BEÜ Fen Bilimleri Dergisi
4(1), 12-20, 2015

BEU Journal of Science
4(1), 12-20, 2015

Araştırma Makalesi / Research Article

Dual Eliptik Birim Küre $\grave{E}_D^2$ Üzerindeki Çemberin E.Study Dönüşümü

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Özet
Bu çalışmada, ilk olarak genelleştirilmiş E.Study dönüşümü üç boyutlu $\mathbb{R}^3_{\alpha\beta}$ uzayındaki doğrular için tanımlandı. Sonra, $D^3_{\alpha\beta}$ dual uzayındaki birim dual eliptik küre $\grave{E}_D^2$ üzerinde ki çemberin E.Study dönüşümü incelendi. Ayrıca, herbiri bir geometrik sonuç olan bazı özel durumlar araştırıldı.

Anahtar Kelimeler: Dual eliptik birim küre, genelleştirilmiş iç çarpım, eğimli kongruans

The E.Study Map of Circle on Dual Eliptic Unit Sphere $\grave{E}_D^2$

Abstract
In this paper, first the generalized E. Study mapping is defined for the lines in 3-space $\mathbb{R}^3_{\alpha\beta}$. Then, the E. Study map of circle which lie on the dual elliptical unit sphere $\grave{E}_D^2$ at the dual space $D^3_{\alpha\beta}$ is studied. Furthermore, some special case is examined, each of which is a geometrical result.

Keywords: Dual elliptical unit sphere, generalized inner product, inclined congruence

1. Introduction

Dual numbers were introduced in the 19th century by W.K. Clifford (1849-79), as a tool for his geometrical investigation. After him E. Study [5] and Kotelnikov [3] systematically applied the dual number and dual vector in their studies of line geometry and kinematics and independently discovered the transfer principle. Study devoted special attention to the representation of directed line by dual unit vectors and defined the mapping that is said with his name; he proved that there exists one-to-one correspondence between the points of the dual unit sphere S_D^2 and the directed lines of Euclidean 3-space $\mathbb{R}^3_{\alpha\beta}$(E. Study's mapping). Hence, a differentiable curve on the sphere S_D^2 corresponds to a ruled surface in the line space [1]. Hacisalihoglu [2] showed that the E. Study map of a circle on a dual unit sphere S_D^2 is a family of hyperboloids of on sheet with two parameters. In [4], by taking the Minkowski 3-space $\mathbb{R}^3_1$ instead of $\mathbb{R}^3$; Ugurlu and Çaliskan gave a correspondence of E. Study mapping as follows: There exists one to one correspondence between the dual time-like and space-like unit vectors of dual hyperbolic and Lorentzian unit sphere H_0^2 and S_1^2 at the dual Lorentzian space D_1^3 and the directed time-like and space-like lines of the Minkowski 3-space $\mathbb{R}^3_1$. Subsequently, Yaylı and et. al [6] investigated the E. Study maps of circles on dual hyperbolic and Lorentzian unit spheres H_0^2 and S_1^2 at is the dual

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Lorentzian space D_1^3. By this mapping, a curve on a dual hyperbolic unit sphere H_0^2 corresponds to a timelike ruled surface in the Lorentzain line space $\mathbb{R}_1^3$, that is, there exists a one-to-one correspondence between the geometry of curves on H_0^2 and the geometry of timelike ruled surfaces in $\mathbb{R}_1^3$. This paper is organized as follows: In the first part, the authors consider a generalized inner product on a real 3-dimensional vector space. In the second part, we take a 3-space $\mathbb{R}_{\alpha\beta}^3$ instead of $\mathbb{R}^3$, then, the E. Study maps is generalized. In doing so, E. Study map of circle which lie on the dual elliptical unit sphere $\grave{E}_D^2$ at the dual space $D_{\alpha\beta}^3$ is studied.

2. Basic Concepts

In this section, we define a new inner product and give a brief summary of the dual numbers and a new dual vector space.

