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η-Ricci-Yamabe Solitons on K-Contact Manifolds under D-Homothetic Deformation

H G Nagaraja[1] **and Pavithra R C**[1*]

[1] *Department of Mathematics, Bangalore University, Jnanabharathi,, Bengaluru-560056, Karnataka, India.*
[*] *Corresponding author*

Abstract

This article presents a study on D-homothetically deformed K-contact manifolds. If a contact metric obtained by a D-homothetic deformation of M is a η-Ricci-Yamabe soliton with point-wise collinear then M reduces to η-Einstein have been established. Furthermore, we characterise an η-Ricci-Yamabe soliton, and two more solitons, on Ricci flat, concircularly flat, M-projectively flat K-contact manifold under D-homothetic deformation.

Keywords: *concircularly flat, K-contact manifold, D-homothetic deformation, M-projectively flat, Ricci flat, η-Ricci-Yamabe soliton.*
2010 Mathematics Subject Classification: *53D10, 53C21, 53C25.*

1. Introduction

The concepts of Ricci flow and Yamabe flow were simultaneously introduced by Hamilton in 1988 [10]. Ricci soliton and Yamabe soliton emerged as limits of solutions of Ricci flow and Yamabe flow, respectively. These solitons are equivalent in dimension two but in greater dimensions, these two do not agree since Yamabe soliton preserves the conformal class of the metric, whereas the Ricci soliton does not, in general.

Over the past twenty years, the theory of geometric flows, such as Ricci flow and Yamabe flow, has been the focus of attraction of many geometers. Recently, in 2019, Güler and Crasmăreanu [9] introduced a new geometric flow called the Ricci-Yamabe flow which is a scalar combination of Ricci and Yamabe flow. This is also called the Ricci Yamabe flow of type (p,q). The Ricci-Yamabe flow is an evolution for the metric on the Riemannian *or* semi-Riemannian manifolds, as defined in [9]

$$\frac{\partial}{\partial t}g(t) = -2pRic(t) + qr(t)g(t), \quad g(0) = g_0,$$

due to the sign of the involved scalars p and q, the Ricci-Yamabe flow can also be Riemannian *or* Semi-Riemannian *or* singular Riemannian flow. This kind of multiple choice can be useful in some geometrical *or* physical models, such as those encountered in relativistic theories. Naturally, the Ricci-Yamabe soliton emerges as the limit of the soliton of the Ricci-Yamabe flow.

A solution to the Ricci-Yamabe flow is called Ricci-Yamabe soliton if it moves only by one parameter group of diffeomorphism and scaling. To be precise, a Ricci-Yamabe soliton on Riemannian manifold (M,g) is a data set (g,V,λ,p,q) satisfying

$$\pounds_V g + 2pRic + (2\lambda - qr)g = 0,$$

where Ric is the Ricci tensor, r is the scalar curvature, and $\pounds_V g$ is the Lie derivative of the Riemannian metric g along the vector field V, and λ, p, q are scalars. This soliton is called shrinking, steady, *or* expanding according to $\lambda < 0$, $\lambda = 0$, *or* $\lambda > 0$, respectively. Ricci-Yamabe soliton of type $(0,q)$ and $(p,0)$ are called $q-$Yamabe soliton and $p-$Ricci soliton, respectively.

In 2009 [5] Cho and Kimura introduced the notion of η-Ricci soliton as an advanced extension of the Ricci soliton. Analogously, in 2020 [14] Siddiqi and Akyol, introduced the concept of $\eta-$Ricci-Yamabe soliton as a generalization of Ricci-Yamabe soliton.

