Hacettepe Journal of Mathematics & Statistics

Hacet. J. Math. Stat.
Volume 54 (3) (2025), 1049 – 1061
DOI : 10.15672/hujms.1538345

Research Article

Option price computation under binary control regime switching triple-factor stochastic volatility model

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Abstract

This study presents an efficient pricing framework for European call options under a binary control regime that switches to a triple-factor stochastic volatility model, tailored for recessionary and stable market phases. The model captures regime transitions via binary controls and incorporates triple volatility sources. We derive the characteristic function and implement a semi-analytical pricing formula using trapezoidal and Gauss-Laguerre quadrature in MATLAB. The economic recovery process is influenced by the control parameter α, while the impacts θ_3 are considered secondary to other factors driving recovery. The results show that the option prices under recessionary conditions were lower compared to the recession-free regime, thereby validating the model's sensitivity to macroeconomic uncertainty. It further confirms that the binary control regime switching triple-factor stochastic volatility model offers greater accuracy and adaptability across economic states, making it a promising tool for option pricing in dynamic financial environments.

Mathematics Subject Classification (2020). 91G30, 62M20, 60E10

Keywords. Binary control, economic recession, option computation, triple factor stochastic volatility.

1. Introduction

Uncertainty study in financial market models becomes prominent as risky financial securities exhibit uncertain characteristics. Following the observed limitations of the Black and Scholes deterministic volatility model [6], there has been a paradigm shift in the formulation and application of stochastic volatility models in financial markets. The economic state of a nation determines the financial well-being of the people and the expected development of the nation [23, 24]. The United Nations provides an analysis of the world economic situation and prospects for 2025, examining macroeconomic factors such as youth employment, inflation in most developing countries, and fiscal policy challenges, and how these affect economic growth [26]. The analysis supports evidence-based decision-making

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Received: 25.08.2024; Accepted: 22.04.2025

to promote economic stability and growth. Hence, in an unstable economy, such as a recessed economy, a stochastic model description of asset prices is indispensable.

Future returns of investments and various financial securities have been tailored to the application of mathematical model formulations suitable for forecasting asset prices, especially risky assets such as stock. Several mathematical models are reported in the literature to address option valuation ([2], [7], [11], [14]). Recently, Liu and Lio [21] applied the uncertain canonical space of Liu [19] to study the power option pricing problem of the uncertain exponential Ornstein-Uhlenbeck model. Gao et al. [12] extended the uncertainty theory of Liu [20] to the calibration of the European option pricing model. In the literature, a high level of contributions to option valuation is reported involving stochastic volatility modeling.

Several stochastic volatility models are formulated for option pricing by [15, 17]; a univariate stochastic volatility Heston model by [16], and the double Heston model by [9]. Some authors added jumps to stochastic volatility models, such as Jiexiang et al. [18] and Naik [22] to mention a few. Charlotte et al. [8] recently modified the single stochastic volatility model of Heston to forecast stock prices. The concept of stochastic volatility induced by economic recession was introduced by [3, 4]. Bankole and Adinya [5] proposed a model for option pricing in which the underlying stock asset is driven by a stochastic interest rate and recession-induced stochastic volatility. In this paper, our attention is given to the formulation of a new class of stochastic volatility models in which a binary control parameter is incorporated. The binary control parameter is to ensure that one could transit between economy recession state and economy recession-free state.

The paper is structured as follows: The background to the model formulation is given in Section 2. The binary control regime switching triple factor stochastic volatility (BCRSTSV) model is defined and we obtain a partial differential equation representation for the BCRSTSV model followed by the model characteristic function derivation in Section 3. In Section 4, we obtain the European call option pricing formula. The application to option prices forecast under the BCRSTSV model is presented and sample paths are given under the proposed model in Section 5. The conclusion is given in Section 6.

2. Background

Christoffersen et al. [9] added a second source of variance to the univariate version of the Heston model [16] driven by its own stochastic differential equation (SDE). The set of the SDEs emerged are given as

$$\begin{cases} \frac{dS(t)}{S(t)} = (r-q)dt + \sqrt{v_1(t)}dW_1(t) + \sqrt{v_2(t)}dW_2(t), \quad S(0) = S_0 > 0 \\ dv_1(t) = \kappa_1(\theta_1 - v_1(t))dt + \sigma_1\sqrt{v_1(t)}d\widehat{W}_1(t), \quad v_1(0) = v_{1_0} > 0. \\ dv_2(t) = \kappa_2(\theta_2 - v_2(t))dt + \sigma_2\sqrt{v_2(t)}d\widehat{W}_2(t), \quad v_2(0) = v_{2_0} > 0. \end{cases} \tag{2.1}$$

subject to the following stochastic correlation structure

$$\begin{aligned} \operatorname{cor}(dW_1, dW_2)_t &= \operatorname{cor}(dW_1, d\widehat{W}_2)_t = \operatorname{cor}(dW_2, d\widehat{W}_1)_t = \operatorname{cor}(d\widehat{W}_1, d\widehat{W}_2)_t = 0, \\ \operatorname{cor}(dW_1, d\widehat{W}_1)_t &= \rho_1 dt, \quad \operatorname{cor}(dW_2, d\widehat{W}_2)_t = \rho_2 dt, \end{aligned}$$

where r is the *interest rate*, q is the *dividend rate*, $\kappa_j, j = 1,2$ are the *mean reverting rate*, $\theta_j, j = 1,2$ are the *volatility of variance* (**vol of vol**) constant.