Definition 1. For the vectors $\vec{x}=(x_1,x_2,x_3)$ and $\vec{y}=(y_1,y_2,y_3)$, the generalized inner product on $\mathbb{R}^3$ is given by

$$g(\vec{x},\vec{y})=\alpha x_1y_1+\beta x_2y_2+\alpha\beta x_3y_3, \tag{1}$$

where α and β are positive numbers. If $\alpha>0$ and $\beta<0$ then $g(\vec{x},\vec{y})$ is called the generalized Lorentzian inner product. The vector space on $\mathbb{R}^3$ equipped with the generalized inner product is called 3-dimensional generalized space and denoted by $\mathbb{R}_{\alpha\beta}^3$.
The cross product in $\mathbb{R}_{\alpha\beta}^3$ is defined by

$$\vec{x}\times\vec{y}=\beta(x_2y_3-x_3y_2)\vec{i}+\alpha(x_3y_1-x_1y_3)\vec{j}+(x_1y_2-x_2y_1)\vec{k}.$$

It is clear $g(\vec{x},\vec{x}\times\vec{y})=0$ and $g(\vec{y},\vec{x}\times\vec{y})=0$.

Special cases:

1. If $\alpha=\beta=1$, then $\mathbb{R}_{\alpha\beta}^3$ is a Euclidean 3-space $\mathbb{R}^3$.
2. If $\alpha=1,\beta=-1$, then $\mathbb{R}_{\alpha\beta}^3$ is a semi-Euclidean 3-space $\mathbb{R}_2^3$.

Proposition 1. For $\alpha,\beta\in\mathbb{R}^+$ the inner and vector products satisfy the following properties;

1. $\vec{u}\times\vec{v}=-\vec{v}\times\vec{u}$.
2. $g(\vec{u}\times\vec{v},\vec{w})=g(\vec{v}\times\vec{w},\vec{u})=g(\vec{u}\times\vec{w},\vec{v})=\det(\vec{u},\vec{w},\vec{v})$.
3. $\vec{u}\times(\vec{v}\times\vec{w})=g(\vec{u}\times\vec{w})\vec{v}-g(\vec{u},\vec{v})\vec{w}$.

Definition 2. Each element of the set

$$D=\{A=a+\varepsilon a^*: a,a^*\in\mathbb{R}\ and\ \varepsilon\neq 0,\varepsilon^2=0\}=\{A=(a,a^*): a,a^*\in\mathbb{R}\}$$

is called a dual number. Summation and multiplication of two dual numbers are defined as similar to the complex numbers but it is must be forgotten that $\varepsilon^2=0$. Thus, D is a commutative ring with a unit element. The set

$$D^3=\{\tilde{a}=(A_1,A_2,A_3)\,|\,A_i\in D, 1\leq i\leq 3\}$$

is a module over the ring D which called a D-module or dual space. The elements of D^3 are called dual vectors. Thus a dual vector $\tilde{a}$ can be written

$$\tilde{a}=\vec{a}+\varepsilon\vec{a}^*$$

where $\vec{a}$ and $\vec{a}^*$ are real vector at $\mathbb{R}^3$.

Definition 3. Let $\tilde{a} = \vec{a} + \varepsilon\vec{a}^*$ and $\tilde{b} = \vec{b} + \varepsilon\vec{b}$ be in D^3 and $\alpha, \beta \in \mathbb{R}$. The generalized inner product of $\tilde{a}$ and $\tilde{b}$ is defined by

$$g(\tilde{a}, \tilde{b}) = g(\vec{a}, \vec{b}) + \varepsilon(g(\vec{a}, \vec{b}^*) + g(\vec{a}^*, \vec{b})). \qquad (2)$$

We put $\mathrm{D}^3_{\alpha\beta} = (\mathrm{D}^3, g(-,-))$. The norm of dual vector $\tilde{a} = \vec{a} + \varepsilon\vec{a}^* \in \mathrm{D}^3_{\alpha\beta}$ is a dual number given by

$$\|\tilde{a}\| = \sqrt{g(\tilde{a}, \tilde{a})} = \|\vec{a}\| + \varepsilon \frac{g(\vec{a}, \vec{a}^*)}{\|\vec{a}\|^2}, \ \|\vec{a}\| \neq 0.$$

For $\alpha, \beta \in \mathbb{R}^+$, the set of all unit dual vector in $\mathrm{D}^3_{\alpha\beta}$ is said the dual elliptical unit sphere and is denoted by $\widehat{\mathrm{E}}^2_{\mathrm{D}}$. The vector product in $\mathrm{D}^3_{\alpha\beta}$ is

$$\tilde{a} \times \tilde{b} = \vec{a} \times \vec{b} + \varepsilon(\vec{a} \times \vec{b}^* + \vec{a}^* \times \vec{b})$$

