Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 19.07.2025
DOI : 10.15672/hujms.1541132

Research Article

Efficient techniques for designing and evaluating multivariate Hotelling control chart with generalized sensitizing rules

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Abstract

In this study, efficient techniques are utilized to design the multivariate Hotelling control chart with sensitizing rules for detecting small-to-moderate variations. The control limit of the proposed chart is derived relative to probability of a single point and number of process characteristics. To calculate probability of a single point for sustained in-control average run length, a generalized single polynomial equation is derived. For evaluation, performance measures are considered based on the average, the median, and the percentile run length. These measures are calculated using Monte Carlo simulation and numerical integration. The results indicate that the proposed control chart has consistent behavior when a process is in control. The in-control average run length is obtained equal to prefixed level which remains valid for all choices of sensitizing rules. This implies that the proposed control chart can resolve the issue of existing control chart in terms of sustained behavior. The effectiveness of sensitizing rules is dependent on process characteristics and variations of mean vector. A comparative analysis of different choices of sensitizing rules is conducted to locate optimal choices of process characteristics. Real-life example, dowel-pin manufacturing, shows that proposed control chart with sensitizing rules is efficient for diagnosing small variations.

Mathematics Subject Classification (2020). 62H10 , 62H12, 62P30, 62Q05

Keywords. Average run length, Hotelling, median run length, percentile run length, performance measures, sensitizing rules, single polynomial approach.

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Received: 11.09.2024; Accepted: 04.07.2025

1. Introduction

A production or manufacturing process consists of several characteristics (variables of interest) that need to be stable over time, space, or sample number. Process variables are affected by two sources of variations, known as common cause and special cause [1]. The common cause variations are small, whereas the special cause variations are large in magnitude. A process in which variables of interest contain common cause variations is called statistically in control, otherwise out-of-control. To determine whether a process is in-control or out-of-control, statistical process control (SPC) methods are applied. The SPC methods are composed of seven tools: cause and effect diagram, control chart, stratification, check sheet, histogram, Pareto chart, and scatter diagram. Among various listed SPC tools, more importance is generally given to quality control charts due to their easy implementation, understandability, and meaningful interpretation.

The quality control chart was first introduced by Shewhart [1] in 1931 to independently monitor multiple process characteristics. A control chart mechanism in which a single process characteristic can be monitored is known as a univariate control chart. The well-known univariate control chart for monitoring location parameters is the mean control chart [1]. For the simultaneous monitoring of multiple characteristics, Hotelling [2] designed multivariate T^2 control chart.

The univariate and multivariate control charts are based on one point decision rule to declare a process in-control or out-of-control. The one point decision rule (also known as classical rule and denoted as $1|1$) is defined as a process is said to be out-of-control if any point triggered outside the control limit(s). Based on, $1|1$ rule, diversified control charts were developed and applied over past decades. Park and Jun [3] designed the exponentially weighted moving average (EWMA) control chart for the identification of special cause variation (shift) in bivariate signal data from a steel company manufacturing process. In addition, Moore et al. [4] employed a multivariate T^2 control chart to investigate oil quality and debris monitoring during wear out process. For intermittent fault detection, Zhao et al. [5] developed moving average multivariate T^2 control chart with multiple window lengths. Furthermore, Oliveira et al. [6] established the modified Hotelling control chart to monitor the dynamics of batch processes. Western [7] demonstrated that the $1|1$ rule with the Shewhart-type control charts [1,2,4,6] is considered efficient for the detection of large variations. He advocated the simultaneous implementation of classical and additional sensitizing rules.

The simultaneous approach benefits from the timely detection of variations, but creates the issue of inflating the false alarm rate (FAR) [7] and/or in-control average run length (IARL). The false alarm rate (FAR) and IARL are well known performance measures used to evaluate a control chart. The FAR is a probability of declaring an in-control process out-of-control provided that the process is in-control. The ARL represent the average number of sample points that should be declared in-control before declaring an in-control process out-of-control. Thus, the issue of sensitizing rules was resolved by various authors by considering two procedures, that is, (i) independent implementation of the sensitizing rules instead of simultaneous and (ii) construction of control limits such that FAR or IARL is sustained at a prefixed level.

Some relevant studies to support the aforementioned discussions are cited as [8–13] and references therein. To achieve (i), Champ and Woodall [8] stated $2|3$ (two out of three consecutive points outside the upper control limit or lower control limit), and $2|4$ (two out of four consecutive points outside the upper control limit or lower control limit). In addition, independent implementation of the rules $2|3$, $2|4$, $3|3$, and $3|4$ can be seen in the study by [9]. For the construction of control limits to meet (ii), incorporate a desired value of the probability of a single point (PSP) (for further details, see [9,14]). The PSP is a probability that a single point lies outside the control limit given that a process is in control. It is decided according to a choice of sensitizing rule and prefixed IARL. For instance, PSP is 0.0027 when the 1|1 rule and prefixed IARL is 370 under consideration [15].

For the calculation of the correct value of the PSP, Champ and Woodall [8] and Klein [9] provided ARL based Markov chain approach (MCA). In more detail, Champ and Woodall [8] applied the ARL based MCA to calculate the PSP for 2|2, and 2|3 rules. Klien [9] presented MCA to calculate PSP with a graphical technique for more choices of sensitizing rules such as 2|4 and 3|4. The ARL based MCA to calculate the PSP consists of a system of linear equations depending on the choice of the sensitizing rules. For example, four and seven sets of linear equation are required for 2|2 and 2|3 sensitizing rules respectively with supplemental constraints. The number of linear equations increases as r or w increases. The solution of the linear equations to calculate PSP often becomes tedious for a quality engineer when the number of equations is large (cf.[9, 16]). Riaz et al. [14], and Mehmood et al. [17] derived FAR based single polynomial equation (SPE) to attain PSP. The application of the FAR based SPE [14] can be seen in the development of the power calculation code [18], the design of the dispersion control chart [17], the dual auxiliary control chart [19], the control chart of skewness correction [20], control charts for known and unknown parameters [21], the bivariate Hotelling control chart [22]. It is essential to mention here that FAR based SPE does not serve the purpose to sustain the in-control run length properties of the multivariate control charts at their desired level [16, 23]. Thus, the prescribed limitations in existing control chart methods are considered motivations for the present study.

The current study proposes efficient techniques for developing and evaluating the ARL based multivariate Hotelling control chart with generalized sensitizing rules. The general form of the sensitizing rules is as follows: a process is said to be statistically out-of-control if r out of w (denoted as $r|w$, $r \leq w$) consecutive points fall outside the control limit. One of the general forms objectives is to explore the more optimal choices of the sensitizing rules, particularly for the multivariate Hotelling T^2 control chart alternative to existing ones. It is worth mentioning that the existing sensitizing rules are the special cases in the general form. In addition, the control limit of the proposed graph is derived as a function of PSP and the number of process characteristics. To determine the PSP, ARL based generalized SPE (GSPE) is considered taking into account limitations of the existing SPE and MCA as mentioned earlier. To evaluate the performance of the proposed control chart, existing and alternative performance measures are considered. The ARL, MRL, PRL, extra quadratic loss EQL based on ARL, relative ARL, performance comparision index (PCI) based on ARL are existing measures whose application can be seen in various studies [24–27]. The alternative overall performance measures cover EQL based on MRL, EQL based on PRL, performance comparison index (PCI) based on MRL, PCI based PRL, relative MRL and relative PRL.

To highlight the practical significance of the current study, proposed methods are applied to monitor several variations in dowel pin characteristics (diameter and length). The rest of the study is organized as follows: In Section 2, ARL based Hotelling T^2 control chart with generalized sensitization rules is developed. In Section 3, the significance and interpretation of performance measures are described. In Section 4, computational procedures are presented to obtain the values of individual and overall performance measures. Section 5 includes results, discussion, and comparative analysis followed by the real-life example in Section 6. The summary and conclusion of the current study are given in Section 7.

2. Proposed multivariate control chart with generalized sensitizing rules

In this section, we define a plotting statistic and also design control limit with generalized sensitizing rules.

2.1. Plotting statistic

Let X_1, X_2, $\cdots$, X_t be t process characteristics which follow multivariate normal distribution with known in-control mean vector U, and variance-covariance matrix Σ are given as follows:

$$U = \begin{pmatrix} U_1 \\ U_2 \\ U_3 \\ \vdots \\ U_t \end{pmatrix},$$

and,

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} & \cdots & \sigma_{1t} \\ \sigma_{21} & \sigma_2^2 & \sigma_{23} & \cdots & \sigma_{2t} \\ \sigma_{31} & \sigma_{32} & \sigma_3^2 & \cdots & \sigma_{3t} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \sigma_{t1} & \sigma_{t2} & \sigma_{t3} & \cdots & \sigma_t^2 \end{pmatrix},$$

where diagonal of Σ represents variance of X_1, X_2, $\cdots$, and X_t, respectively and off diagonal denoted covariance between X_i and X_l $(i \neq l)$. An alternate form of Σ is given as follows:

$$\Sigma = \begin{pmatrix} \sigma_1^2 & \rho_{12}\sigma_1\sigma_2 & \rho_{13}\sigma_1\sigma_3 & \cdots & \rho_{1t}\sigma_1\sigma_t \\ \rho_{21}\sigma_2\sigma_1 & \sigma_2^2 & \rho_{23}\sigma_2\sigma_3 & \cdots & \rho_{2t}\sigma_2\sigma_t \\ \rho_{31}\sigma_3\sigma_1 & \rho_{32}\sigma_3\sigma_2 & \sigma_3^2 & \cdots & \rho_{3t}\sigma_3\sigma_t \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \rho_{t1}\sigma_t\sigma_1 & \rho_{t2}\sigma_t\sigma_2 & \rho_{t3}\sigma_t\sigma_3 & \cdots & \sigma_t^2 \end{pmatrix}.$$

where ρ_{il} donates the correlation between X_i and X_l. Let x_{ijk}, where $i = 1, 2, \ldots, t$ denotes the i^{th} variable, $j = 1, 2, \ldots, m$ denotes the j^{th} sample, and $k = 1, 2, \ldots, n$ denotes the k^{th} observation, and T_j^2 be the plotting statistics for the j^{th} sample is defined as follows:

$$T_j^2 = n(\widehat{U}_j - U)'\Sigma^{-1}(\widehat{U}_j - U), \tag{2.1}$$

where $\widehat{U}_j$ represents sample mean vector to estimate U, $(\widehat{U}_j - U)'$ symbolized transpose of $\widehat{U}_j - U$, and Σ^{-1} is the inverse of Σ. The $\widehat{U}_j$ is given as

$$\widehat{U}_j = \begin{pmatrix} \bar{x}_{1j} \\ \bar{x}_{2j} \\ \bar{x}_{3j} \\ \vdots \\ \bar{x}_{tj} \end{pmatrix},$$

where,

$$\bar{x}_{ij} = \frac{1}{n}\sum_{k=1}^{n} x_{ijk}, \quad i = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, m.$$

2.2. Control limit

Let ARL be the prefixed in-control average run length. Thus, a process is said to be out-of-control if r out of w (denoted as $r|w$) statistic T_j^2 crosses the control limit (denoted as H), and it is defined as follows:

$$H = \chi^2_{1-p,t}, \tag{2.2}$$

where $\chi^2_{1-p,t}$ denotes $(1-p)^{\text{th}}$ percentile of the chi-squared distribution with t degrees of freedom, and p is the PSP such that IARL is equal to the ARL. The value of p depends on the choice of r, w and ARL. To find required value of p, we have proposed a generalized single polynomial equation (GSPE) as follows:

$$E(R) = \frac{(1-p^r)^{w-r+1}\,(w-r)!\,r!}{p^r(1-p)^{w-r}\,(r-wp)\,(w-1)!}. \tag{2.3}$$

The derivation of the Eq.qrefkeyU is given in Appendix **??**. It is important to mention that Eq.(2.3) has the relationship with classical geometric, and generalized geometric distribution, as proved and shown by [37]. In more detail, substituting $r = w = 1$ in Eq.(2.3) leads to $E(R) = 1/p$, which mean of classical geometric distribution with parameter p. This is interpreted as in-control run length distribution of Hotelling T^2 control chart is geometric [16]. Likewise, setting $r = w$, Eq.(2.3) reduces to $E(R) = (1-p^r)/(p^r(1-p))$ which represent the mean of the generalized geometric distribution with parameter p, [38].This means that in-control run length distribution of Shewhart-type upper-sided mean control chart with $r|w$ $(r = w)$ rules is generalized geometric of order w, [37].

Regarding the computation procedure, above equation is solved for p by providing values of r, w and equating $E(R)$ to ARL. As an explanation, we have calculated values of p for certain choices of r, w, and ARL and ultimately presented in the Table 1, see Appendix A3 for code.

Table 1. p values at different choices of ARL

ARL	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
500	0.0020	0.0457	0.0329	0.0272	0.0238	0.1319	0.0943	0.1657	0.3044	0.4089	0.5471
370	0.0027	0.0533	0.0385	0.0319	0.0280	0.1466	0.1052	0.1804	0.3217	0.4285	0.5686
200	0.0050	0.0732	0.0534	0.0446	0.0394	0.1825	0.1324	0.2159	0.3623	0.4735	0.6165

3. Performance measures with significance and interpretations

The current section illustrates various new existing and alternative performance measures to compare the detection capability of a control chart. The existing measures include ARL, MRL, and PRL, extra quadratic loss (EQL) based on ARL, relative ARL, performance comparision index (PCI) based on ARL. The alternative overall performance measures cover EQL based on MRL, EQL based on PRL, performance comparison index (PCI) based on MRL, PCI based PRL, relative MRL and relative PRL. Thus, complete descriptions and interpretations of each performance measure are given in subsequent subsections.

