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BOUNDEDNESS OF INVOLUTIONS IN TOPOLOGICAL ALGEBRAS WITH ORTHONORMAL GENERALIZED BASES

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Abstract. A main result of this article establishes that every involution in a sequentially complete locally convex topological algebra with an orthonormal generalized basis is bounded. This result is obtained through a representation for involutions in these algebras.

1. Introduction

It was known that every multiplicative linear functional on a real or complex Banach algebra is continuous. It is being stated that Mazur questioned in his lectures about the extension of this result to general complex complete metrizable topological algebras. However, Michael [12] raised the following two particular questions: (a) Is every complex multiplicative linear functional on a commutative complex complete metrizable locally multiplicatively convex topological algebra continuous?; (b) Is every complex multiplicative linear functional on a commutative complex complete locally multiplicatively convex topological algebra bounded? All these questions are still being open problems. To obtain some partial answers to these questions, a concept of orthonormal bases in topological algebras was introduced and studied [1, 3, 4, 6, 7, 8, 9, 10, 15], and many successful partial answers were obtained. This article generalizes the concept of orthonormal bases . This article presents a representation theorem for involutions in topological algebras with orthonormal generalized bases, and then obtains boundedness of these involutions in sequentially complete locally convex algebras with orthonormal generalized bases. For the purpose of boundedness, a technique of Dixon and Fremlin [5] is used. Dixon and Fremlin proved in [5] that the questions (a) and (b) of Michael [12], which are given above, are equivalent. Only Hausdorff topological spaces are considered in this article. Vector spaces to be considered are vector spaces over the

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real field or over the complex field. Algebras to be considered will be commutative, because topological algebras [2] with orthonormal generalized bases are commutative. All algebraic operations of topological algebras are jointly continuous in this article. A complete metrizable topological vector space is called an $F-$space [14]. A locally convex $F-$space is called a B_0-space. A complete metrizable topological algebra is called an $F-$algebra. A locally convex $F-$algebra is called a B_0-algebra. A topological algebra $(A, (p_i)_{i\in I})$ is called a locally multiplicatively convex algebra, if the topology on the algebra A is generated by a family $(p_i)_{i\in I}$ of submultiplicative semi norms. A locally multiplicatively convex $F-$algebra is called a Frechet algebra. A sequence $(x_n)_{n=1}^{\infty}$ in a topological vector space X is called a basis [16], if to each $x \in X$, there is a unique sequence of scalars $(f_n(x))_{n=1}^{\infty}$ such that $x = \sum_{n=1}^{\infty} f_n(x)x_n$. Then the sequence $(f_n)_{n=1}^{\infty}$ of linear functionals are called the sequence of coefficient functionals associated with the basis $(x_n)_{n=1}^{\infty}$. It is known that every coefficient functional is continuous on an $F-$space with a basis. A sequence $(e_n)_{n=1}^{\infty}$ in a topological algebra A is called an orthonormal basis, if $e_ne_m = 0$, for $m \neq n$, and $e_ne_n = e_n$ for every n. It is known that every coefficient functional is continuous on a topological algebra with an orthonormal basis [17].

2. Uniform boundedness principle

The next result is known [14] for sequences of linear mappings. The usual arguments for sequences are applicable even for nets.

Theorem 2.1. *Let $(T_\alpha)_{\alpha\in D}$ be a net of continuous linear mappings from an $F-$space X into a topological vector space Y. Let $T : X \to Y$ be a linear mapping such that $(T_\alpha(x))_{\alpha\in D}$ converges to $T(x), \forall x \in X$, and such that the set $\{T_\alpha(x) : \alpha \in D\}$ is bounded, for every $x \in X$. Then $T : X \to Y$ is a continuous linear mapping.*

Proof: By the Banach Stainhaus theorem, the family $\{T_\alpha : \alpha \in D\}$ is an equicontinuous family on X. Fix an open neighbourhood U of 0 in Y. Find an open neighbourhood V of 0 in Y such that $\overline{V} \subset U$. Then there is an open neighbourhood W of 0 in X such that $T_\alpha(W) \subset V$, $\forall \alpha \in D$. Then $T(W) \subset \overline{V} \subset U$. Thus, for a given neighbourhood U of 0 in Y, there is a neighbourhood W of 0 in X such that $T(W) \subset U$. That is, T is continuous on X.