Lemma 1. Let $\tilde{u}, \tilde{v} \in \mathrm{D}^3_{\alpha\beta}$ and $\alpha, \beta \in \mathbb{R}^+$. In this case, we have

$$\tilde{u} \times \tilde{v} = \|\tilde{u}\|.\|\tilde{v}\| \sin \phi . \tilde{s}$$

where $\phi = \varphi + \varepsilon\varphi^*$ is the dual angle subtended by the two axes, and $\tilde{s}$ is the unit dual vector which is orthogonal to both $\tilde{u}$ and $\tilde{v}$.

Definition 4. An oriented line L in the three-dimensional Euclidean space E^3 can be determined by a point $p \in \mathrm{L}$ and a normalized direction vector $\vec{a}$ of L, *i.e.* $\|\vec{a}\| = 1$. To obtain components for L, one forms the moment vector $\vec{a}^* = \vec{p} \times \vec{a}$ with respect to the origin point in E^3. If p is substituted by any point $\vec{q} = \vec{p} + \lambda\vec{a}, (\lambda \in \mathbb{R})$ on L then the above equation implies that $\vec{a}^*$ is independent of p on L. The two vectors $\vec{a}$ and $\vec{a}^*$ are unindependent, they satisfy the following relationships;

$$\langle \vec{a}, \vec{a} \rangle = 1, \ \langle \vec{a}, \vec{a}^* \rangle = 0.$$

The six components a_i, a_i^* $(i = 1, 2, 3)$ of $\vec{a}$ and $\vec{a}^*$ are called the normalized Plücker coordinates of the line L. Hence, the two vectors $\vec{a}$ and $\vec{a}^*$ determine the oriented line L.
Conversely, any six-tuple a_i, a_i^* $(i = 1, 2, 3)$ with relations

$$a_1^2 + a_2^2 + a_3^2 = 1, \ \ a_1 a_1^* + a_2 a_2^* + a_3 a_3^* = 0,$$

represents a line in the three-dimensional space E^3.

Theorem 1. (E. Study) The oriented lines in $\mathbb{R}^3$ are in one-to-one correspondence with the points of the dual unit sphere in D^3 [1].

3. Generalized E. Study Map

In this section, we generalize the theorem which is attributed to Eduard Study [5].

Theorem 2. The set of all oriented lines in $\mathbb{R}^3_{\alpha\beta}$ is in one-to-one correspondence with the set of the points of the dual unit sphere in the dual 3-space $\mathrm{D}^3_{\alpha\beta}$.

Proof: First let $\alpha, \beta > 0$. In $\mathbb{R}^3_{\alpha\beta}$ a directed line can be given by $\vec{y} = \vec{x} + \lambda\vec{a}$, where $\vec{x}$ and $\vec{a}$ are the position vector and the direction vector of the line, respectively. The moment vector $\vec{a}^* = \vec{x} \times \vec{a}$ is not depending on the chosen point on the line. For this reason, by the help of ordered pair of vectors ($\vec{a},\vec{a}^*$), a directed line is determined by one unique point in $D^3_{\alpha\beta}$ and the following conditions are satisfied:

$$g(\vec{a},\vec{a})=1,\ g(\vec{a},\vec{a}^*)=0.$$

Suppose that $\tilde{a}=\vec{a}+\varepsilon\vec{a}^*$ is a dual unit vector in $D^3_{\alpha\beta}$. Taking $\tilde{b}=\tilde{a}$ in equation (2) we obtain:

$$g(\tilde{a},\tilde{a})=g(\vec{a},\vec{a})+2\varepsilon(g(\vec{a},\vec{a}^*))=1$$

where the dual unit vector $\tilde{a}$ represents the unique directed line ($\vec{a},\vec{a}^*$).
Note that this theorem holds for $\alpha>0$ and $\beta<0$.

Special cases:

1) If $\alpha=\beta=1$, then we get E. Study mapping in Euclidean 3-space $\mathbb{R}^3$.
2) If $\alpha=1,\ \beta=-1$, then we will have E. Study mapping for a space which is isomorphic to Minkowski 3-space $\mathbb{R}^3_1$ [4].