A manifold M is said to be η-Ricci-Yamabe soliton of type (p,q) if it satisfies:

$$\pounds_V g + 2pRic + (2\lambda - qr)g + 2\mu\eta\otimes\eta = 0. \tag{1.1}$$

In [3], Blaga studied η-Ricci solitons on Para-Kenmotsu manifolds. Ghosh studied Yamabe soliton and quasi Yamabe soliton on Kenmotsu manifold and also proved that if a Kenmostu metric is a Yamabe soliton, then it has constant scalar curvature [8]. De, Sardar and De [6]

Email addresses: **hgnraj@yahoo.com (H G Nagaraja), Pavithrarc91@gmail.com (Pavithra R C)**

classified three-dimensional Riemannian manifold endowed with a special type of vector field with the Riemannian metrics as Ricci-Yamabe solitons and gradient Ricci-Yamabe solitons. Yoldas [18] measured Kenmotsu metric in terms of η-Ricci-Yamabe soliton.
Extensive research on D-homothetic deformation in contact geometry has been carried out in recent years. In 1968, a serious study in the literature was introduced by Tanno [15]. In [16] the authors used D-homothetic deformation on Sasakian and K-contact structures to get results on the first Betti number, second Betti number, and hormonic forms and gave several examples for manifolds of dimension 3. Ricci solitons and gradient Ricci solitons in D- homothetically deformed K-contact manifold are studied by Nagaraja et al [17]. De and Ghosh [7], explored D-homothetic deformation in almost normal contact metric manifolds and prove that the $Q\phi - \phi Q$ is invariant under such transformations.
Ayar and Yildirim [1] studied geometry and topology of η-Ricci solitons satisfying Ricci semisymmetry condition, on nearly Kenmotsu manifolds. Yildirim and Ayar [20] examined Ricci solitons and gradient Ricci solitons on nearly Cosymplectic manifolds, while Ayar and Demirhan [2] analyzed Ricci solitons on nearly Kenmotsu manifolds with Semi-symmetric metric connections. Pankaj and Chaubey [12] focused on 3-dimensional hyperbolic kenmotsu manifolds endowed with Yamabe and gradient Yamabe metrics.
Curvature is the central subject in Riemannian geometry. Yano introduced the notion of concircular curvature tensor C of type $(1,3)$ on Riemannian manifold for an n-dimensional manifold M as [19]

$$C(D,E)F = \Re(D,E)F - \frac{r}{n(n-1)}[g(E,F)D - g(D,F)E],$$

for all smooth vector fields D, E, $F \in \chi(M)$, where $\Re$ is the Riemannian curvature tensor of type $(1,3)$ and r is the scalar curvature.
Hence if we consider $C^\sharp$ as the concircular curvature tensor under D-homothetic deformation, then for a $(2n+1)$-dimensional manifold M, we have

$$C^\sharp(D,E)F = \Re^\sharp(D,E)F - \frac{r^\sharp}{2n(2n+1)}[g^\sharp(E,F)D - g^\sharp(D,F)E], \tag{1.2}$$

where $\Re^\sharp$ is the curvature tensor and $r^\sharp$ is the scalar curvature under D-homothetic deformation.

Definition 1.1. *A $(2n+1)$-dimensional manifold M is called Ricci-flat under D-homothetic deformation if $Ric^\sharp(D,E) = 0\ \forall\ D,\ E \in \chi(M)$, where $Ric^\sharp$ is the Ricci curvature tensor under D-homothetic deformation.*

Definition 1.2. *A $(2n+1)$-dimensional manifold M is called concircularly flat under D-homothetic deformation if $C^\sharp(D,E)F = 0\ \forall$ $D,E,F \in \chi(M)$.*

In 1971 [13] Pokhariyal and Mishra introduced the notion of M-projective curvature tensor on a Riemannian manifold. The M-projective curvature tensor $\overline{M}$ of rank $(2n+1)$ on n-dimensional manifold M is given by

$$\overline{M}(D,E)F = \Re(D,E)F - \frac{1}{2(n-1)}[Ric(E,F)D - Ric(D,F)E] - \frac{1}{2(n-1)}[g(E,F)QD - g(D,F)QE],$$

for all smooth vectors fields $D,E,F \in \chi(M)$ where Q is the Ricci operator.
Thus for a $(2n+1)$-dimensional manifold, considering $\overline{M}^\sharp$ as the M-projective curvature tensor under D-homothetic deformation we get

