1 - (2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2}\right)\|f\|_{L^2(\mathbb{R}^n)} &\leq \left(1 - \|P_\Lambda Q_\Gamma\|\right)\|f\|_{L^2(\mathbb{R}^n)} \\ &\leq \left(\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2\right)^{1/2}\|f\|_{L^2(\mathbb{R}^n)},\end{aligned}$$

i.e.,

$$(2\pi)^{n/2}|\Omega(B,n)||\Lambda|^{1/2}|\Gamma|^{1/2} \geq 1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2},$$

and so,

$$|\Lambda||\Gamma| \geq \left(\frac{1 - \sqrt{\varepsilon_\Lambda^2 + \varepsilon_\Gamma^2}}{(2\pi)^{n/2}|\Omega(B,n)|} \right)^2 .$$

□

4.2. **L^p-type Donoho-Stark uncertainty principles, with $1 \leq p \leq 2$.** In this subsection we will study certain Donoho-Stark uncertainty principles in the context of $L^p(\mathbb{R}^n)$ spaces, for which, as preparatory results, we will obtain new inequalities that can also be compared, in a certain sense, with the Hausdorff-Young inequality already obtained for $\mathcal{Q}_M$ in the previous section. Those inequalities will also involve the essential supports ("$\operatorname{ess\,supp}$") of $f \in L^p(\mathbb{R}^n)$ and its MQFT.

Proposition 4.1. *If $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{1/p'} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'},$$

where $\frac{1}{p} + \frac{1}{p'} = 1$.

Proof. By the Riemann-Lebesgue Lemma and Hölder's inequality, we have

$$\begin{aligned}\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq & \|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'} \\ \leq & \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^1(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'} \\ \leq & \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{1/p'} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{1/p'}.\end{aligned}$$

□

Proposition 4.2. *If $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, with p' being such that $1/p + 1/p' = 1$, then*
(4.19)

$$\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{(2-p)/2p} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'}.$$

Proof. By the Hausdorff-Young inequality and generalised Hölder's inequality, we obtain

$$\begin{aligned}\|\mathcal{Q}_M(f)\|_{L^2(\mathbb{R}^n)} \leq & \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \\ \leq & \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \\ \leq & \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f\|_{L^2(\mathbb{R}^n)} |\operatorname{ess\,supp} f|^{(2-p)/2p} |\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'}.\end{aligned}$$

□

Corollary 4.2. *If $f \in L^2(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$|\operatorname{ess\,supp} \mathcal{Q}_M(f)|^{(p'-2)/2p'} \geq |\det(B)|^{1/2-1/p} |\operatorname{ess\,supp} f|^{(p-2)/2p},$$

where p' is the conjugate exponent of p.

Proof. We only have to consider the Plancherel Theorem for the MQFT, together with (4.19), to obtain

$$|\det(B)|^{1/p-1/2}|\text{ess supp}\, f|^{(2-p)/2p}|\text{ess supp}\, \mathcal{Q}_M(f)|^{(p'-2)/2p'} \geq 1.$$

□

Lemma 4.5. *Let* Λ, Γ *be two measurable subsets of* $\mathbb{R}^n$ *such that* $0 < |\Lambda|, |\Gamma| < \infty$ *and* $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$, *with* $1/p + 1/p' = 1$. *Then,*

(i) $\|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)}$;

(ii) $\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \leq |\Omega(B,n)||\Lambda|^{1/p'}|\Gamma|^{1/p'}\|f\|_{L^p(\mathbb{R}^n)}$.

Proof. By the Hausdorff-Young inequality for $\mathcal{Q}_M$, we have

(i)

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} =& \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'}\, dx\right)^{1/p'}\\ \leq& \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)}\\ \leq& \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)};\end{aligned}$$

(ii)

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} =& \left(\int_\Gamma |[\mathcal{Q}_M(P_\Lambda f)](x)|^{p'}\, dx\right)^{1/p'}\\ =& \left(\int_\Gamma \left|\int_\Lambda f(y)\mathcal{K}_M^{\mathcal{Q}}(x,y)\, dy\right|^{p'}\, dx\right)^{1/p'}.\end{aligned}$$

In addition, it holds

$$\begin{aligned}\left|\int_\Lambda f(y)\mathcal{K}_M^{\mathcal{Q}}(x,y)\, dy\right| \leq& \left(\int_\Lambda |f(y)|^p\, dx\right)^{1/p}\left(\int_\Lambda |\mathcal{K}_M^{\mathcal{Q}}(x,y)|^{p'}\, dy\right)^{1/p'}\\ \leq& \|f\|_{L^p(\mathbb{R}^n)}|\Lambda|^{1/p'}|\Omega(B,n)|.\end{aligned}$$