Example 1. In $D^3_{\alpha\beta}$, the unit dual vector corresponding to the line $L=\{x=y,z=2\}\subset\mathbb{R}^3_{\alpha\beta}$ is $\tilde{u}=\frac{1}{\sqrt{\alpha+\beta}}((1,1,0)+2\varepsilon(-\beta,\alpha,0))$, where $P=(0,0,2)$ and $\vec{u}=(1,1,0)$ are the point of vector and the direction vector of line L, respectively. The moment vector is $\vec{u}^*=\overrightarrow{op}\times\vec{u}=2(-\beta,\alpha,0)$.
Let us give a point of $\hat{E}^2_D$, for instance, $\tilde{u}=\frac{1}{\sqrt{2}}(0,\frac{1}{\sqrt{\beta}},\frac{1}{\sqrt{\alpha\beta}})+\varepsilon(0,\sqrt{\alpha\beta},\sqrt{\beta})$. It determines the line

$$L=\left\{x=\sqrt{2}\beta,\frac{y-\sqrt{\frac{\alpha}{2\beta}}}{1/\sqrt{2\beta}}=\frac{z-\frac{1}{\sqrt{2\beta}}}{1/\sqrt{2\alpha\beta}}\right\}\subset\mathbb{R}^3_{\alpha\beta}$$

Definition 5. The six components a_i, a_i^* $(i=1,2,3)$ of $\vec{a}$ and $\vec{a}^*$ are called Pluckerian homogenous coordinates of the directed line.

Let $\hat{E}^2_D, O$ and $\{o,\tilde{e}_1,\tilde{e}_2,\tilde{e}_3\}$ denote the dual elliptic unit sphere, the center of $\hat{E}^2_D$ and the dual orthonormal system at O respectively, where we have

$$\tilde{e}_i=\vec{e}_i+\varepsilon\vec{e}_i^*,\quad 1\le i\le 3 \tag{3}$$

$$\tilde{e}_1\times\tilde{e}_2=\tilde{e}_3,\qquad \tilde{e}_2\times\tilde{e}_3=\tilde{e}_1,\qquad \tilde{e}_3\times\tilde{e}_1=\tilde{e}_2 \tag{4}$$

and

$$\vec{e}_1\times\vec{e}_2=\vec{e}_3\quad,\quad\vec{e}_2\times\vec{e}_3=\vec{e}_1\quad,\quad\vec{e}_3\times\vec{e}_1=\vec{e}_2 \tag{5}$$

for

$$\vec{e}_1=(\frac{1}{\sqrt{\alpha}},0,0)\quad,\quad\vec{e}_2=(0,\frac{1}{\sqrt{\beta}},0)\quad,\quad\vec{e}_3=(0,0,\frac{1}{\sqrt{\alpha\beta}}). \tag{6}$$

In this case, the orthonormal system $\{o,\vec{e}_1,\vec{e}_2,\vec{e}_3\}$ is the system of the space of lines in $\mathbb{R}^3_{\alpha\beta}$. The moment vectors $\vec{e}_i^*$ can be written as

$$\vec{e}_i^*=MO\times\vec{e}_i,\quad 1\le i\le 3,\quad MO=(\lambda_3,\lambda_2,\lambda_1) \tag{7}$$

Since these moment vectors are the vectors of $\mathbb{R}^3_{\alpha\beta}$ we may write

$$e_i^*=\sum_{j=1}^3\lambda_{ij}e_j,\ \ \lambda_{ij}\in\mathbb{R},1\le i\le 3 \tag{8}$$

Hence, the relations (7) and (8) give us

$$\begin{bmatrix} \vec{e}_1^* \\ \vec{e}_2^* \\ \vec{e}_3^* \end{bmatrix} = \begin{bmatrix} 0 & \lambda_1\sqrt{\alpha\beta} & -\lambda_2\sqrt{\beta} \\ -\lambda_1\sqrt{\alpha\beta} & 0 & \lambda_3\sqrt{\alpha} \\ \lambda_2\sqrt{\beta} & -\lambda_3\sqrt{\alpha} & 0 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}$$