$$\begin{aligned}\overline{M}^\sharp(D,E)F &= \Re^\sharp(D,E)F - \frac{1}{4n}[Ric^\sharp(E,F)D - Ric^\sharp(D,F)E] \\ &- \frac{1}{4n}[g^\sharp(E,F)Q^\sharp D - g^\sharp(D,F)Q^\sharp E],\end{aligned} \tag{1.3}$$

where $Q^\sharp$ is the Ricci operator under D-homothetic deformation.

Definition 1.3. *[11] A $(2n+1)$-dimensional manifold M is called M-projectively flat under D-homothetic deformation if $\overline{M}^\sharp(D,E)F = 0$, $D,E,F \in \chi(M)$.*

Definition 1.4. *A vector field V on a Riemannian manifold is said to be concurrent if*

$$\nabla_D V = \rho D, \quad \forall D, \tag{1.4}$$

where ρ is a constant.

2. Preliminaries

A $(2n+1)$-dimensional smooth manifold M is said to be a contact manifold if it carries a global 1-form η such that $\eta \wedge (d\eta)^n \neq 0$ everywhere on M, for a given contact 1-form η there exists a unique vector field ξ the Reeb vector field such that $d\eta(\xi,D) = 0$ and $\eta(\xi) = 1$. Polarizing $d\eta$ on the contact subbundle $\eta = 0$, one obtains a Riemannian metric g and a $(1,1)$-tensor field ϕ such that

$$d\eta(D,E) = g(D,\phi E),\ \ \eta(D) = g(D,\xi), \ \ \phi^2 = -I + \eta \otimes \xi, \tag{2.1}$$

g is called an associated metric of η and (ϕ,η,ξ,g) a contact metric structure. The tensor $h = \frac{1}{2}\pounds_\xi \phi$ is known to be self-adjoint, anti-commutes with ϕ, and satifies $Tr.h = Tr.h\phi = 0$.
A contact metric structure is said to be K-contact if ξ is killing with respect to g, equivalently, $h = 0$. For a K-contact manifold,

$$\nabla_D \xi = -\phi D, \ \ (\nabla_D \phi)E = \Re(\xi,D)E, \tag{2.2}$$

$$(\nabla_D \eta)(E) = -g(\phi D,E), \ \ Ric(D,\xi) = g(QD,\xi) = 2n\eta(D), \tag{2.3}$$

$$\Re(D,E)\xi=\eta(E)D-\eta(D)E, \tag{2.4}$$

$$\Re(\xi,D)E=g(D,E)\xi-\eta(E)D, \tag{2.5}$$

for any vector fields D and E, on M, where $\Re$ and Ric denote, respectively, the curvature tensor of type $(1,3)$ and the Ricci tensor of type $(0,2)$.
A contact metric manifold M is said to be η-Einstein, if the Ricci tensor Ric satisfies $Ric=ag+b\eta\otimes\eta$, where a and b are some smooth functions on the manifold.
In particular if $b=0$, then M becomes an Einstein manifold.

3. D-homothetic deformation and η-Ricci-Yamabe soliton on K-contact manifold:

Let $M(\phi,\xi,\eta,g)$ be a $(2n+1)$-dimensional almost contact metric manifold. A D-homothetic deformation is defined by

$$\phi^\sharp=\phi,\ \ \xi^\sharp=\frac{1}{a}\xi,\ \ \eta^\sharp=a\eta,\ \ g^\sharp=ag+a(a-1)\eta\otimes\eta, \tag{3.1}$$

for a positive constant $a(\neq 1)$.
Then $(M,\phi^\sharp,\xi^\sharp,\eta^\sharp,g^\sharp)$ is also a $(2n+1)$-dimensional almost contact metric manifold. Let K-contact manifold under D-homothetic is invariant. The Riemannian curvature $\Re^\sharp$ of deformed K-contact metric manifold is given by

$$\begin{aligned}\Re^\sharp(D,E)F &= \Re(D,E)F-(a-1)(g(\phi E,F)\phi D+g(D,\phi F)\phi E+2g(D,\phi E)\phi F)\\ &+ (a-1)\{[g(E,F)\xi-\eta(F)E]\eta(D)-[g(D,F)\xi-\eta(F)D]\eta(E)\}\\ &+ a(a-1)\{\eta(E)D-\eta(D)E\}\eta(F).\end{aligned} \tag{3.2}$$