3.1. Average run length

Average run length is defined as the average number of samples that must be arranged on a control chart before an out-of-control signal is triggered. It is categorized as in-control average run length (ARL_0) and out-of-control average run length (ARL_1). In addition, ARL is a measure to represent the mean value instead of middle value as the run length distribution of the Shewhart-type is asymmetric. A control chart with minimum ARL_1 is preferred provided that all control charts under consideration have same ARL_0 value [15].

3.2. Median run length

The median run length (MRL) for a control chart is another essential measure to represent a middle value of the run length distribution and it is considered as alternative to ARL [28]. The MRL_0 and MRL_1 illustrate the median of run length distribution for in-control and out-of-control processes, respectively. A control chart with minimal MRL_1 is recommended given that all control chart has equal MRL_0.

3.3. Percentile Run length

In addition to the MRL, multiple PRLs were utilized by [24] such as 25th, 75th and 90th. The $PRL_{(0)25}$, $PRL_{(0)75}$, and $PRL_{(0)90}$ illustrate the in control PRL. In contrast, out-of-control PRL are represented by $PRL_{(1)25}$, $PRL_{(1)75}$ or $PRL_{(1)90}$. Moreover, a control chart is said to be outstanding if it attains a minimum value of the out-of-control PRL ($PRL_{(1)25}$, $PRL_{(1)75}$ or $PRL_{(1)90}$) given that all control charts have similar value of the corresponding in-control PRL ($PRL_{(0)25}$, $PRL_{(0)75}$, or $PRL_{(0)90}$) .

3.4. Extra quadratic loss

Extra quadratic loss (EQL) is another overall performance measure that encapsulates the anticipated loss because of the associated poor quality. It illustrates the overall effectiveness of a control chart and interpreted as a weighted average of out-of-control run length for a complete range of λ. Thus, EQR_{ARL}, EQR_{MRL}, and EQL_{PRL_q} $(q = 25, 75, 90)$ are given as follows:

$$EQL_{ARL} = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \lambda^2 ARL_1(\lambda) d\lambda.$$

$$EQL_{MRL} = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \lambda^2 MRL_1(\lambda) d\lambda.$$

$$EQL_{PRL_q} = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \lambda^2 PRL_{1(q)}(\lambda) d\lambda.$$

where, λ_{min} and λ_{max} represents the minimum and maximum values of the interval (λ), respectively. A control chart with minimal EQL_{ARL}, EQL_{MRL}, and EQL_{PRL_q} is declared best in comparison with other control charts provided that all control charts have similar ARL_0, MRL_0, and $PRL_{0(q)}$ values.

3.5. Relative average run length, median run length and percentile run length

Relative average run length (RARL), median run length (RMRL), and percentile run length (RPRL) are considered overall performance indicators which elaborate the exact departure of the control chart from established benchmark over a complete range of λ. The RARL, RMRL, and RPRL are effective methods which decides the overall potential of a control chart relative to the benchmark. Thus, RARL, is formulated as follows:

$$RARL = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \frac{ARL_1(\lambda)}{ARL_1(\lambda)_{BM}} d\lambda.$$

where λ_{min} denotes the minimum value of interval, λ_{max} denotes maximum value of interval, $ARL_1(\lambda)$ denotes out-of-control ARL of specific control chart, and $ARL_1(\lambda)_{BM}$ denotes out-of-control ARL of benchmark (BM) control chart. The BM control chart is the one that has minimum EQL_{ARL} value. In the similar manners, $RMRL$, and $RPRL$ are presented as follows:

$$RMRL = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \frac{MRL_1(\lambda)}{MRL_1(\lambda)_{BM}} d\lambda.$$

$$RPRL_q = \frac{1}{\lambda_{max} - \lambda_{min}} \int_{\lambda_{min}}^{\lambda_{max}} \frac{PRL_{(1)q}(\lambda)}{PRL_{(1)q}(\lambda)_{BM}} d\lambda.$$

3.6. Performance comparison index

To evaluate the relative efficacy of two control charts, the concept of performance comparison index (PCI) was coined by [29]. Thus, the simplest form of PCI based on ARL (denoted as PCI_{ARL}) is as follows:

$$PCI_{ARL} = \frac{EQR_{ARL}}{EQL_{ARL_{(BM)}}},$$

where $EQL_{ARL_{(BM)}}$ symbolized extra quadratic loss of the BM control chart (minimum EQL_{ARL}). A control chart is considered best when PCI_{ARL} value is greater than one ($PCI_{ARL} > 1$). On a similar guidelines, PCI_{MRL} and PCI_{PRL_q} ($q = 25, 75, 90$) are defined as follows:

$$PCI_{MRL} = \frac{EQR_{MRL}}{EQL_{MRL_{(BM)}}}.$$

$$PCI_{PRL_q} = \frac{EQR_{PRL_q}}{EQL_{PRL_{q(BM)}}}.$$

4. Computation of performance measures

In this section we calculate the values of the performance measures (see Section 3) for proposed Hotelling T^2 control chart with generalized sensitizing rules $(r|w)$. Let δ_1, δ_2, δ_3,…,δ_t be the magnitude of shift (special cause of variations) which may occur in mean level U_1, U_2, U_3, …, U_t, respectively. Assume that a shift occurs in at least one element of the known in-control process mean vector U and variance-covariance matrix Σ is in in-control state. In addition, out-of-control mean vector is denoted as V and defined by

$$V = U + \delta$$

where δ denote the vector of shift and it is defined as

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \\ \delta_3 \\ \vdots \\ \delta_t \end{pmatrix}.$$

In multivariate control chart procedure, Mahalanobis distance (denoted as λ) is considered to represent the amount of shift, that is,

$$\lambda = \sqrt{(V-U)^t \Sigma^{-1} (V-U)},$$

$$\lambda = \sqrt{\delta^t \Sigma^{-1} \delta}.$$

The Mahalanobis distance is always non-negative, the requirement on the covariance matrix to be positive definite ensures that the quadratic form is also positive definite, otherwise we can't take square root of negative quantities. Moreover, the positive definite matrices are non-singular, that also ensures that the covariance matrix is invertible, since we are taking the inverse of it. Prior to calculating the performance measures, it is important to decide the values of λ provided that known Σ. For any value of λ, there are several possible combinations of δ. Here, it is important to mention that the behavior of the performance measures for multivariate control chart is dependent on the choice of λ in respect of the choice of δ and Σ. The aforesaid behavior is known as directional invariance property (for further details, see [24, 30, 31]. Now steps for calculating the performance measures are as follows:

(1) Design the control limit given in Eq.(2.3) for the given choice of ARL, r and w.

(2) For a decided value of λ, generate a sufficient number of random samples (e.g. 7×10^3) of size n from the multivariate normal distribution with out-of-control mean vector V and variance-covariance matrix Σ.
(3) Evaluate the T_j^2 statistic of each sample and store it to utilize for subsequent steps.
(4) Record the sample number (or say run length) at which r consecutive statistics cross the control limit.
(5) Calculate the ARL_0 ($\lambda = 0$) and ARL_1 ($\lambda \neq 0$) by taking the mean of run length vector from Step 5.
(6) Compute the MRL_0 and MRL_1 by taking the median of the run length vector from Step 5. Similarly, calculate the $PRL_{0(25)}$ and $PRL_{1(25)}$ by taking the 25^{th} percentile of run length vector. Likewise, obtain the $PRL_{0(75)}$, $PRL_{1(75)}$, and $PRL_{0(90)}$, $PRL_{1(90)}$ by considering 75^{th} and 90^{th} percentiles of run length vector, respectively.
(7) Compute the overall performance measures using any numerical integration techniques by considering the calculated values of the performance measures in 6 and 7.

Based on Steps 1–7, we have calculated values of individual and overall performance measures (see Section 4) by taking into account number of factors including λ $(0, 0.15, 0.20, 0.30, 0.35, 0.45, 0.55, 0.65, 0.80, 1.00, 1.15, 1.20, 1.25, 1.30, 2.00, 3.00)$, t $(2, 5, 10)$, $ARL(200, 370, 500)$, $r|w$ $(1|1, 2|2, 2|3, 2|4, 2|5, 3|3, 3|4, 4|5, 7|9, 8|9, 9|9)$, and n $(1, 3, 5)$. The different choices of the λ and t are motivated from the study by [31], [32] and [23]. Without loss of generality, one may try the other choices of λ, r, w, and n. Also, by considering the directional in-variance property as discussed earlier, we assume equal shifts $\delta_i = a$, variances $\sigma_i^2 = 1$ for $i = 1, \cdots, t$, and covariance $\sigma_{ij} = 0$ for $i \neq j$,. Thus, a relationship between a and λ can be derived as follows: $a = \lambda/\sqrt{t}$. For instance, actual shift amounts correspond to the λ are 0.00, 0.067, 0.089, 0.11, 0.13, 0.15, 0.17, 0.20, 0.22, 0.24, 0.26, 0.29, 0.31, 0.33, 0.35, 0.44, 0.47, 0.49, 0.51, 0.53, 0.55, 0.58, 0.67, 0.76, 0.89, 1.34 for $t = 5$. Finally, results are summarized in Tables 2–10 and Figures 1–5. Note that the Simpson rule is used as numerical integration technique for calculating the overall performance measures over the domain of small-to-moderate shifts $(0.10 \leq \lambda \leq 2.00)$. The implementation of the Simpson rule to calculate overall performance measures can be seen in different studies [25, 31].

5. Results and discussion

In this section, we are concerned with examining and presenting the behavior of the proposed control chart with respect to t, ARL, r, and w. In this regard, we employ the calculated values of the performance measures (Tables 2–10 and Figures 1–5).

5.1. Sustained behavior

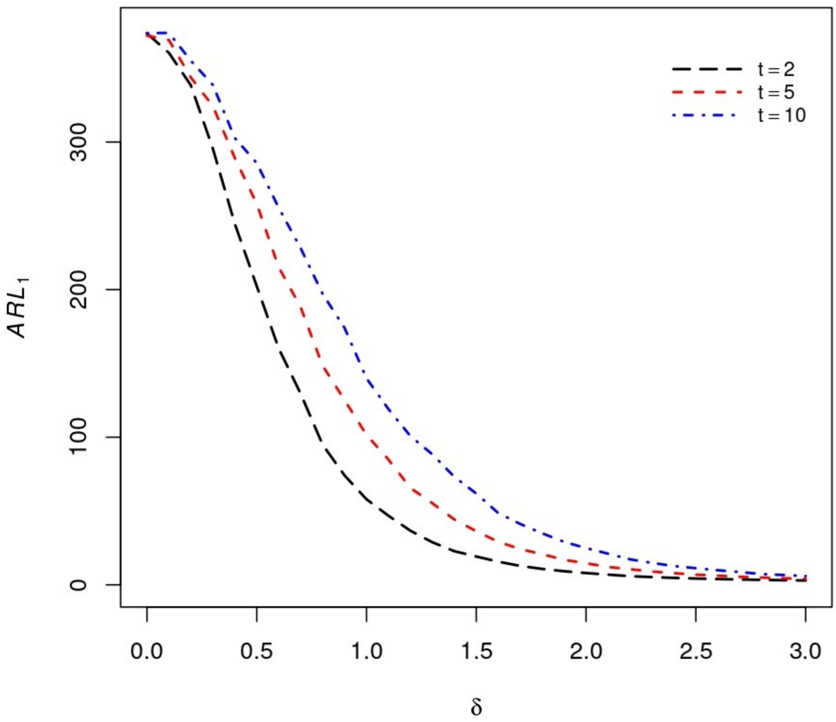
The calculated values of ARL_0, MRL_0, $PRL_{(0)25}$, $PRL_{(0)75}$, and $PRL_{(0)90}$ are recorded close to their desired values for various choices of the $r|w$ and t (see Tables 2–4). For example, in Table 2 at $t = 2$ and $\gamma_1 = 200$, ARL_0 value of the proposed control chart for 2|3 is 197.12, 2|4 is 198.24, 3|4 is 197.06, 4|5 is 197.85, and 7|9 is 199.48, whereas desired value is 200. Similarly, in Tables 3 ($ARL = 370$) and 4 ($ARL = 500$), ARL_0 values at distinct choices of t exhibit comparable behavior. Furthermore, MRL_0, $PRL_{(0)25}$, $PRL_{(0)70}$, and $PRL_{(0)90}$ values of the proposed control chart show sustained behavior with each choice of $r|w$ rule and t (see Tables 2–4). For instance, in Table 2 at $t = 2$ and $\gamma_1 = 200$, MRL_0 values for the proposed control chart with 2|3, 2|4, 3|4, 4|5 and 7|9 rules are 138.00, 135.00, 138.00, 137.00, and 141.00 respectively, whereas desired value is 138. Additionally, with MRL_0, the same behavior is seen at $ARL = 370$ and $ARL = 500$ with different t values. Furthermore, at various selections of $r|w$, t, and ARL, the behavior of $PRL_0(25)$, $PRL_0(75)$, and $PRL_0(90)$ values of control charts is observed to be comparable. This means that the proposed control chart resolves the issue of the existing method in terms of maintaining the in-control run length properties at desired level as discussed in Section 1.