Hence the E. Study mapping can be given as a mapping from the dual orthogonal system, in $\widehat{\mathrm{E}}_D^2$, to the real orthogonal system, in $\mathbb{R}^3_{\alpha\beta}$. Using the above relations; we can express the E. Study mapping in the matrix form as follows:

$$\begin{bmatrix} \tilde{e}_1 \\ \tilde{e}_2 \\ \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & \varepsilon\lambda_1\sqrt{\alpha\beta} & -\varepsilon\lambda_2\sqrt{\beta} \\ -\varepsilon\lambda_1\sqrt{\alpha\beta} & 1 & \varepsilon\lambda_3\sqrt{\alpha} \\ \varepsilon\lambda_2\sqrt{\beta} & -\varepsilon\lambda_3\sqrt{\alpha} & 1 \end{bmatrix} \begin{bmatrix} \vec{e}_1 \\ \vec{e}_2 \\ \vec{e}_3 \end{bmatrix}$$

which says the Study mapping corresponds with a dual orthogonal matrix.
Since we know that the linear mappings are in one-to-one correspondence with the matrices, then we may give the following theorem.

Theorem 3. The E. Study mapping is a linear isomorphism.

A ruled surface in 3-space $\mathbb{R}^3_{\alpha\beta}$ is a differentiable one-parameter set of straight lines, which is generated by the motion of a straight line.
Let $\tilde{x}$ and $\tilde{y}$ denote any two different points at $\widehat{\mathrm{E}}_D^2$ and $\tilde{\Phi}$ denote the dual elliptic angle $(\tilde{x}, \tilde{y})$.The elliptical angle $\tilde{\Phi}$ has a value $\varphi + \varepsilon\varphi^*$ which is a dual number, where φ and φ^* are the elliptic angle and the minimal distance between directed lines $\tilde{x}$ and $\tilde{y}$, respectively.

Theorem 4. Let $\tilde{x}, \tilde{y} \in \widehat{E}_D^2$, then we have $g(\tilde{x}, \tilde{y}) = \cos\tilde{\Phi}$, where

$$\cos\tilde{\Phi} = \cos\varphi - \varepsilon\varphi^* \sin\varphi.$$

Proof: Moment vectors $\vec{x}^*$ and $\vec{y}^*$ are independent of choice of the points p and q on the directed lines L_1 and L_2 which correspond to $\tilde{x}$ and $\tilde{y}$ in $\mathbb{R}^3_{\alpha\beta}$. Thus the points p and q can be thought of feet points of common perpendicular line of L_1 and L_2. The unit vector of common perpendicular is

$$\vec{n} = \mp \frac{\vec{x} \times \vec{y}}{\|\vec{x} \times \vec{y}\|}.$$

If we show the shortest distance between L_1 and L_2 by φ^*, we get

$$\vec{p} - \vec{q} = \mp \frac{\vec{x} \times \vec{y}}{\|\vec{x} \times \vec{y}\|} \varphi^*$$

Now we consider following equations;

$$\begin{aligned} g(\vec{x}, \vec{y}^*) &= g(\vec{x}, \vec{q} \times \vec{y}) = -g(\vec{q}, \vec{x} \times \vec{y}), \\ g(\vec{x}^*, \vec{y}) &= g(\vec{p} \times \vec{x}, \vec{y}) = g(\vec{p}, \vec{x} \times \vec{y}), \end{aligned}$$

where $\vec{x}^* = \vec{p} \times \vec{x}$ and $\vec{y}^* = \vec{q} \times \vec{y}$.
By adding the last two equations, we got

$$\begin{aligned} g(\vec{x},\vec{y}^*)+g(\vec{x}^*,\vec{y}) &= g(\vec{p}-\vec{q},\vec{x}\times\vec{y}) \\ &= g(\mp\frac{\vec{x}\wedge\vec{y}}{\|\vec{x}\wedge\vec{y}\|}\varphi^*,\vec{x}\times\vec{y}) \\ &= \mp\varphi^*\|\vec{x}\times\vec{y}\| \\ &= \mp\varphi^*\sin\varphi \end{aligned}$$

If we choose the minus (sign), we get

$$g(\tilde{x},\tilde{y})=\cos\varphi-\varphi^*\sin\varphi,$$

and from the Taylor formula

$$g(\tilde{x},\tilde{y})=\cos\tilde{\Phi}.$$

Special cases:

i) The condition $g(\tilde{x},\tilde{y})\neq 0$ means that the lines $\tilde{x}$ and $\tilde{y}$ cannot be orthogonal.

ii) If $g(\tilde{x},\tilde{y})=$pure real and $\varphi^*=0$, then the lines $\tilde{x}$ and $\tilde{y}$ intersect each other.

iii) And if $g(\tilde{x},\tilde{y})=\pm 1$, $\varphi=0$ and $\varphi^*=0$, then the line $\tilde{x}$ and $\tilde{y}$ are coincident.