By using above equation, we get

$$Ric^\sharp(E,F)=Ric(E,F)-2(a-1)g(E,F)+2(a-1)(na+n+1)\eta(E)\eta(F), \tag{3.3}$$

$$Q^\sharp E=QE-2(a-1)E+2(a-1)(na+n+1)\eta(E)\xi, \tag{3.4}$$

$$r^\sharp=r+2n(a-1)^2, \tag{3.5}$$

where $Ric^\sharp$ is the Ricci tensor, $Q^\sharp E$ is the Ricci operator, $r^\sharp$ is the scalar curvature of $M(\phi^\sharp,\xi^\sharp,\eta^\sharp,g^\sharp)$ and r is the scalar curvature of $M(\phi,\xi,\eta,g)$.
Let us assume that $g^\sharp$ is a η-Ricci-Yamabe soliton on M.

$$(\pounds_V g^\sharp)(D,E)+2pRic^\sharp(D,E)+(2\lambda-qr)g^\sharp(D,E)+2\mu\eta^\sharp(D)\eta^\sharp(E)=0. \tag{3.6}$$

Now, take the Lie-derivative of $g^\sharp=ag+a(a-1)\eta\otimes\eta$ along V, using the hypothesis, and (3.1), (3.3) in (3.6), we have

$$\begin{aligned}a(\pounds_V g)(D,E) &+ a(a-1)\{(\pounds_V\eta)(D)\eta(E)+\eta(D)(\pounds_V\eta)(E)\}+2pRic(D,E)\\ &- 4p(a-1)g(D,E)+4p(a-1)(na+n+1)\eta(D)\eta(E)\\ &+ (2\lambda-qr)\{ag(D,E)+a(a-1)\eta(D)\eta(E)\}+2\mu a^2\eta(D)\eta(E)=0.\end{aligned} \tag{3.7}$$

Employing $(\pounds_V\eta)(D)=(\lambda-2n)\eta(D)$ and $Ric(D,E)=g(QD,E)$, the equation (3.7) becomes

$$\begin{aligned}a(\pounds_V g)(D,E) &+ [(2\lambda-qr)a-4p(a-1)]g(D,E)+2pRic(D,E)\\ &+ \{(a-1)[4p(2na+n+1)-2qra\\ &+ 2a(\mu+2\lambda)]-2\mu a^2\}\eta(D)\eta(E)=0.\end{aligned} \tag{3.8}$$

Theorem 3.1. *If a K-contact metric $g^\sharp$ obtained by a D-homothetic deformation of (M,g) is a η-Ricci-Yamabe soliton with V point-wise collinear with ξ, then V is a constant multiple of ξ and (M,g) is η-Einstein.*

Proof. Let V be point wise collinear with ξ, that is $V=\alpha\xi$ where α is a smooth function on M. Then from (3.8), we have

$$\begin{aligned}a(D\alpha)\eta(E) &+ a(E\alpha)\eta(D)+[(2\lambda-qr)a-4p(a-1)]g(D,E)\\ &+ 2pg(QD,E)+\{(a-1)\{4p(2na+n+1)\\ &- 2qra+2a(2\lambda+\mu)\}-2\mu a^2\}\eta(D)\eta(E)=0.\end{aligned} \tag{3.9}$$