5.2. Role of t, n and λ

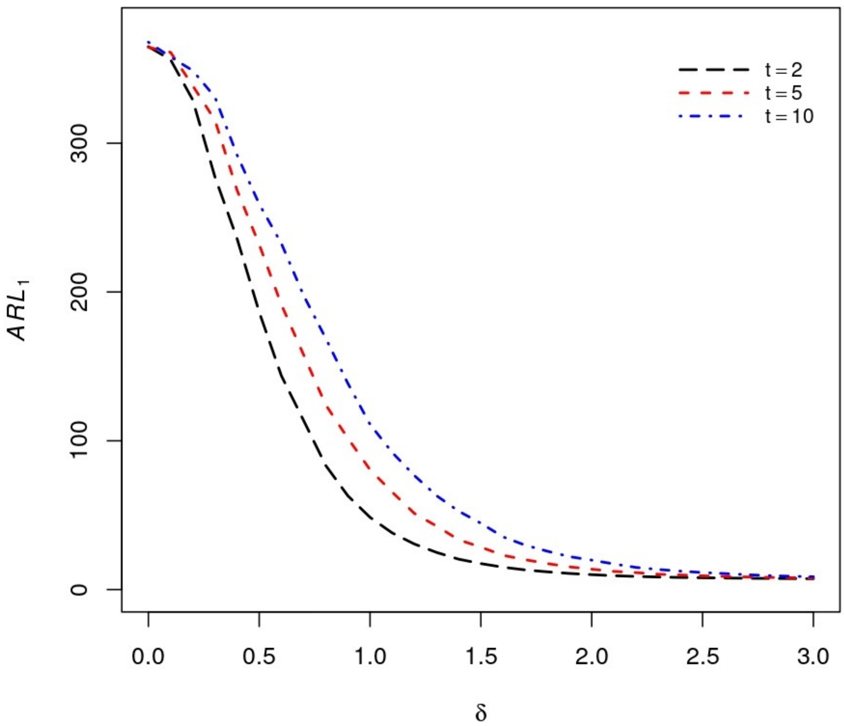
ARL_1, MRL_1, $PRL_{(1)25}$, $PRL_{(1)75}$, and $PRL_{(1)90}$ are observed to be generally decreasing with an increase in λ, and decrease in the number of variables t (see Tables 6–9 and Figure 1). Lower values of the ARL_1, MRL_1, $PRL_{(1)25}$, $PRL_{(1)75}$, and $PRL_{(1)90}$ translate into early detection of the out-of-control signals. This indicates that the proposed control chart performance improves with increasing and decreasing values of λ and t receptively. For example, in Table 5, for $2|3$ and $\lambda = 0.15$, ARL_1 values are 345.04 for $t = 2$, 351.67 for $t = 5$, and 358.14 for $t = 10$.

Additionally, in Table 6, with $2|3$ rule and $\lambda = 0.15$, MRL_1 values are 241.00 at $t = 2$, 242.00 at $t = 5$, and 248.00 at $t = 10$. This trend is sustained for all other values of ARL_1, MRL_1, $PRL_{(1)25}$, $PRL_{(1)75}$, and $PRL_{(1)90}$ at $t = 2, 5$, and $t = 10$ with different values of λ and $r|w$ rule. The following outcomes are in accordance with the existing studies [31, 32], which validate the proposed method.

In Figure 2, it is clear that ARL_1 value decreases as sample size increases. It is interpreted that as performance of control chart increases as sample size increases. Similar approach for evaluating a control chart performance at different choices of sample size can be found in various existing studies such as [16], [37], and [38]. These studies are used as the motivation and guidance for the current study.



(a) 2|2 rule



(b) 7|9 rule

Figure 1. Effect of t on ARL_1 values for various choices of λ, $r|w$ and $ARL = 370$

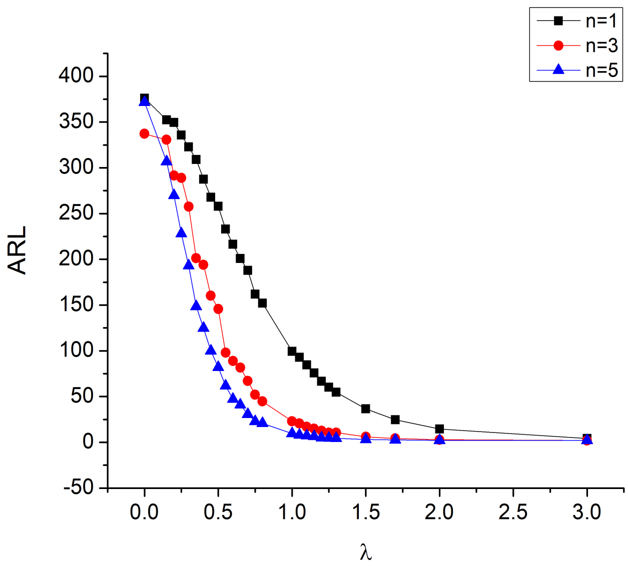


Figure 2. Effect of n on ARL_1 values for various choices of λ, $r|w$ and $ARL = 370$ at $t = 5$.

Table 2. ARL_0, MRL_0, $PRL_{25(0)}$, $PRL_{75(0)}$ and $PRL_{90(0)}$ for different choices of t, $r|w$ and $ARL = 200$

Measures	t	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
ARL_0	2	199.03	202.59	197.12	198.24	194.69	200.89	197.06	197.85	199.48	192.76	198.67
	5	195.56	200.58	199.05	196.40	195.43	199.33	199.50	200.06	196.98	195.23	199.81
	10	201.57	202.92	198.58	198.86	196.43	196.78	197.81	195.04	198.29	198.93	200.23
MRL_0	2	138.00	143.00	138.00	135.00	134.00	137.50	138.00	137.00	141.00	136.00	141.00
	5	134.00	138.00	137.00	138.00	134.00	136.00	142.00	140.00	137.00	138.00	143.00
	10	140.00	142.00	142.00	140.00	139.00	136.00	136.00	137.00	137.00	139.50	141.00
$PRL_{25(0)}$	2	56.00	62.00	57.00	57.00	57.00	59.00	59.00	59.00	63.00	62.00	62.00
	5	58.00	57.00	59.00	59.00	55.00	59.00	60.00	60.00	62.00	61.00	63.00
	10	58.00	61.00	58.00	59.00	60.00	57.75	59.00	59.00	59.00	62.00	62.00
$PRL_{75(0)}$	2	278.00	279.00	272.25	276.00	268.00	276.00	273.00	271.00	275.00	265.00	272.00
	5	266.00	279.00	273.00	269.00	271.00	275.00	273.00	272.25	269.00	273.00	276.00
	10	278.00	283.00	272.00	274.00	273.00	272.00	275.00	269.00	274.00	273.00	274.00
$PRL_{90(0)}$	2	458.10	461.10	451.00	455.10	454.00	455.00	444.00	455.10	448.00	435.00	447.00
	5	450.00	464.00	457.10	450.00	451.00	458.10	454.00	457.00	445.00	442.00	452.00
	10	464.00	468.00	451.10	453.10	447.00	449.00	453.00	437.00	454.00	454.00	445.00

Table 3. ARL_0, MRL_0, $PRL_{25(0)}$, $PRL_{75(0)}$ and $PRL_{90(0)}$ for different choices of t, $r|w$ and $ARL = 370$

Measures	t	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
ARL_0	2	364.14	365.33	368.35	373.97	368.90	372.83	367.31	365.64	367.91	368.26	373.14
	5	366.94	376.09	372.75	370.45	366.51	365.18	371.64	368.73	370.82	366.90	372.24
	10	371.61	370.01	372.50	372.82	364.54	372.30	369.09	373.90	367.76	370.05	368.08
MRL_0	2	250.00	255.00	254.00	260.00	254.00	260.00	260.00	254.00	257.00	260.00	264.00
	5	250.00	264.00	260.00	260.00	257.00	253.00	258.00	258.00	261.00	257.00	261.00
	10	254.00	257.00	262.00	258.00	254.00	257.00	257.00	258.00	261.00	256.00	257.00
$PRL_{25(0)}$	2	103.00	107.00	109.00	108.00	107.00	110.00	111.00	105.00	113.00	111.00	113.00
	5	103.00	113.00	111.00	109.00	108.00	110.00	106.00	111.75	112.00	112.00	115.00
	10	106.75	109.00	110.00	111.75	106.75	105.00	106.00	111.00	110.00	110.00	112.00
$PRL_{75(0)}$	2	507.00	506.00	512.00	514.00	508.00	514.25	512.00	507.00	508.00	507.00	509.00
	5	512.00	514.25	517.00	515.25	502.25	509.00	515.00	512.00	505.00	504.00	513.00
	10	511.00	515.00	517.00	512.25	506.00	515.25	514.00	522.00	515.00	507.00	502.00
$PRL_{90(0)}$	2	850.00	836.00	846.10	856.00	849.00	854.10	834.00	841.10	845.00	844.00	857.00
	5	850.00	855.00	852.20	839.10	840.00	841.10	851.00	844.00	835.00	839.10	839.00
	10	858.00	863.00	850.00	855.00	841.00	862.00	848.00	849.00	836.00	840.00	839.00

Table 4. ARL_0, MRL_0, $PRL_{25(0)}$, $PRL_{75(0)}$ and $PRL_{90(0)}$ for different choices of t, $r|w$ and $ARL = 500$

Measures	t	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
ARL_0	2	499.87	508.21	505.22	504.40	502.16	506.66	498.24	498.19	497.89	497.86	511.39
	5	497.62	497.64	507.62	503.55	509.11	497.59	499.24	499.57	498.01	494.85	498.83
	10	499.42	503.50	506.55	492.61	500.96	496.98	501.42	489.68	495.00	498.07	505.27
MRL_0	2	343.00	349.00	348.00	348.00	342.50	354.00	346.00	349.00	353.50	345.00	355.00
	5	348.00	336.00	350.00	349.00	348.50	347.00	345.00	349.00	350.00	343.00	348.00
	10	348.00	354.00	349.00	339.00	345.50	341.00	343.00	339.00	345.50	353.50	353.00
$PRL_{25(0)}$	2	140.75	148.00	148.00	145.00	146.00	145.00	144.00	148.75	154.00	149.00	153.00
	5	142.00	141.00	148.00	150.00	150.00	146.00	144.00	148.00	148.75	146.00	151.00
	10	140.00	148.00	144.00	135.00	146.00	146.00	142.00	143.00	148.00	152.00	149.00
$PRL_{75(0)}$	2	695.00	700.25	703.00	696.00	694.00	700.25	693.00	696.00	687.25	690.00	703.00
	5	684.00	686.25	710.00	687.00	702.00	692.00	694.00	697.00	691.25	680.00	685.25
	10	691.00	705.00	704.25	677.25	690.00	687.00	697.00	674.00	681.00	692.00	706.00
$PRL_{90(0)}$	2	1153.00	1192.00	1160.00	1166.00	1152.00	1172.20	1141.10	1131.20	1131.10	1127.10	1161.00
	5	1143.00	1151.00	1166.00	1150.10	1173.00	1145.00	1146.10	1151.10	1126.10	1136.20	1134.00
	10	1147.10	1123.00	1169.00	1143.00	1152.10	1148.00	1137.00	1141.00	1126.00	1129.00	1150.00

Table 5. ARL_1 values for different choice of t, λ and $ARL = 370$

t	λ	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
2	0.15	351.01	345.01	345.04	339.87	336.39	346.21	342.59	341.69	342.89	346.11	352.40
	0.20	331.05	336.97	329.81	325.12	323.67	334.30	332.21	333.47	324.40	332.54	333.65
	0.25	306.99	314.19	303.24	301.94	301.92	316.55	306.76	310.19	307.93	311.10	323.12
	0.30	289.91	292.11	288.75	281.94	277.17	296.88	282.42	289.04	285.01	298.52	306.57
	0.35	265.14	264.99	256.53	254.83	253.70	270.63	257.47	262.23	253.07	270.89	289.70
	0.45	225.23	220.82	212.95	206.76	204.35	223.76	212.10	211.42	207.57	221.47	248.99
	0.55	184.91	177.12	168.50	163.74	160.05	180.94	168.45	167.59	163.26	181.11	204.90
	0.65	147.79	142.53	130.40	125.90	124.71	142.41	130.35	131.20	126.77	141.25	165.05
	0.80	105.45	95.36	87.48	84.33	81.39	97.83	84.12	85.23	83.00	93.59	119.06
	1.00	65.46	57.98	51.83	49.92	48.72	58.56	49.93	50.45	48.93	57.57	75.92
	1.15	49.24	40.95	36.57	35.58	34.76	41.33	35.01	35.64	34.87	41.00	56.28
	1.20	43.19	35.90	32.43	30.93	30.01	36.20	30.72	30.82	30.63	35.92	49.49
	1.25	39.75	32.85	29.01	28.01	27.12	32.61	27.72	27.83	27.90	32.96	45.51
	1.30	34.95	28.43	25.48	24.46	23.41	29.46	24.34	24.74	24.57	29.83	40.55
	2.00	9.66	7.91	7.14	7.03	6.93	8.47	7.30	7.93	9.99	11.59	15.26
	3.00	2.58	2.93	2.84	2.84	2.90	3.77	3.58	4.45	7.32	8.25	9.63
5	0.15	357.04	352.36	351.67	353.95	351.39	356.73	357.07	350.63	357.99	350.47	358.79
	0.20	344.25	349.58	340.30	346.76	339.69	348.68	345.64	338.08	344.08	344.43	349.43
	0.25	342.33	335.82	337.37	330.11	333.07	338.13	328.88	331.10	328.58	331.10	342.17
	0.30	321.84	322.84	323.23	315.84	314.72	326.33	323.14	316.21	317.09	316.32	328.12
	0.35	305.77	309.02	299.13	295.57	294.93	300.06	299.27	297.82	290.98	290.78	305.49
	0.45	278.22	267.83	263.39	260.87	258.18	274.76	262.71	256.26	254.15	264.03	278.65
	0.55	242.75	233.02	224.33	225.91	219.38	227.88	218.01	218.41	208.39	214.68	236.10
	0.65	209.27	200.82	191.30	188.68	183.86	197.57	190.09	183.84	178.83	185.03	206.47
	0.80	160.68	152.01	142.01	136.13	134.18	148.28	137.13	132.69	126.56	136.11	155.85
	1.00	114.46	99.37	92.30	90.42	87.98	100.89	89.15	87.08	79.56	90.00	109.01
	1.15	86.72	75.84	71.10	66.46	64.96	74.22	65.61	64.16	59.12	65.77	82.00
	1.20	78.56	66.80	60.99	56.32	56.94	63.65	57.34	54.93	51.07	57.61	72.62
	1.25	71.10	60.33	54.94	51.24	51.39	59.74	50.95	49.67	46.58	53.22	67.09
	1.30	65.51	54.74	50.47	47.59	47.38	53.01	46.00	45.13	42.22	48.55	62.31
	2.00	18.05	14.61	13.22	12.47	12.44	14.58	12.31	12.67	13.74	15.74	20.94
	3.00	4.21	4.07	3.82	3.79	3.86	4.82	4.39	5.16	7.81	8.76	10.53
10	0.15	363.52	360.73	358.14	357.75	353.99	356.58	358.46	357.34	360.22	354.10	362.89
	0.20	358.87	354.37	355.77	352.47	355.34	357.34	351.83	352.66	347.75	350.01	355.97
	0.25	348.79	342.11	346.44	339.40	342.17	343.18	349.37	333.19	334.19	340.26	345.76
	0.30	337.56	337.39	332.84	339.26	331.32	335.82	335.53	333.80	330.26	330.55	336.45
	0.35	327.61	326.08	322.32	319.66	312.75	324.12	320.34	317.55	313.76	316.38	331.50
	0.45	305.65	304.09	297.76	295.84	292.30	303.42	291.04	291.59	280.85	291.30	296.57
	0.55	282.08	275.59	267.60	265.63	264.90	272.07	262.60	258.06	248.55	259.32	271.57
	0.65	246.00	234.54	231.32	226.37	226.30	229.32	222.30	220.47	208.19	215.00	233.70
	0.80	212.65	199.18	191.20	186.83	188.39	192.63	187.07	180.10	168.78	176.00	195.88
	1.00	154.30	143.23	131.86	129.64	128.62	137.28	125.17	121.45	112.48	119.52	139.44
	1.15	128.64	112.36	107.06	102.67	100.30	109.71	100.01	94.64	85.97	93.65	111.78
	1.20	118.06	103.99	93.88	92.02	90.38	98.46	89.42	86.30	75.94	82.96	100.53
	1.25	105.19	92.28	84.49	82.08	80.90	86.84	79.17	75.82	68.23	74.48	89.94
	1.30	102.20	85.84	80.43	77.75	75.37	83.69	73.62	70.67	63.40	71.20	86.29
	2.00	31.64	24.84	22.23	21.30	20.98	23.92	20.20	19.92	19.47	22.36	29.57
	3.00	6.93	5.89	5.44	5.35	5.33	6.55	5.85	6.44	8.63	9.71	12.15