So,

$$\begin{aligned}\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \leq& \left(\int_\Gamma \|f\|^{p'}_{L^p(\mathbb{R}^n)}|\Lambda|\, dx\right)^{1/p'}|\Omega(B,n)|\\ =& \|f\|_{L^p(\mathbb{R}^n)}|\Lambda|^{1/p'}|\Gamma|^{1/p'}|\Omega(B,n)|.\end{aligned}$$

□

Definition 4.3. *(i)* $f \in L^p(\mathbb{R}^n)$ *is said to be* ε_Λ*-concentrated on* Λ *in* L^p*-norm if*

$$\|P_{\Lambda^c}\|_{L^p(\mathbb{R}^n)} = \|f - P_\Lambda f\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Lambda\|f\|_{L^p(\mathbb{R}^n)}.$$

(ii) $\mathcal{Q}_M(f)$ *is is called* ε_Γ*-concentrated on* Γ *in* L^p*-norm if*

$$\|\mathcal{Q}_M(Q_{\Gamma^c} f)\|_{L^p(\mathbb{R}^n)} = \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma\|\mathcal{Q}_M(f)\|_{L^p(\mathbb{R}^n)}.$$

Theorem 4.7. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$. If f is ε_Λ-concentrated on Λ in L^p-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, and $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p}}{1-\varepsilon_\Gamma} \right) \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Consider

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} =& \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma P_\Lambda f) + \mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \\
&+ \|\mathcal{Q}_M(Q_\Gamma f) - \mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} + \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M[Q_\Gamma(f - P_\Lambda f)]\|_{L^{p'}(\mathbb{R}^n)}.
\end{aligned}$$

By (i) in Lemma 4.5 and the fact that f is ε_Λ-concentrated on Λ in the L^p norm, we have

$$\|\mathcal{Q}_M[Q_\Gamma(f - P_\Lambda f)]\|_{L^{p'}(\mathbb{R}^n)} \le \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \|f - P_\Lambda f\|_{L^p(\mathbb{R}^n)} \le \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda \|f\|_{L^p(\mathbb{R}^n)}.$$

By (ii) in Lemma 4.5, we obtain

$$\|\mathcal{Q}_M(Q_\Gamma P_\Lambda f)\|_{L^{p'}(\mathbb{R}^n)} \le |\Omega(B,n)||\Lambda|^{1/p'}|\Gamma|^{1/p'} \|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le& \frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda \|f\|_{L^p(\mathbb{R}^n)} + \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \\
&+ |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p} \|f\|_{L^p(\mathbb{R}^n)},
\end{aligned}$$

which implies that

$$(1-\varepsilon_\Gamma)\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p} \right) \|f\|_{L^p(\mathbb{R}^n)}$$

and so,

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \left(\frac{\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}} \varepsilon_\Lambda + |\Omega(B,n)||\Lambda|^{1-1/p}|\Gamma|^{1-1/p}}{1-\varepsilon_\Gamma} \right) \|f\|_{L^p(\mathbb{R}^n)}.$$

□

Theorem 4.8. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \le p \le 2$. If f is ε_Λ-concentrated on Λ in L^1-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, with $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le \frac{|\Gamma|^{1/p'}|\Lambda|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)(2\pi)^{n/2}} \|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. We have

$$\begin{aligned}
\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \le& \|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} + \|\mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \\
\le& \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'} dx \right)^{1/p'} \\
\le& \varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/p'} \|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)}.
\end{aligned}$$

So, recalling that $0 < \varepsilon_\Gamma < 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}}{1-\varepsilon_\Gamma}\|\mathcal{Q}_M(f)\|_{L^\infty(\mathbb{R}^n)}$$

and, by the Riemann-Lebesgue Lemma for the MQFT, it follows

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(2\pi)^{n/2}}\|f\|_{L^1(\mathbb{R}^n)}.$$

Since f is ε_Λ-concentrated on Λ in L^1-norm, we obtain

$$\begin{aligned}\|f\|_{L^1(\mathbb{R}^n)} \leq &\|P_{\Lambda^c} f\|_{L^1(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^1(\mathbb{R}^n)} \\ \leq &\varepsilon_\Lambda \|f\|_{L^1(\mathbb{R}^n)} + \int_\Lambda |f(x)|\,dx \\ \leq &\varepsilon_\Lambda \|f\|_{L^1(\mathbb{R}^n)} + |\Lambda|^{1/p'}\|f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

by Hölder's inequality. This is equivalent to

$$\|f\|_{L^1(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/p'}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

So, we obtain

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/p'}|\Lambda|^{1/p'}|\det(B)|^{1/2}}{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)(2\pi)^{n/2}}\|f\|_{L^p(\mathbb{R}^n)}.$$

□

Remark 4.4. If $p = p' = 2$, the previous theorem reduces to the classical case

$$|\Gamma|^{1/2}|\Lambda|^{1/2} \geq \frac{(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)}{|\det(B)|^{1/2}}.$$