Let us next apply the technique of Dixon and Fremlin [5] in the proof of the next theorem.

Theorem 2.2. *Let $(T_\alpha)_{\alpha\in D}$ be a net of bounded linear mappings from a sequentially complete locally convex topological vector space $(X, (p_i)_{i\in I})$ into a topological vector space Y, when $(p_i)_{i\in I}$ is a family of seminorms which induce the topology of X. Suppose $(T_\alpha(x))_{\alpha\in D}$ converges to some $T(x)$, $\forall x \in X$, and suppose $\{T_\alpha(x) : \alpha \in D\}$ is bounded, for every $x \in X$. Then $T : X \to Y$ is a bounded linear mapping.*

Proof: Let B be a bounded subset of X. For each $n = 1, 2, 3, \cdots$, let $I_n = \{i \in I : p_i(x) \leq n, \forall x \in B\}$, and let $q_n(x) = \sup\{p_i(x) : i \in I_n\}, \forall x \in X$. Let $X_0 = \{x \in X : q_n(x) < \infty, \forall n = 1, 2, 3, \cdots\}$. Then $(X_0, (q_n)_{n=1}^{\infty})$ is a B_0-space. When the previous theorem is applied to the restricted mappings $T_\alpha : (X_0, (q_n)_{n=1}^{\infty}) \to Y, \alpha \in D$, it can be concluded that the restricted mapping $T : (X_0, (q_n)_{n=1}^{\infty}) \to Y$ is continuous. So, $T(B)$ is a bounded set in Y, because B is a bounded subset of $(X_0, (q_n)_{n=1}^{\infty})$. This proves the theorem.

The same Dixon-Fremlin technique along with the classical closed graph theorem can be applied to derive the following closed graph theorem. This theorem is Theorem 2.1 in [18].

Theorem 2.3. *Let X be a sequentially complete locally convex topological vector space and Y be an $F-$space. Let $T : X \to Y$ be a linear mapping such that its graph $\{(x, Tx) : x \in X\}$ is sequentially closed in $X \times Y$. Then T is a bounded mapping on X.*

The next proposition is useful in applying Theorem 2.1 and Theorem 2.2.

Proposition 2.4. *Let $(D, \leq)$ be a directed set such that $\{\alpha \in D : \alpha \ngeq \beta\}$ is a finite set for every $\beta \in D$. If $(x_\alpha)_{\alpha \in D}$ is a Cauchy net in a topological vector space X, then the set $\{x_\alpha : \alpha \in D\}$ is bounded.*

Proof: Fix a balanced open neighbourhood U of 0 in X. Find a balanced open neighbourhood V of 0 in X such that $V + V \subset U$. Find a $\gamma \in D$ such that $x_\alpha - x_\beta \in V, \forall \alpha, \beta \geq \gamma$ in D. Find a $c > 1$ such that $x_\gamma \in cV$ and such that $x_\alpha \in cV$ whenever $\alpha \ngeq \gamma$ in D. For $\alpha \geq \gamma$ in D, $x_\alpha = (x_\alpha - x_\gamma) + x_\gamma \in V + cV \subset cV + cV \subset cU$. Thus, $x_\alpha \in cU, \forall \alpha \in D$. This proves the proposition.

The condition imposed on the directed sets of the previous proposition is not vague, as it is seen from the next example.

Example 2.1. *For each $n = 1, 2, 3, \cdots$, let E_n be a non empty finite set such that $E_n \cap E_m = \emptyset$, whenever $n \neq m$. Let $D = \bigcup_{n=1}^{\infty} E_n$. Define $\alpha < \beta$ in D if and only if $\alpha \in E_n$ and $\beta \in E_m$ for some m, n satisfying $n < m$. Then $(D, \leq)$ is a directed set such that $\{\alpha \in D : \alpha \ngeq \beta\}$ is a finite set for every $\beta \in D$. Moreover, $\{\alpha \in D : \alpha \leq \beta\}$ is finite and $\{\alpha \in D : \alpha \geq \beta\}$ is infinite, for every $\beta \in D$.*

Proposition 2.5. *Let I be an infinite set. Let D be the collection of all non empty finite subsets of I, which is a directed set under the inclusion relation: $E \leq F$ in D if and only if $E \subset F$. To each $i \in I$, let x_i be a member of a topological vector space X. To each $F \in D$, let $y_F = \sum_{i \in F} x_i$. Suppose $(y_F)_{F \in D}$ is a Cauchy net in X. Then $\{y_F : F \in D\}$ is a bounded set in X.*