4. The E. Study mapping of a circle on $\widehat{\mathrm{E}}_{\mathrm{D}}^2$

Let l be the straight line corresponding to the dual vector $\tilde{e}_3$. If we choose the point P on l then we have $\lambda_2=\lambda_3=0$ and so the matrix (8) reduces to

$$\begin{bmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{bmatrix}=\begin{bmatrix}1 & \varepsilon\lambda_1\sqrt{\alpha\beta} & 0\\ -\varepsilon\lambda_1\sqrt{\alpha\beta} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{bmatrix} \tag{9}$$

The inverse of this mapping

$$\begin{bmatrix}\vec{e}_1\\ \vec{e}_2\\ \vec{e}_3\end{bmatrix}=\begin{bmatrix}1 & -\varepsilon\lambda_1\sqrt{\alpha\beta} & 0\\ \varepsilon\lambda_1\sqrt{\alpha\beta} & 1 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}\tilde{e}_1\\ \tilde{e}_2\\ \tilde{e}_3\end{bmatrix}. \tag{10}$$

Let

$$\widehat{\mathrm{E}}_{\mathrm{D}}^1=\{\tilde{x}\in\widehat{\mathrm{E}}_{\mathrm{D}}^2 \mid g(\tilde{x},\tilde{e}_3)=\cos\tilde{\Phi}=\text{constant}\}$$

be the circle on the sphere $\widehat{\mathrm{E}}_{\mathrm{D}}^2$ and a point of $\widehat{\mathrm{E}}_{\mathrm{D}}^1$ be $\tilde{x}$. Thus the dual vector $\tilde{x}$ can be expressed as

$$\tilde{x}=\frac{1}{\sqrt{\alpha}}\sin\tilde{\Phi}\cos\tilde{\Psi}\tilde{e}_1+\frac{1}{\sqrt{\beta}}\sin\tilde{\Phi}\sin\tilde{\Psi}\tilde{e}_2+\frac{1}{\sqrt{\alpha\beta}}\cos\tilde{\Phi}\tilde{e}_3 \tag{11}$$

where $\tilde{\Phi}=\varphi+\varepsilon\varphi^*$ and $\tilde{\Psi}=\psi+\varepsilon\psi^*$ are the dual elliptic angle and dual angle, respectively. Since we have the relations

$$\tilde{x}=\vec{x}+\varepsilon\vec{x}^*$$

$$\begin{aligned} \sin\tilde{\Phi}&=\sin\varphi+\varepsilon\varphi^*\cos\varphi, \quad \sin\tilde{\Psi}=\sin\psi+\varepsilon\psi^*\cos\psi \\ \cos\tilde{\Phi}&=\cos\varphi-\varepsilon\varphi^*\sin\varphi, \quad \cos\tilde{\Psi}=\cos\psi-\varepsilon\psi^*\sin\psi. \end{aligned} \tag{12}$$

(9) and (11) give us the vector $\vec{x}$ and $\vec{x}^*$ in the matrix form:

$$\vec{x} = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\alpha}} \sin\varphi\cos\psi \\ \frac{1}{\sqrt{\beta}} \sin\varphi\sin\psi \\ \frac{1}{\sqrt{\alpha\beta}} \cos\varphi \end{bmatrix}$$

$$\vec{x}^* = \begin{bmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{\alpha}} \varphi^* \cos\varphi\cos\psi - \left(\frac{\psi^*}{\sqrt{\alpha}} + \lambda_1\sqrt{\alpha} \right) \sin\varphi\sin\psi \\ \frac{1}{\sqrt{\beta}} \varphi^* \sin\psi\cos\varphi + \left(\frac{\psi^*}{\sqrt{\beta}} + \lambda_1\sqrt{\beta} \right) \sin\varphi\cos\psi \\ -\frac{1}{\sqrt{\alpha\beta}} \varphi^* \sin\varphi \end{bmatrix}$$