In the next section, we propose a new class of stochastic volatility models that are dependent on the economic state by incorporating a binary control parameter α.

3. The BCRSTSV model

Let S_t be a stock asset indexed in a regime-switching unstable economy defined in a filtered probability space $(\Omega, \mathcal{F}_t, \mathcal{Q}, \mathbb{F})$. Assume that market filtration is driven by the standard Wiener process in the time horizon $t \in (0, T]$ where $\mathcal{Q}$ is a risk-neutral probability measure. Suppose that the economy obeys the binary switch system between the state of *recession-free* and the state of *recession*.

Suppose further that economic recession induced another source of volatility uncertainty on the stock market driven by its own stochastic differential equation independent of the two sources of volatility emphasized in double Heston model since the uncertainty level of financial assets is inevitably at the high side during recession. This assumption is considered since information flow from a recessed economy influences price instability in the stock market. The proposed BCRSTSV model for stock asset $S(t)$ at time $t \in (0, T]$ is given as

$$\begin{cases} \frac{dS(t)}{S(t)} = (r-q)dt + \sqrt{v_1(t)}dW_1(t) + \sqrt{v_2(t)}dW_2(t) + \alpha\big(\sqrt{v_3(t)}dW_3(t)\big), \\ dv_1(t) = \kappa_1\big(\theta_1 - v_1(t)\big)dt + \sigma_1\sqrt{v_1(t)}d\widehat{W}_1(t), \quad v_1(0) = v_{1_0} > 0 \\ dv_2(t) = \kappa_2\big(\theta_2 - v_2(t)\big)dt + \sigma_2\sqrt{v_2(t)}d\widehat{W}_2(t), \quad v_2(0) = v_{2_0} > 0 \\ dv_3(t) = \alpha\Big(\kappa_3\big(\theta_3 - v_3(t)\big)dt + \sigma_3\sqrt{v_3(t)}d\widehat{W}_3(t)\Big)_{recession}, v_3(0) = v_{3_0} > 0 \end{cases} \tag{3.1}$$

where $S(0) = S_0 > 0$ and α is a binary control parameter defined as

$$\alpha = \begin{cases} 0, & \text{if the economy is not in recession;} \\ 1, & \text{if the economy is in recession.} \end{cases}$$

Here, α is used as a transition between double Heston model in Eq. (2.1) and the proposed BCRSTSV model in Eq. (3.1) depending on the state of the economy. This is so to ensure that the model is applicable in any state of the economy.

The control parameter is considered useful as it ensures the BCRSTSV model in Eq. (3.1) is valid for option valuation even when the economy's recession varnishes. Setting the control parameter to zero (0) will return to the double Heston model presented in Eq. (2.1). The underlying asset at time t is $S(t)$, the interest rate is r, the dividend rate is q, the first two volatilities, v_1, v_2 emanate from the double Heston model while v_3 emerged from the economic recession-induced volatility process, the constants $\kappa_1, \kappa_2, \kappa_3$ are mean reverting rates for the three volatility processes respectively, $\theta_1, \theta_2, \theta_3$ are long-term volatility constants and $\sigma_1, \sigma_2, \sigma_3$ are *volatility of variance* (**vol of vol**) constants which are both positive The economic analysis on the effect of α and θ_3 is given in Section 5.

The Wiener process, W_3 in Eq. (3.1) , describes the Brownian movement in stock prices relative to the volatility of the recession, while W_1 and W_2 originate from the double Heston model. The remaining Wiener processes $\widehat{W_j}, j = 1, 2, 3$, show the stochastic movement of the stock volatilities from the three sources. The model is subjected to the following stochastic correlation structure:

$$\begin{cases} \operatorname{cor}\big(dW_1, dW_2\big)_t = \operatorname{cor}\big(dW_1, dW_3\big)_t = \operatorname{cor}\big(dW_2, dW_3\big)_t = 0, \\ \operatorname{cor}\big(dW_1, d\widehat{W}_2\big)_t = \operatorname{cor}\big(dW_2, d\widehat{W}_1\big)_t = \operatorname{cor}\big(d\widehat{W}_1, d\widehat{W}_2\big)_t = 0, \\ \operatorname{cor}\big(dW_1, d\widehat{W}_3\big)_t = \operatorname{cor}\big(dW_2, d\widehat{W}_3\big)_t = \operatorname{cor}\big(d\widehat{W}_1, d\widehat{W}_3\big)_t = \operatorname{cor}\big(d\widehat{W}_2, d\widehat{W}_3\big)_t = 0, \\ \operatorname{cor}\big(dW_1, d\widehat{W}_1\big)_t = \rho_1 dt, \quad \operatorname{cor}\big(dW_2, d\widehat{W}_2\big)_t = \rho_2 dt, \quad \operatorname{cor}\big(dW_3, d\widehat{W}_3\big)_t = \rho_3 dt. \end{cases} \tag{3.2}$$

Remark 3.1. The inclusion of the third volatility process emanated from economy recession factor which leads to three different correlations given in Eq. (3.2).

4. Main results

In this section, we present the main findings of the paper. First, we give a partial differential equation representation for the BCRSTSV model followed by the derivation of the characteristic function of the model. We also give the European call option pricing formula, the numerical computation of prices under the proposed model, as well as the stock prices sample paths.