By replacing E by ξ and by using (2.3) in foregoing equation we obtain

$$\begin{aligned}a(D\alpha) &+ a(\xi\alpha)\eta(D)+[(2\lambda-qr)a-4p(a-1)]\eta(D)\\ &+ 4pn\eta(D)+\{2(a-1)\{2p(2na+n+1)\\ &- qra+a(2\lambda+\mu)\}-2\mu a^2\}\eta(D)=0.\end{aligned} \tag{3.10}$$

Again putting $D=\xi$ in (3.10), we infer that

$$\begin{aligned}(\xi\alpha) &= -\frac{1}{2a}\{(2\lambda-qr)a-4p(a-1)+4pn \\ &+ \{2(a-1)\{2p(2na+n+1)-qra+a(2\lambda+\mu)\}-2\mu a^2\}\}.\end{aligned} \tag{3.11}$$

Using (3.11) in (3.10), we have

$$\begin{aligned}(D\alpha) &= -\frac{1}{a}\{(2\lambda-qr)a-4p(a-1)+4pn \\ &+ \{2(a-1)\{2p(2na+n+1)-qra+a(2\lambda+\mu)\}-2\mu a^2\}\}\eta(D).\end{aligned} \tag{3.12}$$

Applying exterior derivative on above equation we obtain

$$\begin{aligned}(2\lambda-qr)a &= 4p(a-1)-4pn-2(a-1)\{2p(2na+n+1) \\ &- qra+a(2\lambda+\mu)\}+2\mu a^2.\end{aligned} \tag{3.13}$$

Using (3.13) in (3.12), we obtain

$$D\alpha=0,$$

which implies that α is constant. Then (3.9) becomes

$$\begin{aligned}Ric(D,E) &= \frac{1}{2p}\{[4p(a-1)-(2\lambda-qr)a]g(D,E) \\ &+ \{(a-1)(2qra-4p(2na+n+1)-2a(2\lambda+\mu))+2\mu a^2\}\eta(D)\eta(E)\}.\end{aligned}$$

i.e. M is an η- Einstein manifold. □

Theorem 3.2. *If a K-contact metric $g^\sharp$ obtained by a D-homothetic deformation of (M,g) is a η-Ricci-Yamabe soliton with a concurrent vector field V, then the manifold reduces to η-Einstein with constant scalar curvature.*

Proof. We know that

$$(\pounds_V g)(D,E)=g(\nabla_D V,E)+g(D,\nabla_E V), \tag{3.14}$$

making use of (1.4) in the foregoing equation we get

$$(\pounds_V g)(D,E)=2\rho g(D,E). \tag{3.15}$$

Using (3.15) in (3.8) we obtain

$$\begin{aligned}Ric(D,E) &= \frac{1}{p}[-(\lambda-\frac{qr}{2})a+2p(a-1)-\rho]g(D,E) \\ &- \frac{1}{p}\{(a-1)[2p(2na+n+1)-qra+a(\mu+2\lambda)]-\mu a^2\}\eta(D)\eta(E).\end{aligned} \tag{3.16}$$

Furthermore, by putting $D=e_i$ and $E=e_i$ into the above equation, where e_i indicates a local orthonormal frame, and sum over $i=1,2,3,...2n+1$ we arrive the following relation

$$\begin{aligned}r &= \frac{1}{p}[-(\lambda-\frac{qr}{2})a+2p(a-1)-\rho](2n+1) \\ &- \frac{1}{p}\{(a-1)[2p(2na+n+1)-qra+a(\mu+2\lambda)]-\mu a^2\}.\end{aligned}$$

□

3.1. η-Ricci-Yamabe soliton on Ricci flat K-contact manifold:

Here η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a K-contact manifold is discussed, where M is Ricci flat under D-homothetic deformation.