Theorem 4.9. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n) \cap L^q(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 < q < p < 2$. If f is ε_Λ-concentrated on Λ in L^q-norm and $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, with $1/p + 1/p' = 1$, then*

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)(1-\varepsilon_\Lambda)}\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Since $\mathcal{Q}_M(f)$ is ε_Γ-concentrated on Γ in $L^{p'}$-norm, we have

$$\begin{aligned}\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} = &\|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f) + \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} \\ \leq &\|\mathcal{Q}_M(f) - \mathcal{Q}_M(Q_\Gamma f)\|_{L^{p'}(\mathbb{R}^n)} + \left(\int_\Gamma |[\mathcal{Q}_M(f)](x)|^{p'}\,dx\right)^{1/p'} \\ \leq &\varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/p'-1/q'}\|\mathcal{Q}_M(f)\|_{L^{q'}(\mathbb{R}^n)} \\ \leq &\varepsilon_\Gamma \|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} + |\Gamma|^{1/q-1/p}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|f\|_{L^q(\mathbb{R}^n)},\end{aligned}$$

by the Hausdorff-Young inequality with $1/q + 1/q' = 1$. So, since $0 < \varepsilon_\Gamma < 1$, we have

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)}\|f\|_{L^q(\mathbb{R}^n)}.$$

Since

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \|f - P_\Lambda f\|_{L^q(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^q(\mathbb{R}^n)} \leq \varepsilon_\Lambda \|f\|_{L^q(\mathbb{R}^n)} + |\Lambda|^{1/q-1/p}\|f\|_{L^p(\mathbb{R}^n)},$$

we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently,

$$\|\mathcal{Q}_M(f)\|_{L^{p'}(\mathbb{R}^n)} \leq \frac{|\Gamma|^{1/q-1/p}|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}(1-\varepsilon_\Gamma)} \cdot \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|f\|_{L^p(\mathbb{R}^n)}.$$

□

Let us now prepare to culminate with the last significant result, which will have to do with an uncertainty principle associated with bandlimited functions, in relation to a certain class of functions, invariant under Q_Γ, which we will now formalise. For $1 \leq p \leq 2$ we shall consider $\mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n) := \{h \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n) : Q_\Gamma h = h\}$.

If $f \in L^p(\mathbb{R}^n)$ satisfies

$$\|f-h\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma \|f\|_{L^p(\mathbb{R}^n)}$$

for some $h \in \mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$, then f is said to be ε_Γ-bandlimited on Γ in L^p-norm.

Lemma 4.6. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$. If $h \in \mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$, $1 \leq p \leq 2$, then*

$$\|P_\Lambda h\|_{L^p(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.$$

Proof. By the Hölder inequality, the Hausdorff-Young inequality and the definition of the $\mathcal{B}^p_{Q_\Gamma}(\mathbb{R}^n)$ space, we have

$$\begin{aligned}\|\mathcal{Q}_M(h)\|_{L^1(\mathbb{R}^n)} =& \|\mathcal{Q}_M(Q_\Gamma h)\|_{L^1(\mathbb{R}^n)}\\ =& \|\chi_\Gamma \mathcal{Q}_M(h)\|_{L^1(\mathbb{R}^n)}\\ \leq& |\Gamma|^{1/p}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)}\\ \leq& |\Gamma|^{1/p}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|h\|_{L^p(\mathbb{R}^n)}\end{aligned}$$

and

$$\begin{aligned}\|\mathcal{Q}_M(h)\|_{L^2(\mathbb{R}^n)} =& \|\mathcal{Q}_M(Q_\Gamma h)\|_{L^2(\mathbb{R}^n)}\\ =& \|\chi_\Gamma \mathcal{Q}_M(h)\|_{L^2(\mathbb{R}^n)}\\ \leq& |\Gamma|^{1/2-1/p'}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)}\\ \leq& |\Gamma|^{1/p-1/2}\frac{|\det(B)|^{1/p-1/2}}{(2\pi)^{n/2}}\|h\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

which implies that $\mathcal{Q}_M(h) \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$. Therefore, we have

$$h(t) = (Q_\Gamma h)(t) = \mathcal{Q}_M^{-1}[\chi_\Gamma \mathcal{Q}_M(h)](t) = \int_\Gamma [\mathcal{Q}_M(h)](x)\overline{\mathcal{K}^{\mathcal{Q}}_M(x,t)}\,dx.$$