Table 6. MRL_1 values for different choice of t, λ and $ARL = 370$

t	λ	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
2	0.15	244.00	242.00	241.00	233.00	231.00	240.50	236.00	235.00	241.00	243.00	250.00
	0.20	229.50	234.00	232.00	226.00	225.00	232.00	225.00	230.50	225.00	234.00	229.00
	0.25	215.00	216.00	214.00	212.00	211.00	217.00	213.00	211.00	212.00	216.00	227.00
	0.30	202.00	203.00	205.00	195.00	193.00	205.00	194.00	200.50	200.00	210.00	218.00
	0.35	185.00	183.00	178.00	177.00	177.00	186.00	177.00	183.00	178.00	190.00	203.00
	0.45	156.00	150.00	145.00	144.00	143.00	155.00	148.00	147.00	146.00	156.00	175.50
	0.55	130.00	124.50	118.00	114.00	113.00	128.00	118.00	115.00	115.00	128.00	141.00
	0.65	102.00	101.00	91.00	87.00	88.00	100.00	91.00	92.00	90.00	99.00	117.00
	0.80	72.00	66.00	61.00	59.00	57.00	69.00	60.00	60.00	59.00	66.00	86.00
	1.00	45.00	41.00	37.00	35.00	35.00	41.00	35.00	36.00	36.00	41.00	55.00
	1.15	34.00	29.00	26.00	26.00	25.00	30.00	25.00	26.00	26.00	30.00	41.00
	1.20	31.00	25.00	23.00	22.00	21.00	26.00	22.00	22.00	23.00	27.00	36.00
	1.25	28.00	23.00	21.00	20.00	20.00	23.00	20.00	20.00	21.00	25.00	33.00
	1.30	24.00	21.00	18.00	18.00	17.00	21.00	18.00	18.00	19.00	22.00	30.00
	2.00	7.00	6.00	5.00	5.00	5.00	6.00	6.00	6.00	9.00	9.00	12.00
	3.00	2.00	2.00	2.00	2.00	3.00	3.00	3.00	4.00	7.00	8.00	9.00
5	0.15	245.00	246.00	242.00	251.00	241.00	254.00	243.00	245.00	254.00	244.00	255.00
	0.20	237.00	243.00	238.50	240.00	239.00	243.00	239.00	233.00	241.00	240.00	245.00
	0.25	236.00	229.00	234.00	226.00	229.00	237.00	229.00	232.00	231.00	231.00	240.00
	0.30	226.00	224.00	221.00	223.00	218.00	224.00	227.00	219.00	224.00	224.00	231.00
	0.35	211.00	218.00	206.50	206.00	208.00	208.00	211.00	207.00	203.00	202.50	215.00
	0.45	192.00	190.00	184.00	184.00	179.00	192.00	183.00	179.50	180.00	185.00	197.00
	0.55	171.00	162.00	157.00	157.00	151.00	156.00	153.00	153.00	146.00	149.00	168.00
	0.65	143.00	138.00	133.00	134.00	128.00	139.00	132.00	127.00	126.00	132.00	145.00
	0.80	112.00	105.00	98.00	98.00	93.00	103.00	95.00	92.00	91.00	95.00	109.00
	1.00	82.00	69.00	66.00	64.00	62.00	71.00	63.00	61.00	57.00	64.00	76.00
	1.15	60.00	53.00	50.00	47.00	46.00	51.00	46.00	45.00	43.00	47.00	60.00
	1.20	55.00	47.00	43.00	39.00	40.00	45.00	40.00	39.00	37.00	42.00	52.00
	1.25	50.00	42.00	38.50	36.00	36.50	42.00	37.00	36.00	34.00	39.00	48.00
	1.30	46.00	38.00	35.00	34.00	34.00	38.00	33.00	33.00	31.00	36.00	45.00
	2.00	13.00	11.00	10.00	9.00	9.00	11.00	9.00	10.00	10.00	12.00	16.00
	3.00	3.00	3.00	3.00	3.00	3.00	4.00	4.00	5.00	7.00	8.00	9.00
10	0.15	256.00	247.00	248.00	253.00	244.00	248.00	249.00	251.00	250.00	249.00	255.50
	0.20	248.00	243.00	242.00	245.00	247.00	246.00	246.50	244.50	244.00	245.00	248.00
	0.25	240.00	237.50	239.00	234.00	243.00	239.00	245.00	229.00	234.00	238.00	242.00
	0.30	232.00	235.00	229.00	236.00	230.00	230.00	237.00	234.00	233.00	235.00	233.00
	0.35	225.00	223.00	224.00	225.00	221.00	224.00	226.00	221.00	221.00	222.00	231.00
	0.45	215.00	208.00	211.00	207.00	203.00	212.00	203.00	206.00	195.00	205.00	207.00
	0.55	194.00	193.00	185.00	183.00	183.00	187.00	186.00	180.00	171.00	183.00	191.00
	0.65	170.00	161.00	162.00	157.00	157.00	162.00	155.00	155.00	147.00	151.00	166.00
	0.80	147.00	138.00	133.00	130.00	130.00	134.00	130.00	127.00	119.00	126.00	137.00
	1.00	107.00	101.00	91.00	90.00	90.00	95.00	87.00	85.00	80.50	85.00	98.00
	1.15	90.00	78.00	74.00	72.50	69.00	77.00	71.00	67.00	61.00	67.00	79.00
	1.20	82.00	73.00	65.00	64.00	64.00	69.00	63.00	61.00	54.00	61.00	72.00
	1.25	74.00	65.00	59.00	58.00	56.00	61.00	56.00	54.00	49.00	54.00	64.00
	1.30	71.00	60.00	55.00	54.00	52.00	59.00	53.00	50.00	46.00	52.00	63.00
	2.00	22.00	17.00	16.00	15.00	15.00	17.00	15.00	15.00	15.00	17.00	22.00
	3.00	5.00	4.00	4.00	4.00	4.00	5.00	4.00	5.00	8.00	9.00	9.00

Table 7. $PRL_{1(25)}$ values for different choice of t, λ and $ARL = 370$

t	λ	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
2	0.15	101.00	101.00	100.00	99.00	98.00	100.00	104.00	100.00	104.00	105.00	107.00
	0.20	95.00	97.00	98.00	94.00	92.00	99.00	96.00	99.00	99.00	103.00	102.00
	0.25	88.00	91.00	88.00	91.00	89.00	92.00	92.00	89.00	92.00	94.00	98.00
	0.30	85.00	86.00	87.00	81.75	83.00	86.00	82.00	86.00	88.00	92.00	96.00
	0.35	78.00	77.00	74.00	74.00	75.00	78.00	74.00	78.00	78.00	84.00	91.00
	0.45	65.00	63.00	64.00	63.00	59.00	66.00	64.00	64.00	64.00	69.00	77.00
	0.55	53.00	52.00	49.00	49.00	49.00	55.75	51.00	51.00	52.00	59.00	63.00
	0.65	42.00	43.00	39.00	39.00	38.00	43.00	39.00	41.00	41.00	44.00	51.00
	0.80	31.00	28.00	26.00	26.00	24.00	29.00	27.00	28.00	28.00	32.00	41.00
	1.00	19.00	18.00	16.00	16.00	16.00	19.00	16.00	17.00	18.00	21.00	26.00
	1.15	14.00	13.00	12.00	12.00	12.00	14.00	12.00	13.00	14.00	16.00	21.00
	1.20	13.00	11.00	11.00	11.00	10.00	12.00	11.00	11.00	12.00	15.00	19.00
	1.25	12.00	10.00	10.00	9.00	9.00	11.00	10.00	10.00	12.00	14.00	18.00
	1.30	10.00	9.00	9.00	8.00	8.00	10.00	9.00	10.00	11.00	13.00	16.00
	2.00	3.00	3.00	3.00	3.00	3.00	4.00	4.00	5.00	8.00	8.00	9.00
	3.00	1.00	2.00	2.00	2.00	2.00	3.00	3.00	4.00	7.00	8.00	9.00
5	0.15	101.00	101.00	102.00	106.00	102.00	108.00	104.00	106.00	112.00	108.00	108.00
	0.20	100.00	102.00	98.00	100.00	100.00	102.00	99.75	97.00	103.00	102.00	104.00
	0.25	97.00	96.75	98.00	95.00	96.00	100.00	96.00	97.00	99.00	97.00	103.00
	0.30	94.00	93.00	94.00	95.00	94.00	94.75	96.00	94.00	97.00	97.00	99.00
	0.35	89.00	94.00	84.00	87.00	90.00	89.00	88.00	89.00	90.00	88.00	96.00
	0.45	79.00	79.00	78.00	77.00	75.00	82.00	77.00	75.00	78.00	82.00	88.00
	0.55	71.00	67.00	66.00	67.00	65.00	66.00	66.00	64.00	65.00	66.00	74.00
	0.65	59.00	58.00	57.00	56.00	54.00	59.00	56.00	55.00	57.00	59.00	63.00
	0.80	46.00	44.00	42.00	41.00	40.00	43.00	42.00	41.00	42.00	43.75	49.00
	1.00	34.00	29.75	29.00	28.00	28.00	32.00	27.00	28.00	27.00	30.00	36.00
	1.15	26.00	23.00	22.00	20.00	21.00	23.00	21.00	21.00	21.00	23.00	29.00
	1.20	23.00	20.00	19.00	18.00	18.00	20.00	18.00	18.00	19.00	21.00	26.00
	1.25	21.00	18.00	17.00	16.00	16.00	19.00	17.00	17.00	18.00	20.00	24.00
	1.30	19.00	17.00	16.00	15.00	16.00	17.00	15.00	15.00	17.00	18.00	23.00
	2.00	6.00	5.00	5.00	5.00	5.00	6.00	5.00	6.00	9.00	9.00	9.00
	3.00	2.00	2.00	2.00	2.00	2.00	3.00	3.00	4.00	7.00	8.00	9.00
10	0.15	104.00	105.00	105.00	105.00	104.00	103.00	105.00	108.00	107.00	107.00	110.00
	0.20	106.00	100.75	102.75	103.00	104.00	106.00	104.00	105.00	102.00	106.00	107.00
	0.25	100.00	102.00	102.00	99.00	105.00	99.00	101.00	98.00	101.00	105.00	104.00
	0.30	97.00	98.00	97.00	99.00	96.00	100.00	101.00	102.00	100.00	103.00	101.00
	0.35	96.00	95.00	93.00	97.00	92.00	95.00	96.00	94.00	98.75	97.75	101.00
	0.45	89.00	88.00	89.00	87.00	85.00	89.00	86.00	89.00	84.00	87.00	93.75
	0.55	83.00	79.00	79.00	78.00	78.00	80.00	78.75	77.00	75.00	80.00	83.00
	0.65	69.00	68.00	68.00	67.00	67.00	67.00	67.00	66.00	65.00	67.00	75.00
	0.80	61.00	58.00	55.00	56.00	55.00	59.00	55.00	56.00	53.00	55.00	61.00
	1.00	45.00	42.00	39.00	39.00	39.00	41.00	38.00	37.00	37.00	39.00	45.00
	1.15	38.00	32.75	33.00	31.00	30.00	34.00	31.00	30.00	30.00	31.00	37.00
	1.20	35.00	31.00	28.00	27.00	28.00	31.00	28.00	28.00	26.00	29.00	34.00
	1.25	31.00	27.00	25.00	25.00	24.00	26.00	25.00	25.00	24.00	26.00	30.00
	1.30	30.00	26.00	24.00	23.00	23.00	26.00	23.00	23.00	23.00	26.00	29.00
	2.00	9.00	8.00	8.00	7.00	8.00	8.00	8.00	8.00	9.00	10.00	13.00
	3.00	2.00	2.00	3.00	3.00	3.00	3.00	4.00	4.00	7.00	8.00	9.00