On the other hand the point $\tilde{x}$ is on the circle whose center is a point of the axis $\tilde{e}_3$.Thus we may write

$$g(\tilde{x}, \tilde{e}_3) = \cos\tilde{\Phi} = (\cos\varphi - \varepsilon\varphi^* \sin\varphi) = \text{constant}$$

which means that

$$\varphi = c_1(\text{constant}) \text{ and } \varphi^* = c_2(\text{constant})$$

The above equations permit us to write the following relation:

$$\begin{aligned} &g(\vec{x}, \vec{x}) = 1, \\ &g(\vec{x}, \vec{x}^*) = 0, \\ &g(\vec{x}, \vec{e}_3) - \cos\varphi = 0, \\ &g(\vec{x}, \vec{e}_3^*) + g(\vec{x}^*, \vec{e}_3) + \varphi^* \sin\varphi = 0. \end{aligned}$$

The last equations have only two parameters φ and φ^*. So it represents a line Congruence in $\mathbb{R}^3_{\alpha\beta}$. Now we may calculate the equations of this congruence in Plucker coordinates. Let $\vec{y}$ denote a point of this congruence then we have

$$\vec{y} = \vec{x}(\psi, \psi^*) \wedge \vec{x}^*(\psi, \psi^*) + v\,\vec{x}(\psi, \psi^*)$$

If the coordinates of $\vec{y}$ are (y_1, y_2, y_3), then this gives us

$$\begin{aligned} y_1 &= -\frac{1}{\sqrt{\alpha}} \varphi^* \sin\psi - \frac{1}{\sqrt{\alpha}} \left(\psi^* + \lambda_1\beta \right) \sin\varphi\cos\varphi\cos\psi + v\frac{1}{\sqrt{\alpha}} \sin\varphi\cos\psi \\ y_2 &= \frac{1}{\sqrt{\beta}} \varphi^* \cos\psi - \frac{1}{\sqrt{\beta}} \left(\psi^* + \lambda_1\alpha \right) \sin\varphi\cos\varphi\sin\psi + v\frac{1}{\sqrt{\beta}} \sin\varphi\sin\psi \\ y_3 &= \frac{\psi^*}{\sqrt{\alpha\beta}} \sin^2\varphi + \lambda_1 \sin^2\varphi \left(\sqrt{\frac{\alpha}{\beta}} \sin^2\psi + \sqrt{\frac{\beta}{\alpha}} \cos^2\psi \right) + v\frac{1}{\sqrt{\alpha\beta}} \cos\varphi. \end{aligned}$$

Case 1: if $\alpha = \beta = 1$, we have

$$\begin{aligned} y_1 &= -\varphi^* \sin\psi - \left(\psi^* + \lambda_1 \right) \sin\varphi\cos\varphi\cos\psi + v\sin\varphi\cos\psi \\ y_2 &= \varphi^* \cos\psi - \left(\psi^* + \lambda_1 \right) \sin\varphi\cos\varphi\sin\psi + v\sin\varphi\sin\psi \\ y_3 &= (\psi^* + \lambda_1) \sin^2\varphi + v\cos\varphi \end{aligned}$$

In the case, we have a hyperbolic of one sheet

$$\frac{y_1^2}{c_2^2}+\frac{y_2^2}{c_2^2}-\frac{\left[y_3-\left(\psi^*+\lambda_1\right)\right]^2}{\left[c_2\cot c_1\right]^2}=1 \text{ [2].}$$

Case 2: if $\alpha=\beta$, we have

$$\begin{aligned} y_1 &= -\frac{1}{\sqrt{\alpha}}\varphi^*\sin\psi-\left(\frac{\psi^*}{\sqrt{\alpha}}+\sqrt{\alpha}\lambda_1\right)\sin\varphi\cos\varphi\cos\psi+v\frac{1}{\sqrt{\alpha}}\sin\varphi\cos\psi \\ y_2 &= \frac{1}{\sqrt{\alpha}}\varphi^*\cos\psi-\left(\frac{\psi^*}{\sqrt{\alpha}}+\sqrt{\alpha}\lambda_1\right)\sin\varphi\cos\varphi\sin\psi+v\frac{1}{\sqrt{\alpha}}\sin\varphi\sin\psi \\ y_3 &= (\frac{\psi^*}{\alpha}+\lambda_1)\sin^2\varphi+v\frac{1}{\alpha}\cos\varphi \end{aligned} \tag{18}$$