4.1. Partial differential equation of the BCRSTSV model

Let $f(y_t, v_1, v_2, v_3)$ be a twice-continuously differentiable function with respect to Itô's Calculus. Suppose further that $f(y_t, v_1, v_2, v_3)$ satisfies the BCRSTSV model in Eq. (3.1), then the partial differential equation (PDE) representation is given as

$$\begin{aligned}\frac{\partial f}{\partial t} =& \tfrac{1}{2}(v_1+v_2+v_3)\frac{\partial^2 f}{\partial y^2} + \Big(r-q-\tfrac{1}{2}(v_1+v_2+v_3)\Big)\frac{\partial f}{\partial y} + \tfrac{1}{2}\sigma_1^2 v_1\frac{\partial^2 f}{\partial v_1^2}\\ &+\tfrac{1}{2}\sigma_2^2 v_2\frac{\partial^2 f}{\partial v_2^2} + \tfrac{1}{2}\sigma_3^2 v_3\frac{\partial^2 f}{\partial v_3^2} + \rho_1\sigma_1 v_1\frac{\partial^2 f}{\partial y\partial v_1} + \rho_2\sigma_2 v_2\frac{\partial^2 f}{\partial y\partial v_2} + \rho_3\sigma_3 v_3\frac{\partial^2 f}{\partial y\partial v_3}\\ &+\kappa_1(\theta_1-v_1)\frac{\partial f}{\partial v_1} + \kappa_2(\theta_2-v_2)\frac{\partial f}{\partial v_2} + \kappa_3(\theta_3-v_3)\frac{\partial f}{\partial v_3}.\end{aligned} \tag{4.1}$$

We state the Feynman-Kac formula for the proposed BCRSTSV-model as follows:

Proposition 4.1 (Feynman-Kac formula for the BCRSTSV-model)**.** *Let $f(y,t)$ be $C^{2,1}$-differentiable function with respect to some Itô diffusion processes. The PDE of $f(y,t)$ is given by*

$$\frac{\partial f}{\partial t} + \mathcal{L}f(y,t) - r(y,t) = 0 \tag{4.2}$$

subject to the boundary condition (f_τ, τ). The solution is given in the form

$$f(y_t,t) = \mathbb{E}_Q\Big[\exp\Big(\int_t^\tau r(y_u,u)du\Big)f(y_\tau,\tau)\Big|\mathcal{F}_t\Big]$$

where $\mathcal{F}_t$ is the filtration up to time t.

$\mathcal{L}$ is an *infinitesimal generator* of the BCRSTSV-PDE defined by

$$\mathcal{L} := \sum_{i=1}^{n}\mu_i\frac{\partial}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n}\left(\sigma\sigma^T\right)_{ij}\frac{\partial^2}{\partial y_i\partial y_j} \tag{4.3}$$

with $y = \ln S(t)$, μ and $(\sigma\sigma^T)$ are defined in Equations (4.9) and (4.11). This implies that $\mathcal{L}$ could generate the right hand side of the BCRSTSV-PDE and Eq. (4.1) is equivalently stated as

$$\frac{\partial f}{\partial t} = \mathcal{L}_{y_t,v_j(t)}f, \quad j=1,2,3 \tag{4.4}$$

subject to terminal condition, $f(\omega,0,y) = \exp(i\omega y)$, with

$$\begin{aligned}\mathcal{L}_{y_t,v_j(t)}f =& \frac{1}{2}(v_1+v_2+v_3)\frac{\partial^2 f}{\partial y^2} + \Big(r-q-\frac{1}{2}(v_1+v_2+v_3)\Big)\frac{\partial f}{\partial y} + \frac{1}{2}\sigma_1^2 v_1\frac{\partial^2 f}{\partial v_1^2}\\ &+\frac{1}{2}\sigma_2^2 v_2\frac{\partial^2 f}{\partial v_2^2} + \frac{1}{2}\sigma_3^2 v_3\frac{\partial^2 f}{\partial v_3^2} + \rho_1\sigma_1 v_1\frac{\partial^2 f}{\partial y\partial v_1} + \rho_2\sigma_2 v_2\frac{\partial^2 f}{\partial y\partial v_2}\\ &+\rho_3\sigma_3 v_3\frac{\partial^2 f}{\partial y\partial v_3} + \kappa_1(\theta_1-v_1)\frac{\partial f}{\partial v_1} + \kappa_2(\theta_2-v_2)\frac{\partial f}{\partial v_2} + \kappa_3(\theta_3-v_3)\frac{\partial f}{\partial v_3}\end{aligned} \tag{4.5}$$

4.2. The characteristic function of the BCRSTSV model

Theorem 4.2. *Let the price of the stock asset evolve using the BCRSTSV model (3.1) and let the logarithmic stock price be denoted by* $y_\tau = \ln S(t)$*. The characteristic function* $\varphi(\cdot)$ *of the logarithm stock asset price* $y(t)$ *with respect to time* T*-forward measure* $\mathbf{Q}$ *is given as* $f\big(\omega_0, \omega_1, \omega_2, \omega_3; y_t, v_1(t), v_2(t), v_3(t)\big)$ *for* $\big(y_T, v_1(T), v_2(T), v_3(T)\big)$ *in log linear form as*

$$\begin{aligned} f\big(\omega_0, \omega_1, \omega_2, \omega_3; y_t, v_1(t), v_2(t), v_3(t)\big) &= \mathbb{E}\Big(e^{i\omega_0 y_t + i\omega_1 v_1(t) + i\omega_2 v_2(t) + i\omega_3 v_3(t)}\Big) \\ &= \exp\Big(A(\tau,\omega) + B_0(\tau,\omega) y_t + B_1(\tau,\omega) v_1(t) + B_2(\tau,\omega) v_2(t) + B_3(\tau,\omega) v_3(t)\Big) \end{aligned} \tag{4.6}$$

where $A, B_0, B_1, B_2,$ *and* B_3 *are the coefficients terms of the stochastic processes* $y_t, v_1(t), v_2(t), v_3(t)$ *which depends on the time to expiry* $\tau = T - t$ *and each* $\omega_i, i = 0, 1, \ldots, 3$.