Proof: Let U be a balanced neighbourhood of 0 in X. Find a balanced neighbourhood V of 0 in X such that $V + V \subset U$. Find a set $E \in D$ such that $y_F - y_E \in V$ whenever $F \in D$ and $F \supset E$. Thus $y_F \in V$ whenever $F \in D$ and $F \subset I \setminus E$. Find $c > 1$ such that $y_F \in cV$ whenever $F \in D$ and $F \subset E$. Then, for any $F \in D$, $y_F = y_{F \cap E} + y_{F \cap (I \setminus E)} \in cV + V \subset cV + cV \subset cU$. This proves that $\{y_F : F \in D\}$ is a bounded set in X.

3. Unconditional orthonormal generalized basis

The last result of the previous section gives a motivation for the results of this section.

Definition 3.1. *A collection $(e_i)_{i \in I}$ in a topological algebra A is said to be orthonormal, if $e_i e_j = 0$ whenever $i \neq j$ in I and $e_i e_i = e_i \neq 0, \forall i \in I$. This orthonormal collection $(e_i)_{i \in I}$ is said to be total in A, if the linear span of the set $\{e_i : i \in I\}$ is dense in A.*

Remark.

(a) *If $(e_i)_{i\in I}$ is orthonormal in A, then the set $\{e_i : i \in I\}$ is linearly independent. For, if $\sum_{i\in F} \lambda_i e_i = 0$, for some finite subset F of I and for some scalars λ_i, then $\lambda_j e_j = e_j \sum_{i\in F} \lambda_i e_i = 0$, $\forall j \in F$, when $e_j \neq 0$.*

(b) *If $(e_i)_{i\in I}$ is orthonormal in A, then the linear span of the set $\{e_i : i \in I\}$ is also an algebra.*

(c) *If $(e_i)_{i\in I}$ is a total orthonormal collection in A, then it is a maximal orthonormal collection in A. (Verification: Suppose $(e_i)_{i\in I}$ is a total orthornormal collection in A. If it is not a maximal orthonormal collection in A, then there is an element e_α in A such that $e_\alpha \neq e_i, \forall i \in I$, and such that $\{e_i : i \in I\} \cup \{e_\alpha\}$ is orthonormal. Then there is a net $(x_\delta)_{\delta\in D}$ in B, the linear span of $\{e_i : i \in I\}$, such that $(x_\delta)_{\delta\in D}$ converges to e_α. Let $x_\delta = \sum_{i\in F_\delta} \lambda_{i\delta} e_i$, when F_δ is a finite subset of I and $\lambda_{i\delta}$ are scalars. Then $(x_\delta e_\alpha)_{\delta\in D}$ converges to e_α, when $x_\delta e_\alpha = 0, \forall \delta \in D$ and when $e_\alpha \neq 0$. This is impossible.)*

(d) *Let $(e_i)_{i\in I}$ be a total orthonormal collection in A, and let $B = span\{e_i : i \in I\}$. To each $x = \sum_{i\in F} \lambda_i e_i$ in B, let us define $f_i(x) = \lambda_i, \forall i \in F$, a finite subset of I. This is meaningful in view of the linear independence of $\{e_i : i \in I\}$. Let $(x_\delta)_{\delta\in D}$ be a net in B that converges to zero. Then $(e_i x_\delta)_{\delta\in D} = (f_i(x_\delta)e_i)_{\delta\in D}$ converges to zero for every $i \in I$. Thus $(f_i(x_\delta))_{\delta\in D}$ converges to the scalar zero, for every $i \in I$. Thus each f_i is a continuous linear functional on B. Then there is a unique continuous extension for f_i to A. Let us denote this extension also by the same notation f_i. Let us call the continuous linear functional f_i on A as the coefficient functional corresponding to e_i. Let us call the collection $(f_i)_{i\in I}$ as the collection of coefficient functionals associated with $(e_i)_{i\in I}$.*

Definition 3.2. *Let $(e_i)_{i\in I}$ be an orthonormal collection in a topological algebra A. Let D be the directed set of all non empty finite subsets of the set I. Suppose that for each $x \in A$, there is a unique collection $(\lambda_i)_{i\in I}$ of scalars such that the net $\left(\sum_{i\in F} \lambda_i x_i\right)_{F\in D}$ converges to x; and let us say that the series $\sum_{i\in I} \lambda_i x_i$ converges to x, in this case. Then this collection $(e_i)_{i\in I}$ is called as an orthonormal generalized basis in A.*

Remark.