Theorem 3.3. *If a $(2n+1)$-dimensional K-contact manifold M, is Ricci flat under D-homothetic deformation, then η-Ricci-Yamabe soliton on M is shrinking, steady or expanding according as $\frac{qr}{2}-\mu<2np(1-a^2)$, $\frac{qr}{2}-\mu=2np(1-a^2)$ or $\frac{qr}{2}-\mu>2np(1-a^2)$, respectively.*

Proof. Let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M. Then from (1.1) we have

$$g(\nabla_E\xi,F)+g(\nabla_F\xi,E)+2pRic(E,F)+(2\lambda-qr)g(E,F)+2\mu\eta(E)\eta(F)=0.$$

Using (2.2) in the equation, we get

$$2pRic(E,F)+(2\lambda-qr)g(E,F)+2\mu\eta(E)\eta(F)=0.$$

Setting $F=\xi$ in the foregoing equation, we obtain

$$pRic(E,\xi)=(\frac{qr}{2}-\lambda-\mu)\eta(E). \tag{3.17}$$

Now, if it is Ricci-flat with respect to $\nabla^{\sharp}$, then equation (3.3) becomes

$$Ric(E,F)=2(a-1)g(E,F)-2(na^2-n+a-1)\eta(E)\eta(F).$$

Setting $F=\xi$ and multiplying by p in the above equation we get

$$pRic(E,\xi)=2pn(1-a^2)\eta(E). \tag{3.18}$$

Comparing (3.17) and (3.18) we infer

$$\lambda=\frac{qr}{2}-\mu-2np(1-a^2). \tag{3.19}$$

□

Now from (3.19) we have, $\lambda=\frac{qr}{2}-\mu$, when $p=0$, and $\lambda=-\mu-2np(1-a^2)$, when $q=0$. Thus, from the above theorem we respectively conclude the following results.

Corollary 3.1 If a $(2n+1)$-dimensional K-contact manifold is Ricci-flat under D-homothetic deformation, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady, *or* expanding according to $qr<2\mu, \quad qr=2\mu \quad or \quad qr>2\mu$, respectively.

Corollary 3.2 If a $(2n+1)$-dimensional K-contact manifold is Ricci- flat under D-homothetic deformation, then a p-η-Ricci soliton (g,ξ,λ,μ,q) on M is shrinking, steady, *or* expanding according to $2np(a^2-1)<\mu, \quad 2np(a^2-1)=\mu \quad or \quad 2np(a^2-1)>\mu$, respectively.

3.2. η-Ricci-Yamabe soliton on Concircularly flat K-contact manifold:

This subsection deals with the study of η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional concircularly flat K-contact manifold M under D-homothetic deformation.

Theorem 3.4. *If a $(2n+1)$-dimensional K-contact manifold is concircularly flat under D-homothetic deformation, then an η-Ricci-Yamabe soliton on M is shrinking, steady, or expanding according as* $\frac{qr}{2}-\mu<p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)], \quad \frac{qr}{2}-\mu=p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)] \quad or \quad \frac{qr}{2}-\mu>p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)]$, *respectively.*

Proof. Since M is concircularly flat with respect to $\nabla^{\sharp}$, from (1.2) we have

$$\mathfrak{R}^{\sharp}(D,E)F=\frac{r^{\sharp}}{2n(2n+1)}[g^{\sharp}(E,F)D-g^{\sharp}(D,F)E].$$

Using (3.1), (3.2) and (3.5) in the foregoing equation, we obtain

$$\begin{aligned} Ric(E,F) \quad &= \quad [\frac{a(r+2n(a-1)^2)}{2n+1}+2(a-1)]g(E,F) \\ &+ \quad [\frac{a(a-1)(r+2n(a-1)^2)}{2n+1}-2(a-1)(n(a+1)+1)]\eta(E)\eta(F). \end{aligned} \tag{3.20}$$

Putting $F=\xi$ in (3.20) and multiplying both side by p we get

$$pRic(E,\xi)=p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)]\eta(E). \tag{3.21}$$

Next, let (g,ξ,λ,μ,p,q) be an η-Ricci-Yamabe soliton on M, then equating (3.17) and (3.21) we obtain

$$\lambda=\frac{qr}{2}-\mu-p\{\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)\}. \tag{3.22}$$

□

Now, from (3.22) we have, $\lambda=\frac{qr}{2}-\mu$, when $p=0$, and $\lambda=-\mu-p\{\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)\}$, when $q=0$. Thus, from the above theorem, we respectively conclude the following results.