If $\sqrt{\alpha}y_1=Y_1$, $\sqrt{\alpha}y_2=Y_2$ and $Y_3=\alpha y_3$ then;

$$\frac{Y_1^2}{c_2^2}+\frac{Y_2^2}{c_2^2}-\frac{\left[Y_3-\left(\psi^*+\alpha\lambda_1\right)\right]^2}{\left[c_2\cot c_1\right]^2}=1 \tag{19}$$

Which has two parameters ψ^* and λ_1. So it represents a line congruence with degree 2. The lines of this congruence are located so that

(a) The shortest distance of these lines and line l is $\varphi^*=c_2$.
(b) The angle of these lines and the line l is $\varphi=c_1$.

Hence we can say that the lines of this congruence intersect the generators of a cylinder whose radius is φ^*=constant, and the axis is l, under the angle φ=constant.

Definition 6. If all the lines of a line congruence have a constant angle with a definite line then the congruence is called an inclined congruence.

According to this definition, (19) represents an inclined congruence. Hence we may give the following theorem.

Theorem 5. Let $\hat{E}_D^1$ be a circle with two parameter on the dual elliptic unit sphere $\hat{E}_D^2$.
The Study map of $\hat{E}_D^1$ is an inclined congruence with degree two.

On the other hand we know that the shortest distance of the axis l of the cylinder and the lines of the congruence is c_2.Therefore, this cylinder is the envelope of the lines of the congruence.

Special cases:

(i) The case that $\varphi^*\neq 0$ and $\varphi=\frac{\pi}{2}$.

In this case the lines of the congruence (19) orthogonally intersect the generators of the cylinder whose axis is l and the radius is φ^*. Indeed, in the case (19) reduces to

$$\begin{aligned} Y_1^2+Y_2^2 &= c_2^2+v^2 \\ Y_3 &= \psi^*+\alpha\lambda_1 \end{aligned}$$

(ii) The case that $\varphi^*\neq 0$ and $\varphi=0$.

In this case the lines of the congruence coincide with the generators of the cylinder which is the envelope of the lines congruence. This means that the Study map of reduces to the cylinder whose equations, form (19), are

$$Y_1^2 + Y_2^2 = c_2^2$$
$$Y_3 = v$$

(iii) The case that $\varphi^* = 0$ and $\varphi \neq 0$.

In this case, all of the lines congruence intersect the axis l under the constant elliptic angle φ. We can say that the lines of the congruence are the common lines of two linear line complexes. Form (19) the equation of congruence is

$$Y_1^2 + Y_2^2 - \frac{\left[Y_3 - \left(\psi^* + \alpha\lambda_1\right)\right]^2}{\cot^2 c_1} = 1$$

(iv)The case that $\varphi^* = 0$ and $\varphi = \dfrac{\pi}{2}$.

In this case $\widehat{\mathrm{E}}_{\mathrm{D}}^1$ is a great circle on $\widehat{\mathrm{E}}_{\mathrm{D}}^2$.Then all of the lines of congruence orthogonally intersect the axis l. This means that the inclined congruence reduces to a linear line complex whose axis is l.(19) gives us that equation of congruence is

$$Y_1^2 + Y_2^2 = v^2$$
$$Y_3 = (\psi^* + \alpha\lambda_1)$$

(v) The case that $\varphi^* = 0$ and $\varphi = 0$.

In the case, all of the lines of the congruence are coincide with the line l. Indeed, (19) reduces to the line l:

$$Y_1^2 + Y_2^2 = 0$$
$$Y_3 = v.$$

Definition 7. If all the lines of a line congruence orthogonally intersect a constant line then the congruence is called a recticongruence.

Theorem 6. Let $\widehat{\mathrm{E}}_{\mathrm{D}}^1$ be a great circle on $\widehat{\mathrm{E}}_{\mathrm{D}}^2$, that is,

$$\widehat{\mathrm{E}}_{\mathrm{D}}^1 = \{\tilde{x} \in \widehat{\mathrm{E}}_{\mathrm{D}}^2 \,|\, \langle \tilde{x}, \tilde{e}_3 \rangle = 0\}$$

Then the E. Study mapping of $\widehat{\mathrm{E}}_{\mathrm{D}}^1$ is a recticongruence.

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Geliş Tarihi: 25/01/2015
Kabul Tarihi: 25/03/2015