Hence,

$$\begin{aligned}|h(t)| &\leq \int_\Gamma |[\mathcal{Q}_M(h)](x)||\overline{\mathcal{K}_M^{\mathcal{Q}}(x,t)}|\,dx\\ &= \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}\int_\Gamma |[\mathcal{Q}_M(h)](x)|\,dx\\ &\leq \frac{|\det(B)|^{1/2}}{(2\pi)^{n/2}}|\Gamma|^{1/p}\|\mathcal{Q}_M(h)\|_{L^{p'}(\mathbb{R}^n)}\\ &\leq \frac{|\det(B)|^{1/p}}{(2\pi)^{n}}|\Gamma|^{1/p}\|h\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Consequently,

$$\|P_\Lambda h\|_{L^p(\mathbb{R}^n)} \leq \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.$$

□

Theorem 4.10. *Let Λ, Γ be two measurable subsets of $\mathbb{R}^n$ such that $0 < |\Lambda|, |\Gamma| < \infty$, and $f \in L^1(\mathbb{R}^n)\cap L^q(\mathbb{R}^n)\cap L^p(\mathbb{R}^n)$, $1 < q \leq p < 2$. If f is ε_Λ-concentrated on Λ in L^q-norm and ε_Γ-bandlimited on Γ in L^p-norm, then*

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \left(\frac{\varepsilon_\Gamma|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda} + \frac{|\Gamma|^{1/p}|\Lambda|^{1/q}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n(1-\varepsilon_\Lambda)}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

Proof. Since f is ε_Λ-concentrated on Λ in L^q-norm, we obtain

$$\begin{aligned}\|f\|_{L^q(\mathbb{R}^n)} &\leq \|f - P_\Lambda f\|_{L^q(\mathbb{R}^n)} + \|P_\Lambda f\|_{L^q(\mathbb{R}^n)}\\ &\leq \varepsilon_\Lambda\|f\|_{L^q(\mathbb{R}^n)} + |\Lambda|^{1/q-1/p}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)},\end{aligned}$$

which implies

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \frac{|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)}. \tag{4.20}$$

As f is ε_Γ-bandlimited on Γ in L^p-norm and by the previous lemma, there exists a function $h \in \mathcal{B}^p(\Gamma)$ such that

$$\begin{aligned}\|P_\Lambda f\|_{L^p(\mathbb{R}^n)} &\leq \|P_\Lambda(f-h)\|_{L^p(\mathbb{R}^n)} + \|P_\Lambda h\|_{L^p(\mathbb{R}^n)}\\ &\leq \|f-h\|_{L^p(\mathbb{R}^n)} + \|P_\Lambda h\|_{L^p(\mathbb{R}^n)}\\ &\leq \varepsilon_\Gamma\|f\|_{L^p(\mathbb{R}^n)} + \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}}{(2\pi)^n}\|h\|_{L^p(\mathbb{R}^n)}.\end{aligned}$$

Since

$$\|h\|_{L^p(\mathbb{R}^n)} - \|f\|_{L^p(\mathbb{R}^n)} \leq \|h-f\|_{L^p(\mathbb{R}^n)} \leq \varepsilon_\Gamma\|f\|_{L^p(\mathbb{R}^n)},$$

we have that

$$\|h\|_{L^p(\mathbb{R}^n)} \leq (1+\varepsilon_\Gamma)\|f\|_{L^p(\mathbb{R}^n)}.$$

So,

$$\|P_\Lambda f\|_{L^p(\mathbb{R}^n)} \leq \left(\varepsilon_\Gamma + \frac{(|\Gamma||\Lambda|)^{1/p}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

Consequently, recalling (4.20), we have

$$\|f\|_{L^q(\mathbb{R}^n)} \leq \left(\frac{\varepsilon_\Gamma|\Lambda|^{1/q-1/p}}{1-\varepsilon_\Lambda} + \frac{|\Gamma|^{1/p}|\Lambda|^{1/q}|\det(B)|^{1/p}(1+\varepsilon_\Gamma)}{(2\pi)^n(1-\varepsilon_\Lambda)}\right)\|f\|_{L^p(\mathbb{R}^n)}.$$

□

If $p = q$, then the last result allows us to directly write the following corollary.

Corollary 4.3. *Let* Λ, Γ *be two measurable subsets of* $\mathbb{R}^n$ *such that* $0 < |\Lambda|, |\Gamma| < \infty$, *and* $f \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, $1 \leq p \leq 2$. *If* f *is* ε_Λ*-concentrated on* Λ *and* ε_Γ*-bandlimited on* Γ *in* L^p*-norm, then*

$$|\Gamma||\Lambda| \geq \frac{(1-\varepsilon_\Lambda-\varepsilon_\Gamma)^p(2\pi)^{np}}{|\det(B)|(1+\varepsilon_\Gamma)^p}.$$

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Luís Pinheiro Castro
University of Aveiro
Department of Mathematics
3810-193 Aveiro, Portugal
ORCID: 0000-0002-4261-8699
Email address: castro@ua.pt

Rita Correia Guerra
University of Aveiro
Department of Mathematics
3810-193 Aveiro, Portugal
ORCID: 0000-0001-5821-4843
Email address: ritaguerra@ua.pt