Given that the stock asset $S(t)$ evolves by the BCRSTSV model in Eq. (3.1) and the logarithm stock price is denoted by $x_\tau = \ln S(t)$. Applying the result of [10] which established the fact that a characteristic function could be given as a system of Ricatti equations in [15] and [25]. We express Eq. (3.1) as a system of Ricatti differential equations

$$\begin{cases} \frac{\partial B_0}{\partial t} &= -\mathbf{J}_1^T \beta - \frac{1}{2}\beta^T \mathbf{H}_1 \beta \\ \frac{\partial B_1}{\partial t} &= -\mathbf{J}_2^T \beta - \frac{1}{2}\beta^T \mathbf{H}_2 \beta \\ \frac{\partial B_2}{\partial t} &= -\mathbf{J}_3^T \beta - \frac{1}{2}\beta^T \mathbf{H}_3 \beta \\ \frac{\partial B_3}{\partial t} &= -\mathbf{J}_4^T \beta - \frac{1}{2}\beta^T \mathbf{H}_4 \beta \\ \frac{\partial A}{\partial t} &= -\mathbf{J}_0^T \beta - \frac{1}{2}\beta^T \mathbf{H}_0 \beta \end{cases} \tag{4.7}$$

where $\beta^T := (B_0, B_1, B_2, B_3)$ and the boundary conditions to the above Ricatti equations are given as

$$\begin{cases} B_0(0) &= i\omega_0 \\ B_1(0) &= i\omega_1 \\ B_2(0) &= i\omega_2 \\ B_3(0) &= i\omega_3 \\ A(0) &= 0 \end{cases} \tag{4.8}$$

The coefficient matrices' terms $\mathbf{J_i}$, $\mathbf{H_i}$, $i = 1, 2, \cdots, 4$, emanated from the drift term $\mu(y_t, t)$ and the volatility $\sigma(y_t, t)$. The matrix representation of the drift and volatility terms are given respectively as

$$\mu = \begin{pmatrix} r - q - \frac{1}{2}(v_1 + v_2 + v_3) \\ \kappa_1(\theta_1 - v_1) \\ \kappa_2(\theta_2 - v_2) \\ \alpha\kappa_3(\theta_3 - v_3) \end{pmatrix} \tag{4.9}$$

The volatility is defined as

$$\sigma(y_t, t) = \begin{pmatrix} \sqrt{v_1} & \sqrt{v_2} & \sqrt{v_3} & 0 & 0 & 0 \\ \sigma_1\sqrt{v_1}\rho_1 & 0 & 0 & \sigma_1\sqrt{v_1(1-\rho_1^2)} & 0 & 0 \\ 0 & \sigma_2\sqrt{v_2}\rho_2 & 0 & 0 & \sigma_2\sqrt{v_2(1-\rho_2^2)} & 0 \\ 0 & 0 & \alpha\sigma_3\sqrt{v_3}\rho_3 & 0 & 0 & \alpha\sigma_3\sqrt{v_3(1-\rho_3^2)} \end{pmatrix} \tag{4.10}$$

The product of the volatility matrix in Eq. (4.10) and its transpose matrix σ^T gives the symmetric volatility matrix below

$$\sigma\sigma^T = \begin{pmatrix} v_1+v_2+v_3 & \sigma_1 v_1\rho_1 & \sigma_2 v_2\rho_2 & \sigma_3 v_3\rho_3 \\ \sigma_1 v_1\rho_1 & \sigma_1^2 v_1 & 0 & 0 \\ \sigma_2 v_2\rho_2 & 0 & \sigma_2^2 v_2 & 0 \\ \alpha^2\sigma_3 v_3\rho_3 & 0 & 0 & \alpha^2\sigma_3^2 v_3 \end{pmatrix} \tag{4.11}$$

The symmetric volatility matrix makes the coefficients of terms associated with partial differential equation formulation for the BCRSTSV model in Eq. (3.1) easy to behold. The coefficient matrices terms $\mathbf{J_i}$, $\mathbf{H_i}$, $i = 1, 2, \cdots, 4$ are given as

$$\mathbf{J_0} = \begin{pmatrix} r-q \\ \kappa_1\theta_1 \\ \kappa_2\theta_2 \\ \alpha\kappa_3\theta_3 \end{pmatrix}, \quad \mathbf{J_1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{J_2} = \begin{pmatrix} -\frac{1}{2} \\ -\kappa_1 \\ 0 \\ 0 \end{pmatrix}, \quad \mathbf{J_3} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ -\kappa_2 \\ 0 \end{pmatrix}, \quad \mathbf{J_4} = \begin{pmatrix} -\frac{1}{2} \\ 0 \\ 0 \\ -\kappa_3 \end{pmatrix}$$

satisfying

$$\mu = \mathbf{J_0} + \mathbf{J_1} y + \mathbf{J_2} v_1 + \mathbf{J_3} v_2 + \mathbf{J_4} v_3 \tag{4.12}$$