Table 8. $PRL_{1(75)}$ values for different choice of t, λ and $ARL = 370$

t	λ	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
2	0.15	482.25	478.25	477.00	465.00	471.00	478.25	476.00	478.00	475.00	481.00	487.00
	0.20	462.00	471.25	462.00	452.00	444.00	459.25	460.00	458.00	447.25	458.00	461.00
	0.25	425.25	435.25	422.00	420.25	418.00	436.25	419.25	428.00	421.00	429.00	443.00
	0.30	403.25	404.00	398.00	386.00	381.25	415.00	387.00	401.00	390.00	410.25	424.25
	0.35	367.00	368.00	354.25	355.00	350.00	372.00	356.00	361.00	351.00	375.00	393.25
	0.45	311.00	305.00	294.00	281.25	281.00	310.00	293.00	291.00	285.00	301.00	345.00
	0.55	260.00	247.00	234.00	225.00	219.00	250.00	231.00	230.00	224.00	248.00	282.00
	0.65	206.00	195.00	180.00	174.00	172.00	197.00	178.00	180.00	171.00	192.00	225.00
	0.80	144.00	131.00	120.00	115.00	112.00	134.00	115.00	118.00	113.00	128.00	162.25
	1.00	90.00	80.00	71.00	68.00	67.00	80.00	68.00	68.00	66.00	77.00	102.00
	1.15	68.00	56.00	50.00	48.00	47.00	56.00	47.00	48.00	45.00	54.00	76.00
	1.20	60.00	50.00	44.00	42.00	41.00	49.00	42.00	42.00	40.00	47.00	67.00
	1.25	54.00	45.00	40.00	38.00	37.00	44.00	38.00	37.00	36.00	43.00	61.00
	1.30	48.00	39.00	35.00	33.00	32.00	40.00	33.00	33.00	32.00	39.00	54.00
	2.00	13.00	10.00	9.00	9.00	9.00	11.00	9.00	10.00	10.00	13.00	18.00
	3.00	3.00	4.00	3.00	3.00	3.00	4.00	4.00	5.00	8.00	8.00	9.00
5	0.15	496.00	491.00	488.00	492.00	486.25	490.00	497.00	481.25	486.00	482.00	491.00
	0.20	475.00	485.00	473.00	479.00	474.00	485.25	484.00	467.00	475.00	482.00	480.00
	0.25	475.00	465.00	470.00	460.00	461.00	469.00	456.00	457.00	453.00	456.00	469.00
	0.30	445.25	446.00	444.00	436.25	442.00	454.00	446.00	442.00	440.00	431.00	451.00
	0.35	422.25	429.00	418.00	408.00	408.25	414.00	411.00	410.00	399.00	400.00	420.00
	0.45	386.00	373.00	369.00	360.00	360.00	380.00	369.00	358.00	347.00	361.00	386.00
	0.55	339.00	327.00	309.00	312.00	304.00	316.00	299.00	299.00	283.25	296.25	326.00
	0.65	287.00	277.00	263.00	261.00	256.00	272.00	266.00	252.00	243.00	254.00	285.00
	0.80	223.00	211.00	197.00	187.00	185.00	205.00	189.00	183.00	173.00	187.00	215.00
	1.00	160.00	137.00	128.00	125.00	121.00	137.00	123.00	119.00	109.00	123.00	148.00
	1.15	120.00	105.00	98.00	90.25	88.00	103.00	91.00	88.00	80.00	89.00	111.00
	1.20	110.00	90.00	83.00	77.00	78.00	87.00	77.00	74.00	68.00	78.00	99.00
	1.25	99.00	83.00	75.00	70.00	70.00	81.00	70.00	68.00	62.00	71.00	91.00
	1.30	90.00	75.00	69.00	65.00	64.00	73.00	64.00	61.00	55.00	65.00	84.00
	2.00	25.00	20.00	18.00	17.00	17.00	19.00	16.00	16.00	17.00	19.00	27.00
	3.00	6.00	5.00	5.00	4.00	5.00	6.00	5.00	5.00	8.00	9.00	9.00
10	0.15	501.00	495.00	502.00	499.00	486.00	500.00	499.00	491.00	495.00	486.00	506.00
	0.20	487.00	491.00	489.00	487.00	489.00	490.00	486.00	495.00	481.00	478.00	492.00
	0.25	488.00	473.00	480.00	473.00	474.00	469.00	483.00	458.00	458.00	469.25	475.25
	0.30	468.00	464.00	460.00	467.00	458.00	469.00	464.00	464.00	457.00	453.00	462.00
	0.35	452.00	446.00	446.00	445.00	437.00	446.00	444.25	442.00	432.00	429.25	454.00
	0.45	422.00	424.00	411.00	411.00	406.00	424.00	399.00	403.00	387.00	395.00	406.00
	0.55	389.25	381.00	371.00	366.00	365.00	373.00	362.00	355.25	342.00	354.00	383.00
	0.65	342.00	320.25	322.00	315.00	311.00	316.00	304.00	305.00	286.00	296.00	320.00
	0.80	297.00	275.00	264.00	255.25	261.00	266.00	260.00	249.00	229.00	240.00	270.00
	1.00	213.00	198.00	182.00	179.00	177.00	189.00	170.00	168.00	154.00	162.00	193.00
	1.15	178.00	156.00	148.00	141.00	138.00	153.00	139.00	129.00	117.00	128.00	151.00
	1.20	165.00	142.00	129.00	128.00	125.00	137.00	122.00	118.00	104.00	113.00	139.00
	1.25	145.00	128.00	117.00	112.00	112.00	120.00	109.00	103.00	92.00	101.00	121.00
	1.30	143.00	119.00	110.00	108.00	103.00	115.00	101.00	96.00	85.00	96.00	118.00
	2.00	44.00	34.00	30.00	29.00	28.00	32.00	27.00	27.00	24.00	29.00	38.00
	3.00	9.00	8.00	7.00	7.00	7.00	8.00	7.00	8.00	9.00	9.00	14.00

Table 9. $PRL_{1(90)}$ values for different choice of t, λ and $ARL = 370$

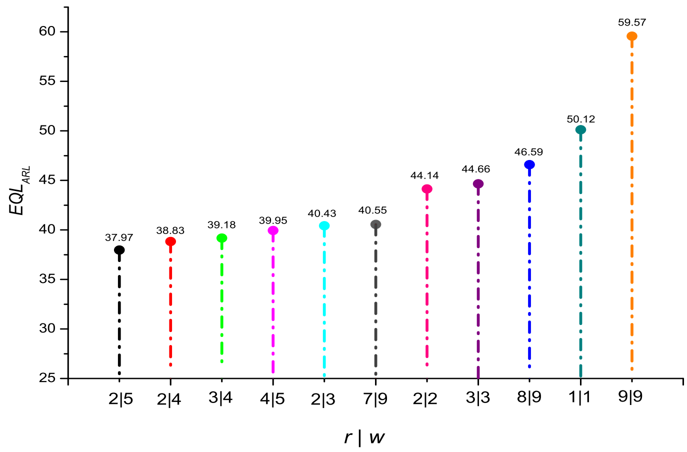
t	λ	1|1	2|2	2|3	2|4	2|5	3|3	3|4	4|5	7|9	8|9	9|9
2	0.15	812.10	791.10	787.10	777.10	774.00	797.00	787.10	787.00	782.00	783.00	800.00
	0.20	751.20	774.00	746.00	745.10	741.10	766.10	767.00	763.00	740.00	758.20	758.10
	0.25	696.00	729.00	689.10	692.10	694.10	731.00	703.00	724.00	703.10	710.00	738.00
	0.30	667.20	673.10	658.00	653.10	628.10	686.00	656.00	671.00	639.10	683.00	701.00
	0.35	606.00	609.00	590.00	585.00	572.10	631.10	589.00	594.00	569.00	612.00	650.00
	0.45	523.00	508.00	489.00	471.10	470.00	514.00	482.00	486.00	470.00	500.00	568.10
	0.55	425.00	409.00	381.10	373.00	368.00	414.00	382.10	378.00	368.00	409.00	468.00
	0.65	340.00	325.00	297.00	287.00	286.00	326.00	294.00	297.00	286.00	318.00	371.00
	0.80	244.00	219.00	200.00	193.00	185.00	224.00	190.00	190.00	186.00	206.00	264.00
	1.00	149.00	131.00	117.00	113.00	110.00	131.00	112.00	111.00	105.00	122.00	167.00
	1.15	114.00	92.00	82.00	79.00	78.00	92.00	78.00	77.00	72.00	86.00	122.00
	1.20	99.00	82.00	72.00	68.00	66.00	80.00	68.00	66.00	63.10	74.00	105.00
	1.25	90.00	73.00	65.00	61.00	60.00	75.00	61.00	60.00	56.00	67.00	97.00
	1.30	80.00	63.00	57.00	54.00	51.00	65.00	53.00	52.00	49.00	60.00	85.00
	2.00	21.00	16.00	14.00	14.00	13.00	17.00	14.00	14.00	15.00	18.00	26.00
	3.00	5.00	5.00	4.00	4.00	4.00	6.00	4.00	5.00	8.00	9.00	10.00
5	0.15	822.10	804.00	809.10	807.10	809.00	810.00	816.00	797.00	809.00	789.00	805.00
	0.20	803.00	806.10	783.00	794.00	779.00	811.00	804.00	775.00	794.00	782.00	806.10
	0.25	791.10	775.00	770.00	761.00	779.00	770.00	752.00	754.10	740.00	759.00	775.10
	0.30	731.00	743.10	746.00	715.00	724.00	749.00	737.10	720.10	711.10	720.00	746.10
	0.35	710.10	705.00	688.10	670.10	672.10	684.00	680.00	685.00	658.00	656.00	688.00
	0.45	637.00	610.00	609.00	600.00	596.00	626.00	605.00	581.00	581.00	600.00	632.00
	0.55	548.00	529.00	513.00	519.00	510.00	521.00	492.00	499.00	471.00	489.10	528.00
	0.65	489.00	465.10	437.10	428.00	422.00	456.00	435.00	425.00	407.00	417.00	466.00
	0.80	369.10	349.00	327.00	306.00	305.00	338.00	310.00	300.00	281.00	303.00	354.00
	1.00	260.00	229.00	208.00	202.00	200.00	229.00	203.00	197.00	174.00	199.00	244.00
	1.15	199.00	173.00	162.00	149.00	145.10	169.00	148.00	143.00	128.00	144.00	178.10
	1.20	179.00	155.00	138.00	126.00	128.00	145.00	129.00	122.00	110.00	124.00	160.00
	1.25	165.00	137.00	125.00	114.00	117.00	137.00	114.00	109.00	101.00	116.00	146.00
	1.30	152.00	124.00	112.10	108.00	104.00	118.00	103.00	100.00	89.00	102.00	134.00
	2.00	40.00	31.00	28.00	27.00	26.00	30.00	25.00	25.00	24.00	29.00	40.00
	3.00	9.00	8.00	7.00	7.00	6.00	8.00	7.00	8.00	9.00	9.00	16.00
10	0.15	823.00	831.10	830.10	815.00	816.00	818.10	814.00	824.20	830.30	815.10	828.00
	0.20	829.10	825.00	826.00	804.00	816.00	810.00	797.00	808.20	788.00	797.10	809.00
	0.25	801.00	783.10	795.00	781.00	766.00	797.00	792.00	758.00	762.10	775.00	779.00
	0.30	789.00	766.00	764.10	790.00	764.20	770.10	765.00	749.10	750.00	746.00	761.00
	0.35	751.00	754.00	744.00	726.00	709.00	752.00	730.10	720.10	707.10	715.10	755.00
	0.45	699.10	698.00	676.00	677.00	674.00	703.00	659.10	668.00	632.00	670.00	668.00
	0.55	641.10	627.00	612.00	608.10	607.00	625.10	593.10	592.00	562.10	586.00	613.00
	0.65	568.00	539.00	528.00	523.00	515.00	517.00	510.10	497.00	471.00	480.10	524.00
	0.80	490.00	460.00	438.00	426.10	438.00	438.00	424.00	408.00	382.00	391.00	444.00
	1.00	354.00	325.00	302.00	299.00	292.00	318.00	285.00	276.00	252.00	271.00	309.00
	1.15	297.00	260.00	241.00	233.00	228.00	248.10	222.00	213.00	188.00	208.00	250.00
	1.20	273.00	240.00	217.00	212.00	202.00	223.10	204.00	196.00	166.10	181.00	222.00
	1.25	240.10	211.00	192.00	186.00	182.00	197.00	178.00	168.00	149.00	165.00	200.00
	1.30	235.00	196.00	185.10	177.00	173.00	191.00	166.00	158.00	138.00	156.00	188.00
	2.00	72.00	55.00	48.00	47.00	46.00	52.00	43.00	41.00	37.00	44.00	60.00
	3.00	15.00	12.00	11.00	10.00	10.00	12.00	10.00	11.00	11.00	14.00	19.00

Table 10. Calculated values of overall performance measures at various choices of t, $r|w$ and $ARL = 370$