(a) *If $(e_i)_{i\in I}$ is an orthonormal generalized basis in A, then each coefficient functional f_i corresponding to e_i is continuous on A. Moreover, $x = \sum_{i\in I} f_i(x)e_i, \forall x \in A$.*

(b) *Let us have a meaning as explained in the previous definition when the expressions of the form "$x = \sum_{i\in I} x_i$" are used.*

Theorem 3.1. *Let $T : A \to B$ be a homomorphism from a topological algebra A into a topological algebra B. Let $(e_i)_{i\in I}$ be an orthonormal generalized basis in A. Suppose $T\left(\sum_{i\in I} \lambda_i e_i\right) = \sum_{i\in I} \lambda_i T(e_i)$ for every $\sum_{i\in I} \lambda_i e_i \in A$. Then T is continuous, if A is an F-algebra. T is bounded, if A is a sequentially complete locally convex topological algebra.*

Proof: Let D be the directed set of all non empty finite subsets of I, directed by the inclusion relation. For each $F \in D$, let us define $T_F : A \to B$ by $T_F\left(\sum_{i\in I} \lambda_i e_i\right) = \sum_{i\in F} \lambda_i T(e_i)$, for every $\sum_{i\in I} \lambda_i e_i \in A$. Then $T\left(\sum_{i\in I} \lambda_i e_i\right)$ is the limit of the net $\left(T_F\left(\sum_{i\in I} \lambda_i e_i\right)\right)_{F\in D}$. Let $(f_i)_{i\in I}$ be the collection of continuous coefficient functionals on A, associated with $(e_i)_{i\in I}$. Let us note that $\{T(e_i) : T(e_i) \neq 0, i \in I\}$ is a total orthonormal set in the closure E of the span of $\{T(e_i) : i \in I\}$ in B. (Take $E = \{0\}$, when $T = 0$.) Let g_j be the coefficient functional on E corresponding to $T(e_j)$ for which $T(e_j) \neq 0$. Each g_j is continuous on E. Let $(x_\delta)_{\delta\in D}$ be a net that converges to zero in A. If $T(e_j) \neq 0$, then $(f_j(x_\delta))_{?\in D}$ converges to zero, and hence $(f_j(x_\delta)T(e_j))_{\delta\in D}$ converges to zero. Thus, $(g_j(T(x_\delta))T(e_j))_{\delta\in D}$ converges to zero, when $T(e_j) \neq 0$. So, each T_F is continuous on A, whenever F is a singleton subset of I. Hence, each T_F is continuous on A, for every $F \in D$. Theorem 2.1 and Theorem 2.2 are now applied along with Proposition 2.5 to get the desired conclusions.

Theorem 3.2. *Let $T : A \to B$ be a homomorphism from a topological algebra A into a topological algebra B. Let $(e_i)_{i\in I}$ be an orthonormal generalized basis in A such that $(T(e_i))_{i\in I}$ is an orthonormal generalized basis in B. Suppose $T\left(\sum_{i\in I} \lambda_i e_i\right) = \sum_{i\in I} \lambda_i T(e_i)$ for every $\sum_{i\in I} \lambda_i e_i \in A$. Then T has a closed graph in $A \times B$.*

Proof: Let f_i and g_i be the continuous coefficient functionals corresponding to e_i and $T(e_i)$, respectively, for every $i \in I$. Then $f_i = g_i \circ T, \forall i \in I$. Let $(x_\delta)_{\delta\in D}$ be a net which converges to zero in A such that $(T(x_\delta))_{\delta\in D}$ converges to some y in B. Then $(f_i(x_\delta))_{\delta\in D}$ converges to 0, and $(g_i(T(x_\delta)))_{\delta\in D}$ converges to $g_i(y)$, for every $i \in I$. Since $g_i(T(x_\delta)) = f_i(x_\delta), \forall i \in I, \forall \delta \in D$, then $g_i(y) = 0, \forall i \in I$. This implies that $y = 0$. Thus T has a closed graph in $A \times B$.