while

$$\mathbf{H_0} = \mathbf{H_1} = \begin{pmatrix} 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix}, \qquad \mathbf{H_2} = \begin{pmatrix} 1 & \sigma_1\rho_1 & 0 & 0 \\ \sigma_1\rho_1 & \sigma_1^2 & 0 & 0 \\ 0&0&0&0 \\ 0&0&0&0 \end{pmatrix},$$

$$\mathbf{H_3} = \begin{pmatrix} 1 & 0 & \sigma_2\rho_2 & 0 \\ 0&0&0&0 \\ \sigma_2\rho_2 & 0 & \sigma_2^2 & 0 \\ 0&0&0&0 \end{pmatrix}, \quad \text{and} \quad \mathbf{H_4} = \begin{pmatrix} 1 & 0 & 0 & \alpha\sigma_3\rho_3 \\ 0&0&0&0 \\ 0&0&0&0 \\ \alpha\sigma_3\rho_3 & 0 & 0 & \sigma_3^3 \end{pmatrix}$$

such that

$$\sigma\sigma^T = \mathbf{H_0} + \mathbf{H_1} y + \mathbf{H_2} v_1 + \mathbf{H_3} v_2 + \mathbf{H_4} v_3. \tag{4.13}$$

Next, upon the use of Eq. (4.8), the first Ricatti equation in Eq. (4.7) with respect to its own boundary condition yields a solution $B_0(\tau) = i\omega_0$. The Ricatti equations in Eq. (4.7) are further transformed upon substituting the boundary condition for other terms and reversing the sign so as to get derivatives in form of time-to-maturity, τ leads to

$$\begin{cases} \dfrac{\partial B_1}{\partial \tau} = \dfrac{1}{2}\sigma_1^2 B_1^2 - (\kappa_1 - i\omega_0\rho_1\sigma_1)B_1 - \dfrac{1}{2}\omega_0(\omega_0 + i) \\ \dfrac{\partial B_2}{\partial \tau} = \dfrac{1}{2}\sigma_2^2 B_2^2 - (\kappa_2 - i\omega_0\rho_2\sigma_2)B_2 - \dfrac{1}{2}\omega_0(\omega_0 + i) \\ \dfrac{\partial B_3}{\partial \tau} = \dfrac{1}{2}\sigma_3^2 B_3^2 - (\kappa_3 - i\omega_0\rho_3\sigma_3)B_3 - \dfrac{1}{2}\omega_0(\omega_0 + i) \\ \dfrac{\partial A}{\partial \tau} = \kappa_1\theta_1 B_1 + \kappa_2\theta_2 B_2 + \alpha\cdot\kappa_3\theta_3 B_3. \end{cases} \tag{4.14}$$

The fourth equation in the above set of transformed Ricatti differential equations only requires a straightforward integration, while the first three equations are just one-dimensional Ricatti equations. The initial conditions for Eq. (4.14) reduce to $B_1(0) = B_2(0) = B_3(0) = 0$ but $B_0(0) = i\omega$ as obtained earlier while $A(0) = 0$ so that we can set $\omega = \omega_0$. The decision is based on the fact that the characteristic function for the log stock price $y_T = \ln S_T$

is of paramount interest to us rather than the joint characteristic function for the stochastic processes $(y_T, v_1(T), v_2(T), v_3(T))$. The solution for $B_1(\tau), B_2(\tau)$ and $B_3(\tau)$ follows the pattern of univariate and bivariate counterparts. Hence, we gave the solution of the Ricatti equations in Eq. (4.14) for the BCRSTSV model as

$$\begin{cases} A(\omega,\tau) = & (r-q)\omega_1(\tau) + \frac{\kappa_1\theta_1}{\sigma_1^2}\left[(\kappa_1 - \rho_1\sigma_1\omega_1 i + d_1)\tau - 2\ln\left(\frac{1-g_1 e^{d_1\tau}}{1-g_1}\right)\right] \\ & +(r-q)\omega_2(\tau) + \frac{\kappa_2\theta_2}{\sigma_2^2}\left[(\kappa_2 - \rho_2\sigma_2\omega_2 i + d_2)\tau - 2\ln\left(\frac{1-g_2 e^{d_2\tau}}{1-g_2}\right)\right] \\ & +\alpha(r-q)\omega_3(\tau) + \frac{\alpha\kappa_3\theta_3}{\sigma_3^2}\left[(\kappa_3 - \rho_3\sigma_3\omega_3 i + d_3)\tau - 2\ln\left(\frac{1-g_3 e^{d_3\tau}}{1-g_3}\right)\right] \\ B_1(\omega,\tau) = & \frac{1}{\sigma_1^2}(\kappa_1 - \rho_1\sigma_1\omega i + d_1)\left[\frac{1-g_1 e^{d_1\tau}}{1-g_1}\right] \\ B_2(\omega,\tau) = & \frac{1}{\sigma_2^2}(\kappa_2 - \rho_2\sigma_2\omega i + d_2)\left[\frac{1-g_2 e^{d_2\tau}}{1-g_2}\right] \\ B_3(\omega,\tau) = & \frac{\alpha}{\sigma_3^2}(\kappa_3 - \rho_3\sigma_3\omega i + d_3)\left[\frac{1-g_3 e^{d_3\tau}}{1-g_3}\right] \end{cases} \tag{4.15}$$