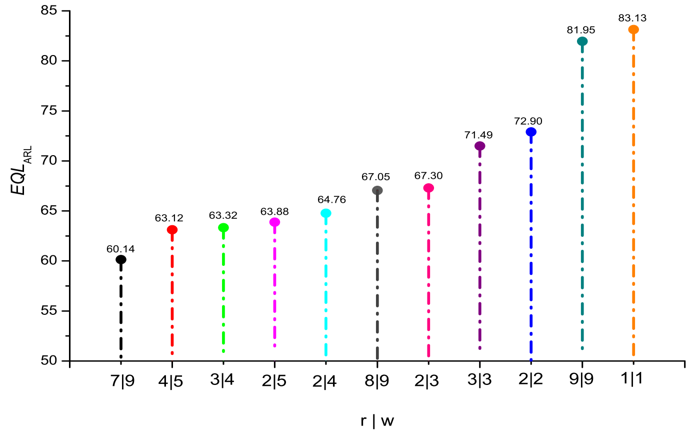
t	λ	$1\mid 1$	$2\mid 2$	$2\mid 3$	$2\mid 4$	$2\mid 5$	$3\mid 3$	$3\mid 4$	$4\mid 5$	$7\mid 9$	$8\mid 9$	$9\mid 9$
2	$RARL$	1.31	1.15	1.06	1.02	1.00	1.17	1.03	1.05	1.08	1.23	1.57
	$RMRL$	1.28	1.15	1.05	1.02	1.00	1.17	1.04	1.06	1.12	1.27	1.62
	RPR_{25}	1.12	1.06	1.00	1.00	1.01	1.13	1.02	1.11	1.35	1.48	1.70
	RPR_{75}	1.33	1.16	1.06	1.03	1.00	1.17	1.02	1.04	1.03	1.19	1.53
	RPR_{90}	1.01	1.19	1.08	1.04	1.01	1.18	1.03	1.04	1.00	1.16	1.51
	EQL_{ARL}	50.12	44.14	40.43	38.83	37.97	44.66	39.18	39.95	40.55	46.59	59.57
	EQL_{MRL}	34.97	31.25	28.68	27.52	27.02	31.78	28.17	28.66	30.05	34.18	43.79
	EQL_{25}	14.93	14.05	13.18	13.14	13.21	15.01	13.47	14.63	17.90	19.86	22.99
	EQL_{75}	68.91	60.30	55.14	53.09	51.55	60.62	52.84	53.77	52.54	60.93	79.05
	EQL_{90}	83.46	99.26	90.16	85.97	83.46	98.71	85.90	86.45	82.52	95.99	125.46
	PCI_{ARL}	1.32	1.16	1.06	1.02	1.00	1.18	1.03	1.05	1.07	1.23	1.57
	PCI_{MRL}	1.29	1.16	1.06	1.02	1.00	1.18	1.04	1.06	1.11	1.27	1.62
	PCI_{25}	1.14	1.07	1.00	1.00	1.01	1.14	1.03	1.11	1.36	1.51	1.75
	PCI_{75}	1.34	1.17	1.07	1.03	1.00	1.18	1.03	1.04	1.02	1.18	1.53
	PCI_{90}	1.01	1.20	1.09	1.04	1.01	1.20	1.04	1.05	1.00	1.16	1.52
5	$RARL$	1.34	1.19	1.10	1.06	1.05	1.17	1.04	1.04	1.00	1.11	1.33
	$RMRL$	1.28	1.14	1.07	1.03	1.02	1.12	1.01	1.02	1.00	1.10	1.32
	RPR_{25}	1.18	1.08	1.03	1.00	1.01	1.10	1.03	1.04	1.13	1.23	1.43
	RPR_{75}	1.40	1.23	1.14	1.09	1.08	1.20	1.07	1.06	1.00	1.11	1.35
	RPR_{90}	1.47	1.27	1.17	1.13	1.11	1.24	1.10	1.08	1.00	1.11	1.36
	EQL_{ARL}	83.13	72.91	67.31	64.77	63.89	71.50	63.33	63.12	60.14	67.06	81.96
	EQL_{MRL}	57.90	51.11	47.75	45.76	45.29	50.21	44.97	45.04	44.10	49.00	59.57
	EQL_{25}	24.58	22.29	21.22	20.48	20.64	22.65	21.01	21.32	23.15	25.39	29.90
	EQL_{75}	115.01	100.21	92.39	88.68	87.30	97.84	86.14	85.40	79.94	89.77	110.12
	EQL_{90}	190.85	165.08	151.20	145.71	143.08	160.86	141.18	139.11	127.55	142.25	176.51
	PCA_{ARL}	1.38	1.21	1.12	1.08	1.06	1.19	1.05	1.05	1.00	1.12	1.36
	PCI_{MRL}	1.31	1.16	1.08	1.04	1.03	1.14	1.02	1.02	1.00	1.11	1.35
	PCI_{25}	1.20	1.09	1.04	1.00	1.01	1.11	1.03	1.04	1.13	1.24	1.46
	PCI_{75}	1.44	1.25	1.16	1.11	1.09	1.22	1.08	1.07	1.00	1.12	1.38
	PCI_{90}	1.50	1.29	1.19	1.14	1.12	1.26	1.11	1.09	1.00	1.12	1.38
10	$RARL$	3.17	2.74	2.48	1.00	2.65	2.38	2.31	2.15	2.36	2.79	2.79
	$RMRL$	3.04	2.64	2.41	1.00	2.58	2.33	2.29	2.15	2.36	2.79	2.79
	RPR_{25}	2.56	2.30	2.18	1.00	2.31	2.14	2.13	2.16	2.37	2.71	2.71
	RPR_{75}	3.28	2.81	2.53	1.00	2.71	2.42	2.36	2.14	2.36	2.81	2.81
	RPR_{90}	3.37	2.88	2.59	1.00	2.77	2.48	2.38	2.14	2.36	2.83	2.83
	EQL_{ARL}	121.53	105.77	96.24	38.07	102.40	92.52	89.84	83.20	90.88	106.97	106.97
	EQL_{MRL}	84.35	73.76	67.36	27.24	72.01	65.14	63.91	59.86	65.56	77.18	77.18
	EQL_{25}	35.39	31.64	29.92	13.09	31.82	29.31	29.14	29.55	32.45	37.24	37.24
	EQL_{75}	168.60	145.81	131.98	51.69	140.87	126.63	123.19	111.87	122.69	144.49	144.49
	EQL_{90}	277.79	240.72	217.78	83.83	231.77	208.63	200.78	180.50	197.89	234.61	234.61
	PCA_{ARL}	3.19	2.78	2.53	1.00	2.69	2.43	2.36	2.19	2.39	2.81	2.81
	PCI_{MRL}	3.10	2.71	2.47	1.00	2.64	2.39	2.35	2.20	2.41	2.83	2.83
	PCI_{25}	2.70	2.42	2.29	1.00	2.43	2.24	2.23	2.26	2.48	2.85	2.85
	PCI_{75}	3.26	2.82	2.55	1.00	2.73	2.45	2.38	2.16	2.37	2.80	2.80
	PCI_{90}	3.31	2.87	2.60	1.00	2.76	2.49	2.40	2.15	2.36	2.80	2.80

5.3. Comparative analysis on the optimal choices of sensitizing rules

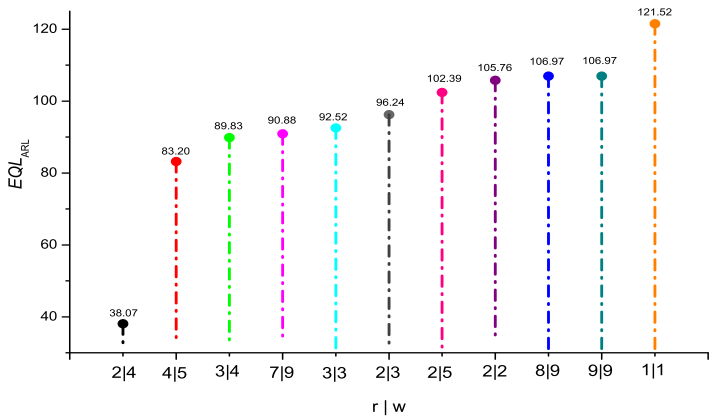
In Figure 3, the behavior of the sensitizing rules in terms of EQL_{ARL} measure is dependent on the number of process characteristics (t). For instance, at $t = 2$, the minimum EQL_{ARL} value is 37.97 for the 2|5 rule. It interprets as a 2|5 rule with the proposed control chart performed efficiently to detect variations relative to the other choices when $t = 2$ is considered. Likewise, 7|9 and 2|4 are counted as excellent with the proposed control chart when $t = 5$ and $t = 10$, respectively. Similarly, Figure 4, shows that the performance of the sensitizing rule concerning $RARL$ is associated with the choice of t. The lower value of $RARL$ illustrates the excellent performance of the proposed control chart at the particular choices of sensitizing rules. Similar effectiveness of the aforesaid sensitizing rules are noted while considering $RMRL$, $RPRL_{25}$, $RPRL_{75}$, $RPRL_{90}$ EQL_{MRL}, $EQL_{PRL(25)}$, $EQL_{PRL(75)}$, $EQL_{PRL(90)}$, PCI_{MRL}, $PCI_{PRL(25)}$, $PCI_{PRL(75)}$, and $PCI_{PRL(90)}$ over selected values of t. The overall performance order of the sensitizing rules on the basis of various measures are 2|5, 2|4, 3|4, 4|5, 2|3, 7|9, 2|2, 3|3, 8|9, 1|1 and 9|9 when $t = 2$, as shown in Figures 3a, 4a, and 5a. Similarly, at $t = 5$ and $t = 10$, performance order is apparent in Figures 3b–3c, 4b– 4c, and 5b–5c.



(a) $t = 2$

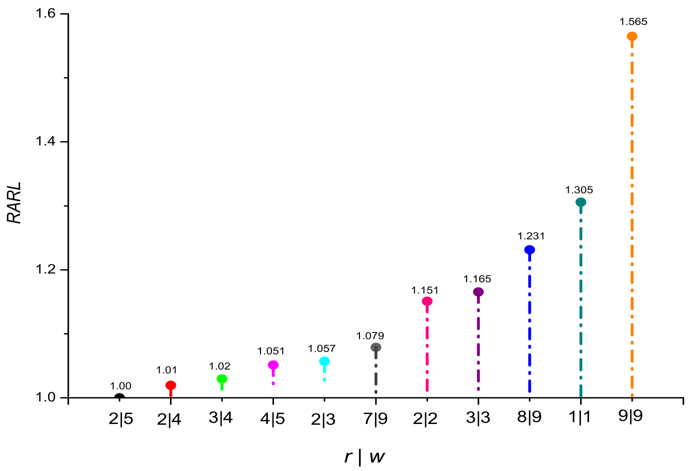


(b) $t = 5$

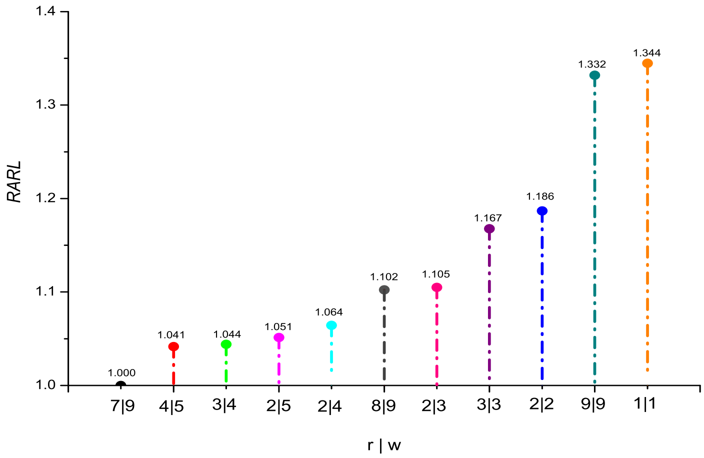


(c) $t = 10$

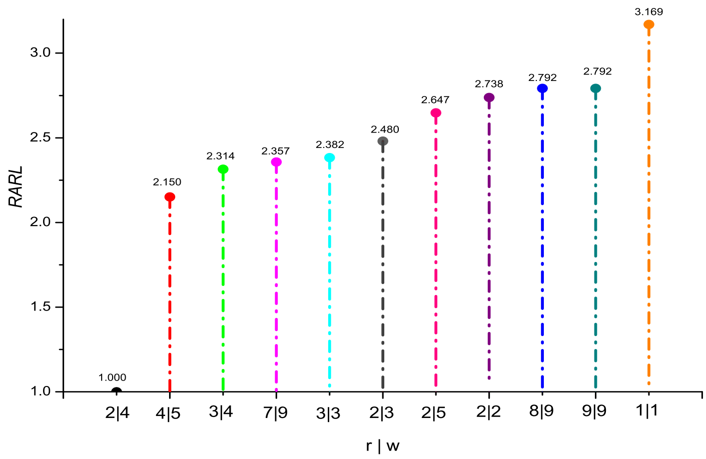
Figure 3. EQL_{ARL} values against $r|w$ at various choices of t and $ARL = 370$