Corollary 3.3 If a $(2n+1)$-dimensional K-contact manifold M of a concircularly flat under D-homothetic deformation, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady *or* expanding according as $qr<2\mu, \quad qr=2\mu \quad or \quad qr>2\mu$ respectively.

Corollary 3.4 If a $(2n+1)$-dimensional K-contact manifold is concircularly flat under D-homothetic deformation, then an p-η-Ricci soliton (g,ξ,λ,μ,p) on M is shrinking, steady *or* expanding according as $-\mu<p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)], \quad -\mu=p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)] \quad or \quad -\mu>p[\frac{a^2(r+2n(a-1)^2)}{2n+1}-2n(a^2-1)]$ respectively.

3.3. η-Ricci-Yamabe soliton on M-Projectively flat K-contact manifold:

Here η-Ricci-Yamabe soliton of type (p,q), q-η-Yamabe soliton and p-η-Ricci soliton on a $(2n+1)$-dimensional M-projectively flat K-contact manifold M under D-homothetic deformation.

Theorem 3.5. *If a $(2n+1)$-dimensional K-contact manifold is M-projectively flat under D-homothetic deformation, then an η-Ricci-Yamabe soliton on M is shrinking, steady or expanding according as $\frac{qr}{2}-\mu < 2p(n-a^2(a-1))$, $\frac{qr}{2}-\mu = 2p(n-a^2(a-1))$ or $\frac{qr}{2}-\mu > 2p(n-a^2(a-1))$ respectively.*

Proof. Since M is M-projectively flat with respect to $\nabla^\sharp$, from (1.3) we have

$$\mathfrak{R}^\sharp(D,E)F = \frac{1}{4n}[Ric^\sharp(E,F)D - Ric^\sharp(D,F)E] + \frac{1}{4n}[g^\sharp(E,F)Q^\sharp D - g^\sharp(D,F)Q^\sharp E].$$

Applying (3.1), (3.2), (3.3), and (3.4), into above equation we get

$$\begin{aligned} Ric(E,F) &= 2[(na+a-1)+\frac{a(a-1)}{2n}(na-n+1)]g(E,F) \\ &- 2(a-1)[(n+1+a(a-1))+\frac{a}{2n}(na+n+1)]\eta(E)\eta(F). \end{aligned} \tag{3.23}$$

Setting $F=\xi$ in (3.23) and multiplying both side by p we get

$$pRic(E,\xi) = 2p\{n-a^2(a-1)\}\eta(E). \tag{3.24}$$

Equating from (3.17) and (3.24) we obtain

$$\lambda = \frac{qr}{2} - \mu - 2p\{n-a^2(a-1)\}. \tag{3.25}$$

□

Now from (3.25) we have, $\lambda = \frac{qr}{2}-\mu$, when $p=0$, and $\lambda = -\mu-2p\{n-a^2(a-1)\}$, when $q=0$. Thus, from the above theorem, we respectively conclude the following results.

Corollary 3.5 If a $(2n+1)$-dimensional K-contact manifold M of a M-projectively flat under D-homothetic deformation, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady, *or* expanding according to $qr < 2\mu$, $qr = 2\mu$ *or* $qr > 2\mu$ respectively.

Corollary 3.6 If a $(2n+1)$-dimensional K-contact manifold M of a M-projectively flat under D-homothetic deformation, then a q-η-Yamabe soliton (g,ξ,λ,μ,q) on M is shrinking, steady, *or* expanding according to $-\mu < 2p\{n-a^2(a-1)\}$, $-\mu = 2p\{n-a^2(a-1)\}$ *or* $-\mu > 2p\{n-a^2(a-1)\}$ respectively.

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