where

$$\begin{cases} d_1 = & \sqrt{(\kappa_1 - \rho_1\sigma_1\omega i)^2 + \sigma_1^2\omega(\omega+i)} \\ d_2 = & \sqrt{(\kappa_2 - \rho_2\sigma_2\omega i)^2 + \sigma_2^2\omega(\omega+i)} \\ d_3 = & \sqrt{(\kappa_3 - \rho_3\sigma_3\omega i)^2 + \sigma_3^2\omega(\omega+i)} \\ g_1 = & \frac{\kappa_1 - \rho_1\sigma_1\omega i - d_1}{\kappa_1 - \rho_1\sigma_1\omega i + d_1} \\ g_2 = & \frac{\kappa_2 - \rho_2\sigma_2\omega i - d_2}{\kappa_2 - \rho_2\sigma_2\omega i + d_2} \\ g_3 = & \frac{\kappa_3 - \rho_3\sigma_3\omega i - d_3}{\kappa_3 - \rho_3\sigma_3\omega i + d_3} \end{cases} \tag{4.16}$$

Applying the [1] representation called "The Little Heston Trap", we give an alternate solution to the solution in Eq. (4.15). This is done by replacing the positive sign attached to the term $d_j, j = 1, \cdots, 3$ by a negative sign. In our own case, we set $c_j = \frac{1}{g_j}$. A comparison can be made between the Albrecher and Little Heston Trap approach in the papers [1] and [13]. Thus, ensuring that one has a well-behaved integrand for the proposed BCRSTSV model requires setting $c_j = \frac{1}{g_j}$. This gives the following result

$$\begin{cases} A(\omega,\tau) = & (r-q)\omega_1(\tau) + \frac{\kappa_1\theta_1}{\sigma_1^2}\left[(\kappa_1 - \rho_1\sigma_1\omega_1 i - d_1)\tau - 2\ln\left(\frac{1-c_1 e^{-d_1\tau}}{1-c_1}\right)\right] \\ & +(r-q)\omega_2(\tau) + \frac{\kappa_2\theta_2}{\sigma_2^2}\left[(\kappa_2 - \rho_2\sigma_2\omega_2 i - d_2)\tau - 2\ln\left(\frac{1-c_2 e^{-d_2\tau}}{1-c_2}\right)\right] \\ & +\alpha\cdot(r-q)\omega_3(\tau) + \frac{\alpha\kappa_3\theta_3}{\sigma_3^2}\left[(\kappa_3 - \rho_3\sigma_3\omega_3 i - d_3)\tau - 2\ln\left(\frac{1-c_1 e^{-d_3\tau}}{1-c_3}\right)\right] \\ B_1(\omega,\tau) = & \frac{1}{\sigma_1^2}(\kappa_1 - \rho_1\sigma_1\omega i + d_1)\left[\frac{1-c_1 e^{-d_1\tau}}{1-c_1}\right] \\ B_2(\omega,\tau) = & \frac{1}{\sigma_2^2}(\kappa_2 - \rho_2\sigma_2\omega i + d_2)\left[\frac{1-c_2 e^{-d_2\tau}}{1-c_2}\right] \\ B_3(\omega,\tau) = & \frac{\alpha}{\sigma_3^2}(\kappa_3 - \rho_3\sigma_3\omega i + d_3)\left[\frac{1-c_3 e^{-d_3\tau}}{1-c_3}\right] \end{cases} \tag{4.17}$$

where

$$\begin{cases} d_1 &= \sqrt{(\kappa_1 - \rho_1\sigma_1\omega i)^2 + \sigma_1^2\omega(\omega+i)} \\ d_2 &= \sqrt{(\kappa_2 - \rho_2\sigma_2\omega i)^2 + \sigma_2^2\omega(\omega+i)} \\ d_3 &= \sqrt{(\kappa_3 - \rho_3\sigma_3\omega i)^2 + \sigma_3^2\omega(\omega+i)} \\ c_1 &= \frac{\kappa_1-\rho_1\sigma_1\omega i-d_1}{\kappa_1-\rho_1\sigma_1\omega i+d_1} \\ c_2 &= \frac{\kappa_2-\rho_2\sigma_2\omega i-d_2}{\kappa_2-\rho_2\sigma_2\omega i+d_2} \\ c_3 &= \frac{\kappa_3-\rho_3\sigma_3\omega i-d_3}{\kappa_3-\rho_3\sigma_3\omega i+d_3}. \end{cases}$$

Hence, the characteristic function for the BCRSTSV model holds from Eq. (4.7) through Eq. (4.17) as

$$\begin{aligned} f(\omega_0,\omega_1,\omega_2,\omega_3;y_t,v_1(t),v_2(t),v_3(t)) &= \mathbb{E}\Big(e^{i\omega_0 y_t+i\omega_1 v_1(t)+i\omega_2 v_2(t)+i\omega_3 v_3(t)}\Big) \\ &= \exp\Big(A(\tau,\omega)+B_0(\tau,\omega)y_t+B_1(\tau,\omega)v_1(t)+B_2(\tau,\omega)v_2(t)+\alpha B_3(\tau,\omega)v_3(t)\Big) \end{aligned} \tag{4.18}$$

In a more similar fashion to double Heston model but differs in terms of additional state variable and volatility imposed owing to recession, the call pricing formula is given as

$$C(K) = S_t e^{-q\tau}P_1 - Ke^{-r\tau}P_2 \tag{4.19}$$

such that

$$P_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\omega \ln K)\, f(\omega - i; y_t, v_1(t), v_2(t), v_3(t))}{i\omega S_t e^{(r-q)\tau}}\right] d\omega,$$

$$P_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty \mathrm{Re}\left[\frac{\exp(-i\omega \ln K)\, f(\omega; y_t, v_1(t), v_2(t), v_3(t))}{i\omega}\right] d\omega$$

where "Re" is the real part of the expression. The characteristic function $f(\omega-i;y_t,v_1(t),v_2(t),v_3(t))$ is exactly the same as the one given in Eq. (4.6).