(a) $t = 2$

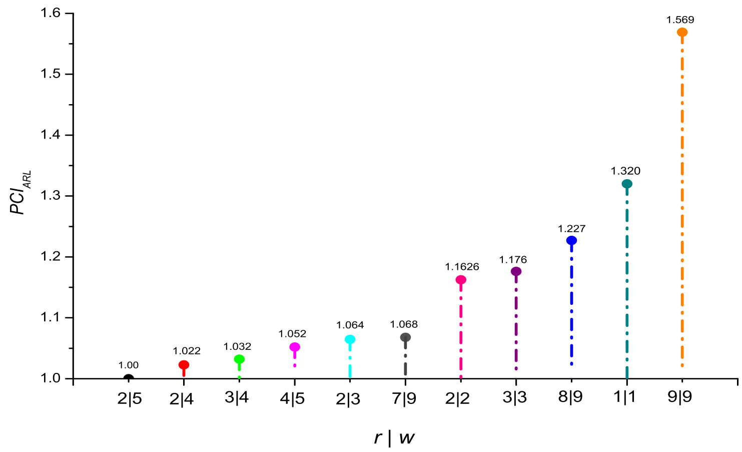


(b) $t = 5$

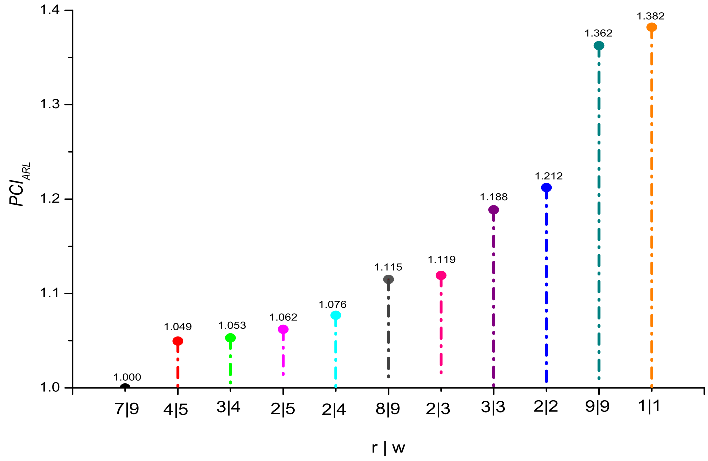


(c) $t = 10$

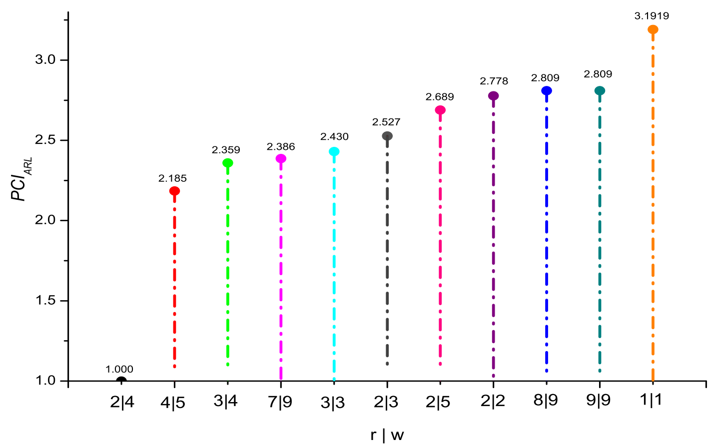
Figure 4. *RARL* values against $r|w$ at various choices of t and $ARL = 370$



(a) $t = 2$



(b) $t = 5$



(c) $t = 10$

Figure 5. PCI_{ARL} values against $r|w$ at various choices of t and $ARL = 370$

5.4. Special cases

The proposed control chart method is considered a generalized form of the existing control charts. In brief, Hotelling T^2 control chart ([2]) is the special case of the proposed control chart when $r|w = 1|1$. Also, multivariate control charts with sensitizing rules [23,24] are special cases when $r = w$.

6. Application

In this section, we show an application and advantages of the proposed multivariate Hotelling T^2 control chart with sensitizing rules by involving a dowel pin manufacturing process. Moreover, we highlight the advantage of sensitizing rules over classical rule by considering both in-control and out-of-control situations. The dowel pins are the simple type of fasteners with multiple applications to various industrial processes, and comprised of simple shape and multiple characteristics such as length and diameter. A real data sets containing 40 samples with two dowel pin characteristics (diameter and length) is taken from [40] and provided in Table 11. For multivariate normal data, marginal distribution and linear combinations should also be normal. This provides a starting point for assessing normality in the multivariate setting. A scatter plot for each pair of variables together with a Gamma plot (Chi-squared Q-Q plot) is used in assessing bivariate normality. For more than two variables, a Gamma plot can still be used to check the assumption of multivariate normality (For details, see [39]). For readers interest, we have analyzed the normality assumption using Q-Q plots. In Figures 6 – 7, data shows normal distribution. Similar data is used by [40] to show the application of multivariate T^2 - control chart. For practitioners concern, steps for applying the proposed control chart at a given choice of sensitizing rule $(r|w)$, and prefixed IARL (ARL) is as follows:

(1) Calculate the p value according to the choice of r, w and ARL by following the procedure given in subsection 2.2. For instance, the p values for selected sensitizing rules such as $1|1$, $2|2$, $3|3$, and $3|4$ at $ARL = 20$ are 0.05, 0.25, 0.432, and 0.355, respectively. Note that selected value of $ARL = 20$ is motivated from the [33].

(2) Calculate of the mean vector and variance-covariance of dowel pin characteristics variables

$$U = \begin{pmatrix} 0.500 \\ 1.002 \end{pmatrix},$$

and

$$\Sigma = \begin{pmatrix} 4.90 \times 10^{-5} & 8.58 \times 10^{-5} \\ 8.58 \times 10^{-5} & 4.199 \times 10^{-4} \end{pmatrix}.$$

(3) Calculate the control chart statistic T_j^2 for each sample. For example, calculated value of T_j^2 for first sample $(j = 1)$ is given below:

$$T_1^2 = n(\widehat{U}_1 - U)'\Sigma^{-1}(\widehat{U}_1 - U) = 1.615,$$

(4) Plot the calculated control chart statistic T_j^2 and control limit H against the sample number.

Based on the Steps 1–4, we have constructed control charts for some selected choices of sensitizing rules ($1|1$, $2|2$, $3|3$ and $3|4$) and $ARL = 20$ (see Figures 8–9). It is quit apparent from Figure 8, the classical rule shows in-control process because all statistics fall within the control region. The classical sensitizing rule indicates that process is in control, and no further action is essential. In contrast, the $2|2$ rule shows out-of-control signal at sample number 23 (see Figure 8b) and a quality control inquisition and curative action are integral to locate and annihilate the reasons liable for this out of control process behavior. In addition, $3|3$ rule in Figure 9a, shows in control process over all samples. Moreover, the control chart for $3|4$ rule shows out of control process at sample number 28 as 3 out of 4 consecutive samples violating the control limit. To further show the advantages of sensitizing rules over classical rule for detection of small-to-moderate variations, a small shift of amount $\lambda = 0.00001$ in each dowel pin characteristics is introduced by considering [22] as guidance, and now process characteristics is considered out-of-control. Afterwards, Steps 1–4 are adopted to develop the control charts for out-of-control process characteristics (see Figures 10–11). In Figure 10a, 2|2 rule shows multiple infringements at samples number 26 and 30, respectively and exhibits that the process is statistically out of-control. In contrast, for 1|1 rule (classical rule), it is observed that all points lies within the control limits and process is statistically in-control. Similarly, the proposed Hoteling T^2 control chart for 3|3 rule shows numerous violations at samples number 10, 28, and 32, respectively (see Figure 11a).

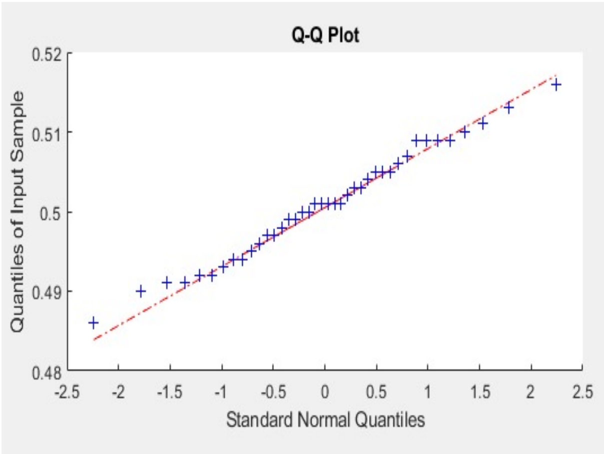


Figure 6. Q-Q plot for the diameter of dowel pin

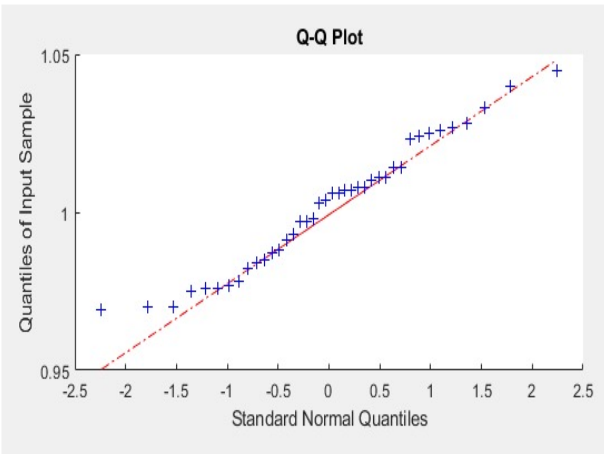
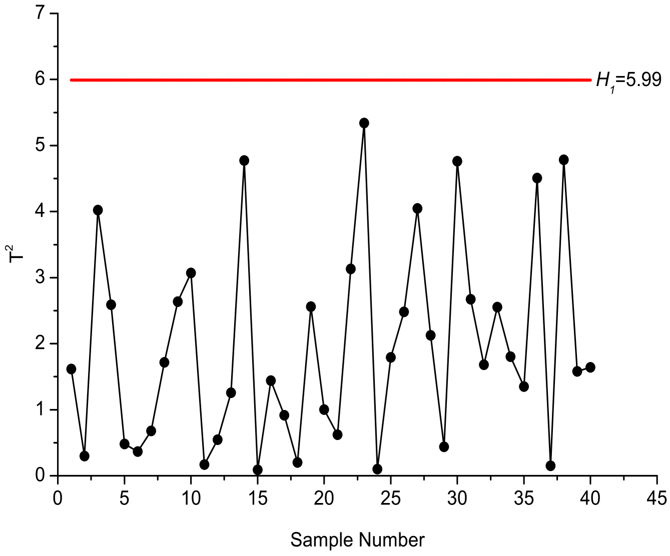


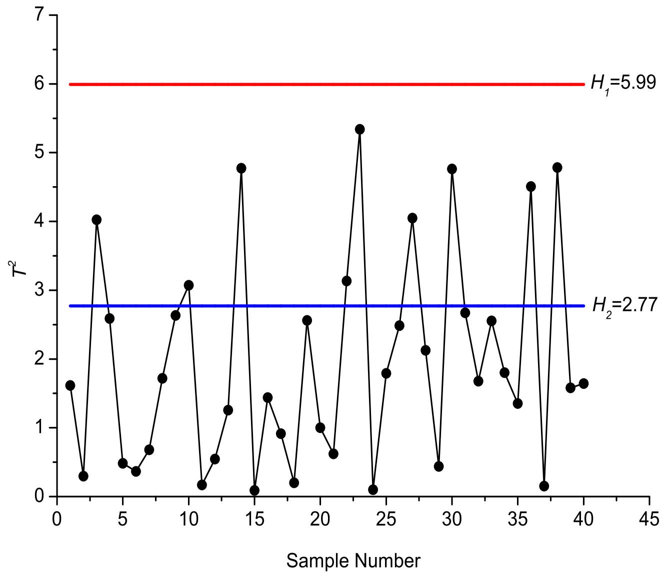
Figure 7. Q-Q plot for the length of dowel pin

Table 11. Ellipsoid contour dowel pin datasets from dowel pins manufacturing process

Sample Number	Diameter	Length
1	0.492	0.988
2	0.501	1.011
3	0.491	1.008
4	0.492	0.97
5	0.505	1.003
6	0.500	1.01
7	0.497	0.985
8	0.509	1.006
9	0.49	0.975
10	0.499	1.027
11	0.498	0.997
12	0.497	0.987
13	0.5	0.982
14	0.503	0.97
15	0.501	1.007
16	0.509	1.011
17	0.495	0.984
18	0.504	1.008
19	0.491	0.997
20	0.494	0.993
21	0.506	1.006
22	0.494	1.014
23	0.501	1.04
24	0.503	1.004
25	0.499	0.977
26	0.507	0.991
27	0.501	0.969
28	0.501	1.026
29	0.505	1.014
30	0.516	1.023
31	0.511	1.007
32	0.505	1.028
33	0.51	1.033
34	0.493	0.976
35	0.496	0.978
36	0.486	0.976
37	0.502	0.998
38	0.513	1.045
39	0.509	1.024
40	0.509	1.025

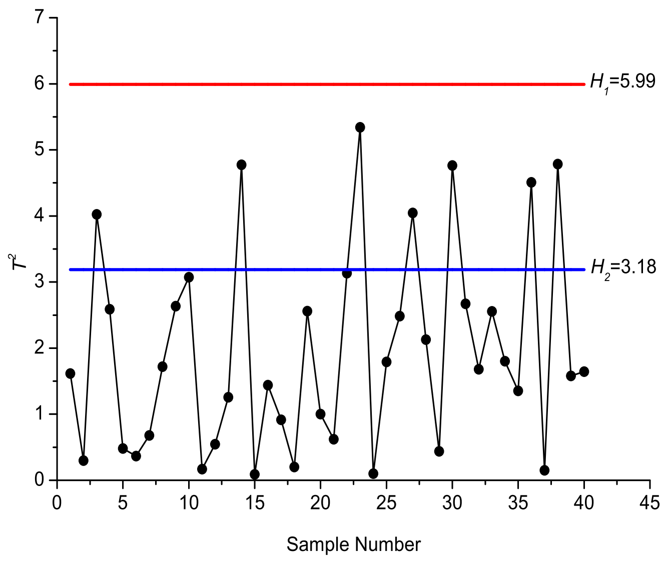


(a) 1|1 rule

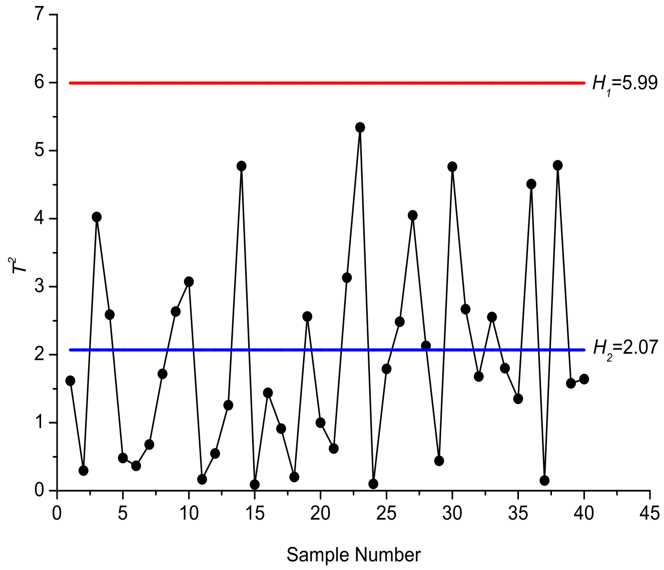


(b) 2|2 rule

Figure 8. Proposed control chart for monitoring the in-control dowel pin process characteristics at 1|1 and 2|2 rules

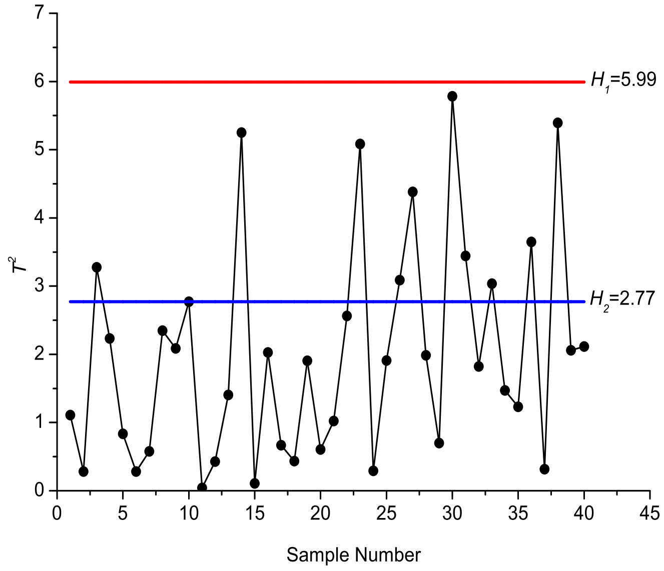


(a) 3|3 rule

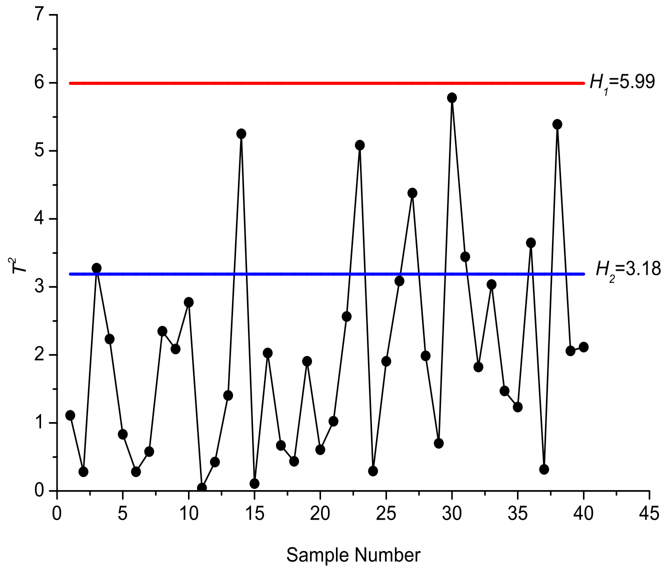


(b) 3|4 rule

Figure 9. Proposed control chart for monitoring the in-control dowel pin process characteristics at 3|3 and 3|4 rules

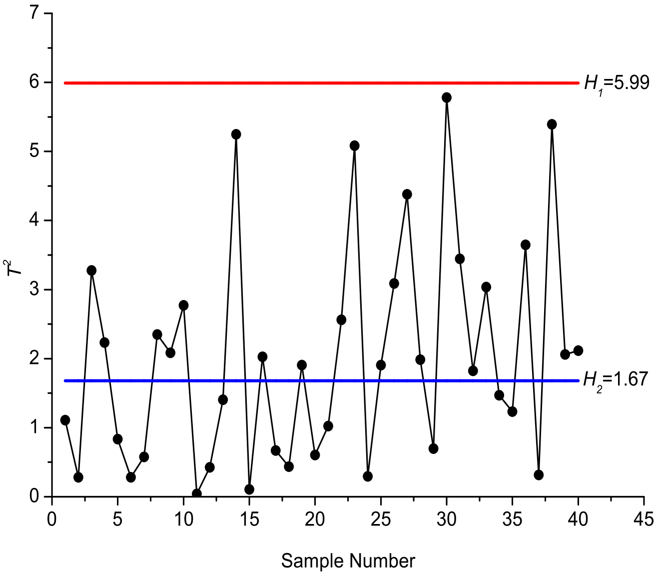


(a) 2|2 rule

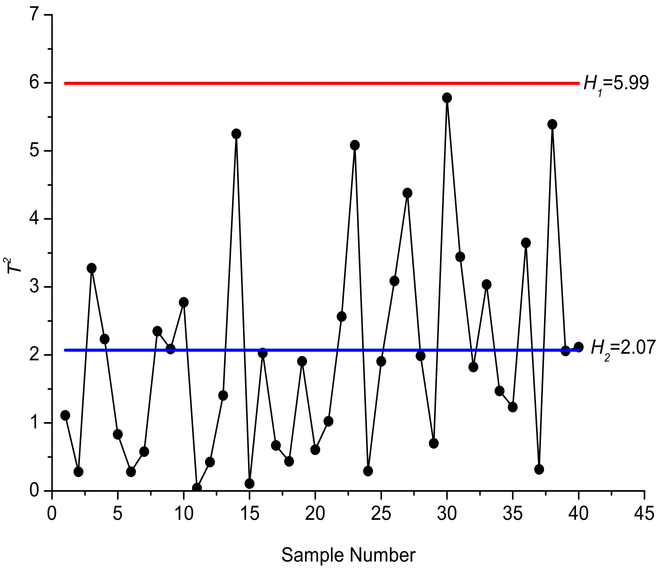


(b) 2|3 rule

Figure 10. Proposed control chart for monitoring the dowel pin process characteristics at 2|2, 2|3 and $\lambda = 0.0001$



(a) 3|3 rule



(b) 3|4 rule

Figure 11. Proposed control chart for monitoring the dowel pin process characteristics at 3|3, 3|4 and $\lambda = 0.0001$

7. Summary and conclusion

The primary objective of this study is to design and evaluate ARL based multivariate Hotelling control chart with sensitizing rules to efficiently detect small to moderate variations in the characteristics of the process. The control limit of the proposed control chart is presented as a function of the probability of a single point (PSP), and the number of process characteristics. To determine the desired value of the PSP which control the in-control average run length in control at the intended level, a generalized single polynomial equation is derived. Existing and alternative performance measures including individual and overall are considered. The behavior of the proposed control chart is assessed considering the variant choices of the special cause variations in the mean vector, the sensitizing rules, and the number of process characteristics. Furthermore, the Monte Carlo simulation mechanism and numerical integration are applied to calculate the values of performance measures.

The results indicate that the multivariate Hotelling control chart with sensitizing rules has invariant behavior when the process is in control. The actual in-control average run length remained stable at a prefixed level. In addition, the detection ability of the proposed control chart for special causes of variations is dependent on the choice of sensitizing rules, the number of process characteristics, and the amount of change. To categorize the optimal choices of the sensitizing rules, a comprehensive analysis is conducted taking into account various control chart factors. A case study on dowel pin manufacturing was adopted to demonstrate the effectiveness of the proposed multivariate Hotelling control chart integrated with the sensitizing rules.

Acknowledgements

Dr. Maysaa express their gratitude to Princess Nourah bint Abdulrahman University Researchers Supporting Project number (PNURSP2025R913),Princess Nourah bint Abdulrahman University, Riyadh,Saudi Arabia

Author contributions. All the co-authors have contributed equally in all aspects of the preparation of this submission.

Conflict of interest statement. The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Funding. Dr. Maysaa recieved funding through supporting project number (PNURSP2025R913),Princess Nourah bint Abdulrahman University, Riyadh,Saudi Arabia.

Data availability. No data was used for the research described in the article.

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APPENDIX

A1. Derivation of the proposed single polynomial equation

Let p be the probability of single point (PSP) lies outside the control limit, and γ_1 be the probability of r points out of w lie outside the control limit, this is,

$$\gamma_1 = \binom{w}{r} p^r \left(1-p\right)^{w-r}. \tag{7.1}$$

Let γ_2 be the probability of r points out of $w-1$ lie outside the control limit, that is,

$$\gamma_2 = \binom{w-1}{r} p^r \left(1-p\right)^{w-r-1}. \tag{7.2}$$

Let γ_3 be the difference between γ_1 and γ_2, then from Eq. (7.1) and Eq. (7.2), we obtain,

$$\gamma_3 = \binom{w}{r} p^r (1-p)^{w-r} - \binom{w-1}{r} p^r (1-p)^{w-r-1}$$
$$= \frac{w!\, p^r (1-p)^{w-r}}{(w-r)!\, r!} - \frac{(w-1)!\, p^r (1-p)^{w-1-r}}{(w-1-r)!\, r!}$$
$$= \frac{w\,(w-1)!\, p^r (1-p)^{w-r}}{(w-r)!\, r!} - \frac{(w-1)!\ (w-r)\, p^r (1-p)^{w-r}}{(w-r)!\, r!\, (1-p)}$$
$$\gamma_3 = \frac{(1-p)^{w-1-r} p^r\, (r-wp)\, (w-1)!}{(w-r)!\, r!}$$

Let R be a random variable which represents the run length (or say sample number) at which r out of w consecutive points lie outside the control limit when actually process is in-control. Now the expected value of R_2 is given as follows:

$$E(R) = C.\frac{1}{\gamma_3}, \tag{7.3}$$

where C denote a correction factor and it is defined as follows:

$$C = \frac{(1-p^r)^{w-r+1}}{(1-p)}$$

Now substitute C and γ_3 in Eq.(7.3),

$$E(R) = \frac{(1-p^r)^{w-r+1}}{(1-p)} . \frac{(w-r)!\, r!}{(1-p)^{w-1-r} p^r\, (r-wp)\, (r-1)!}$$
$$= \frac{(1-p^r)^{w-r+1}\, (w-r)!\, r!}{(1-p)\, (1-p)^{w-1-r} p^r\, (r-wp)\, (w-1)!}$$
$$E(R) = \frac{(1-p^r)^{w-r+1}\, (w-r)!\, r!}{p^r (1-p)^{w-r}\, (r-wp)\, (w-1)!}$$

A2. R code for computing p

```
{
    p=c(1:1000000)/1000000
    N=(2^m-p^m)^(k-m+1)*factorial(k-m)*factorial(m)
    D=2^((m-1)*(k-m))*(2-p)^(k-m)*p^m*(2*m-k*p)*factorial(k-1)
    G=N/D
    H=(abs(G-ARL))
    p=p[H>0&H<1]
    # optimization of p
    N=(2^m-p^m)^(k-m+1)*factorial(k-m)*factorial(m)
    D=2^((m-1)*(k-m))*(2-p)^(k-m)*p^m*(2*m-k*p)*factorial(k-1)
    G=N/D
    K=abs(G-ARL)
    S=p[which(K==min(K))]
    return(S)
  }
onepoint(370,2,3)
```

A3. R code for computing performance measures

```
library(QRM)
t1=2
n=1
p=0.0027        # 370
ucl=round(qchisq(1-p,t1),4)
sigmax=1;sigmay=1;rhoxy=0
meanv2=matrix(0,t1,1)
delta=c(0.00,0.15,0.20,0.25,0.30,0.35,0.40,0.45,0.50,0.55,
    0.60,0.65,0.70,0.75,0.80,1.00,1.05,1.10,1.15,1.20,
    1.25,1.30,1.50,1.70,2.00,3.00)
ARL=c()   SDRL=c()   MRL=c()   PRL25=c()   PRL75=c()   PRL90=c()
for(del in 1:length(delta))
{x=c()
  RL=c()
  simu=10000
    for( t in 1:simu)
  {for(j in 1:70000)
    {if(j==1)
    {meanv=matrix(round(delta[del]/sqrt(t1),2),t1,1) \\
    sigmam= diag(1, nrow=t1, ncol=t1)
        biv1=rmvnorm(n,mean=meanv,sigma=sigmam)
        a=rbind(biv1)
        meanv1=matrix(colMeans(a),t1,1)
        x[1]=n*t(meanv1-meanv2)%*%solve(sigmam)%*%(meanv1-meanv2)
        if(x[j]>ucl)
        {RL[t]=j
        break}}
        if(j>1)
      {meanv=matrix(round(delta[del]/sqrt(t1),2),t1,1)
        sigmam= diag(1, nrow=t1, ncol=t1)
        biv1=rmvnorm(n,mean=meanv,sigma=sigmam)
        a=rbind(biv1)
        meanv1=matrix(colMeans(a),t1,1)
        x[j]=n*t(meanv1-meanv2)%*%solve(sigmam)%*%(meanv1-meanv2)
        if(x[j]>ucl)
        {RL[t]=j
          break}}}
    print(mean(RL)) }
  ARL[del]=mean(RL)
  SDRL[del]=sd(RL)
  MRL[del]=median(RL)
  PRL25[del]=quantile(RL,0.25)
  PRL75[del]=quantile(RL,0.75)
  PRL90[del]=quantile(RL,0.90)}

g1=round(cbind(delta,ARL,SDRL,MRL,PRL25,PRL75,PRL90),3)
write.table(g1, file =
    "C:\\Users\\LaTiTude-E6410\\Desktop\\Results\\Rule_1_1_Side=U.xls",
    append = FALSE, quote = TRUE, sep = "\t",eol = "\n", na = "NA", dec =
    ".", row.names = FALSE,col.names =
    c("delta","ARL","SDRL","MRL","PRL25","PRL75","PRL90"))
```