5. Numerical results

We set the control parameter $\alpha = 1$, indicating that the economy is in a recession state. The parameter values used to generate the following sample paths are specified in Table 1 in addition to the initial asset price $S_0 = 100.$. The sample paths demonstrate the likely returns of assets under the recession state with respect to the BCRSTSV model subject to regular grid points using the number of time steps $N = \{5000, 3000, 1000, 500\}$, respectively.

Table 1. Parameters used for the numerical computation.

Initial volatility (v_{0i})	σ_i	κ_i	ρ_i	θ_i
$v_{01} = 0.6^2$	$\sigma_1 = 0.10$	$\kappa_1 = 0.90$	$\rho_1 = -0.4$	$\theta_1 = 0.10$
$v_{02} = 0.7^2$	$\sigma_2 = 0.15$	$\kappa_2 = 0.80$	$\rho_2 = -0.3$	$\theta_2 = 0.10$
$v_{03} = 0.9^2$	$\sigma_3 = 0.13$	$\kappa_3 = 0.40$	$\rho_3 = -0.3$	$\theta_3 = 0.0001$

In this section, two categories of discussion are presented. We give a detailed economic analysis of the control parameter α, the mean reversion parameter θ_3 on SDEs in Eq. (3.1). Recession is controlled with α as given in the model. The dynamics in the values of α for economic recovery could be influenced by a range of factors that include fiscal policy, technological innovations, monetary policy, and global economic trends, among others. If the mean reversion parameter θ_3 is increased faster, it can lead to a reduction in uncertainty and volatility from the recessed economy, which may improve investors' decisions and contribute to economic recovery.

The decrease in recession-induced volatility can lead to increased investor confidence, and can influence investment and economic growth. A lower θ_3 would imply a slower mean reversion, leading to more persistent and elevated volatility in the state of recessions. Variation in θ_3 would influence option prices, particularly longer-maturity options. An increase in θ_3 could cause an increase in correlation values with the entire model. However, the recovery process is fully influenced by the control parameter α, while impacts θ_3 are considered secondary to other factors that drive recovery.

The numerical results presented in Table 2 show the option prices on the underlying stock based on the trapezoidal and Gauss-Laguerre quadrature for the BCRSTSV model. This shows the effectiveness and efficiency of the two numerical approaches used for the numerical computation of the BCRSTSV model. Setting the control parameter $\alpha = 0$ in the BCRSTSV model in Eq. (3.1) will lead to the double Heston model in Eq. (2.1). This is possible if the economy is fully recovered from the recession.

Table 2. European call option prices based on trapezoidal and Gauss-Laguerre quadrature for the BCRSTSV model ($\alpha = 1$).

	BCRSTSV- Trapezoidal		BCRSTSV- Gauss-Laguerre	
Strike K	Original	Little Trap	Original	Little Trap
71.33	17.0273	17.0273	17.0273	17.0273
71.33	29.3181	29.3181	29.3181	29.3181
86.62	9.7580	9.7580	9.7580	9.7580
86.62	24.6827	24.6827	24.6827	24.6827
89.67	8.3042	8.3042	8.3042	8.3042
89.67	23.7556	23.7556	23.7556	23.7556
96.81	4.9118	4.9118	4.9118	4.9118
96.81	21.5924	21.5924	21.5924	21.5924
101.90	2.4887	2.4887	2.4887	2.4887
101.90	20.0473	20.0473	20.0473	20.0473

A clear variation in option prices was observed in Table 3 as one passes from a recession-free state to a recession-free state. One of the inferences we could draw from Table 3 is that option prices are affected in a recession state compared to the recession-free state. This verifies the uncertainty effect of the volatile economy on the performance of financial derivatives in an unstable market situation.

Table 3. European call option for the BCRSTSV model price comparison in economic state 1 & 2.

	Recession state- (BCRSTSV)		**Recession-free state- (BCRSTSV)**	
Strike K	Original	Little Trap	Original	Little Trap
71.33	17.0273	17.0273	46.5273	46.5273
71.33	29.3181	29.3181	76.4599	76.4599
101.90	2.4887	2.4887	33.3090	33.3090
101.90	20.0473	20.0473	70.1271	70.1271

Figure 1 and Figure 2 show the sample paths of the stock prices S_T and the trends of the stock prices generated at the specified time steps $N = \{5000, 3000, 1000, 500\}$ respectively, under the BCRSTSV-model.

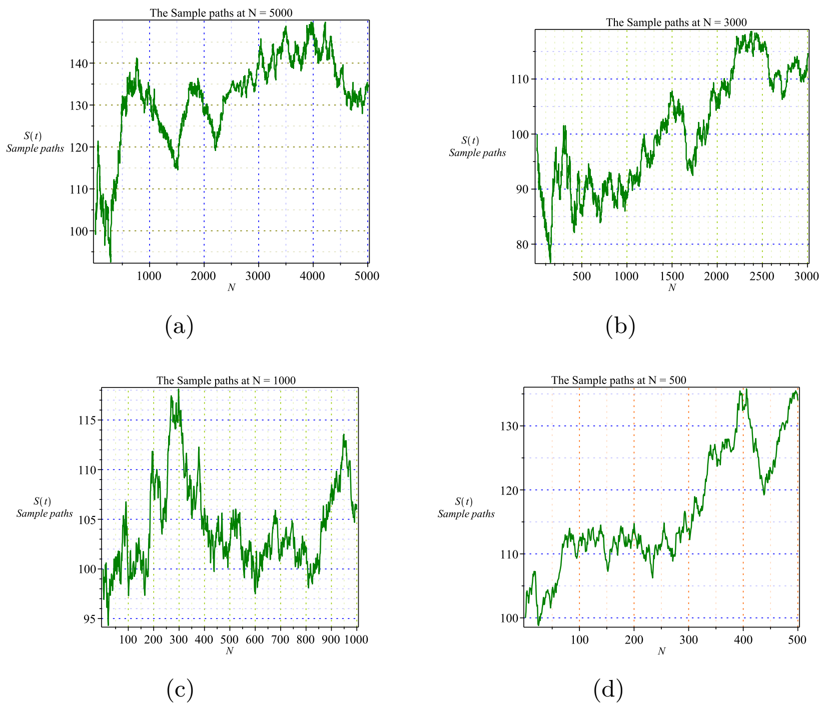

Figure 1. The sample paths of stock prices generated using the BCRSTSV-model.

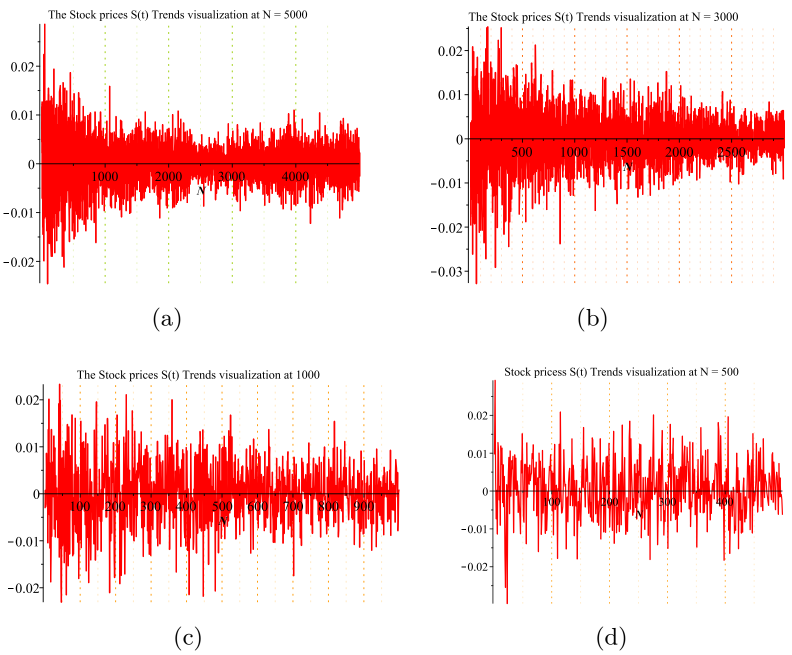

Figure 2. Stock price trends visualization under the BCRSTSV model.

6. Conclusion

This study introduced a novel binary control regime that switches the triple-factor stochastic volatility (BCRSTSV) model for the calculation of option prices under volatile economic conditions, particularly in a recessionary economy. The core innovation lies in the incorporation of a binary control parameter α, which enables dynamic switching between recession and recession-free regimes. This improves the flexibility of the model in capturing macroeconomic shocks and makes it particularly suitable for financial markets experiencing structural economic transitions.

A closed-form characteristic function and a pricing formula for European-style call options were derived and implemented using both the trapezoidal and Gauss-Laguerre quadrature methods. The numerical results clearly demonstrate that the BCRSTSV model produces significantly different option prices depending on the economic regime, as highlighted in Table 3. In particular, option prices under recessionary conditions were lower compared to the recession-free regime, thereby validating the model's sensitivity to macroeconomic uncertainty.

The BCRSTSV model offers a richer volatility structure by introducing a third volatility factor activated only during recessions compared to the classical Heston model [16]. Although the double Heston model of Christoffersen et al. [9] improves the fitting of implied volatility through two variance sources, it lacks regime sensitivity. The proposed BCRSTSV model fills this gap by embedding an economic-state-dependent mechanism, enhancing realism and practical relevance. Similarly, while Liu and Lio [21] developed an option pricing model using uncertainty theory, specifically applying an uncertain exponential Ornstein-Uhlenbeck process within the canonical space framework, their formulation operates outside the traditional probabilistic setting and is thus less adaptable to classical risk-neutral valuation. Their model effectively handles ambiguity but lacks the flexibility to transition between economic regimes. In contrast, the present BCRSTSV model is embedded within a fully stochastic and arbitrage-free framework, facilitating both analytical tractability and consistency with real-world financial market mechanisms, especially under varying macroeconomic conditions.

Acknowledgements

We would like to thank the anonymous referees and editor for their careful reading and helpful comments.

Author contributions. Philip Ajibola Bankole: Conceptualization, Methodology, Writing-Original draft preparation, Software, Visualization. **Olabisi O. Ugbebor:** Methodology, Validation lead & Supervision. **M. E. Egwe:** Formal analysis, Validation support & Supervision.
All authors discussed the results and contributed to the final submission.

Conflict of interest statement. There is no any known conflict of interests or personal relationship that influence this paper.

Funding. No funding was received from any funding organization for this research.

Data availability. No data was used for this research.

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