Corollary 3.3. *If A and B are $F-$algebras in the previous theorem, then T is continuous. If A is a sequentially complete locally convex algebra and B is an $F-$algebra in the previous theorem, then T is bounded.*

Proof: Let us apply the classical closed graph theorem for the first part, and let us apply Theorem 2.3 for the second part.

All the results stated above and all classical theorems used for them are applicable even for conjugate linear mappings. Let us use this fact in the remaining part of this article. Let us observe that an involution in a complex algebra is a conjugate linear mapping.

4. Involutions

Let us first establish a representation theorem for involutions in topological algebras with orthonormal generalized bases. One representation theorem of this type is given in the article [17], but for orthonormal bases.

Theorem 4.1. *Let A be a topological algebra with an orthonormal generalized basis $(e_i)_{i\in I}$. Let $*$ be an involution in A. Then there is a bijection $P : I \to I$ such that $\left(\sum\limits_{i\in I} \lambda_i e_i\right)^* = \sum\limits_{i\in I} \overline{\lambda_i} e_{P(i)} = \sum\limits_{i\in I} \overline{\lambda_{P(i)}} e_i, \forall \sum\limits_{i\in I} \lambda_i e_i \in A$.*

PROOF: For $j \in I$, let $e_j^* = \sum\limits_{i\in I} \lambda_i e_i$. Then $e_j^* e_j^* = e_j^*$ implies $\sum\limits_{i\in I} \lambda_i^2 e_i = \sum\limits_{i\in I} \lambda_i e_i$ so that $\lambda_i^2 = \lambda_i, \forall i$, or equivalently, $\lambda_i = 0$ or $1, \forall i \in I$. Let $I_j = \{i \in I : \lambda_i = 1\} \neq \emptyset$, because $e_j^* \neq 0$. If $k \in I_j$, then $0 \neq e_j^* e_k = \left(\sum\limits_{i\in I_j} e_i\right) e_k = e_k, e_k^* = e_j e_k^*$ so that $0 \neq e_k^* = \sum\limits_{i\in I_k} e_i = e_j \left(\sum\limits_{i\in I_k} e_i\right) = e_j$, and hence $e_k^* = e_j$. Thus, I_j contains only one element, say, $P(j), \forall j \in I$. Note that $P(P(j)) = j$, because $(e_j^*)^* = e_j, \forall j \in I$.

Let $x = \sum\limits_{i\in I} \lambda_i e_i$. Then $xe_j = \lambda_j e_j$ so that $x^* e_j^* = \overline{\lambda_j} e_j^*$ or $x^* e_{P(j)} = \overline{\lambda_j} e_{P(j)}, \forall j \in I$. So, $x^* e_i = \overline{\lambda_{P(i)}} e_i, \forall i \in I$. This proves that $\left(\sum\limits_{i\in I} \lambda_i e_i\right)^* = \sum\limits_{i\in I} \overline{\lambda_i} e_{P(i)} = \sum\limits_{i\in I} \overline{\lambda_{P(i)}} e_i, \forall \sum\limits_{i\in I} \lambda_i e_i \in A$.

Corollary 4.2. *If A in the previous theorem is an $F-$algebra, then the involution is continuous. If A in the previous theorem is a sequentially complete locally convex topological algebra, then the involution is bounded.*

PROOF: Let us apply the classical uniform boundedness theorem for the first part. Let us apply Theorem 2.2 along with Proposition 2.5 for the second part.

5. Orthonormal directed generalized bases

Let us follow the article [13] to consider directed sets $(D, \leq)$ having the properties: (a) $\{\alpha \in D : \alpha \leq \beta\}$ is finite, for every $\beta \in D$; and (b) $\{\alpha \in D : \alpha \geq \beta\}$ is infinite, for every $\beta \in D$. Let us proceed with these additional assumptions on all directed sets. These conditions (a) and (b) are satisfied by the directed set $\mathbb{N} \times \mathbb{N}$ used in the article [11].

Definition 5.1. *Let $(e_\alpha)_{\alpha\in D}$ be a net in a topological algebra A such that $e_\alpha^2 = e_\alpha \neq 0, \forall \alpha \in D$, and such that $e_\alpha e_\beta = 0$ whenever $\alpha \neq \beta$ in D. Then $(e_\alpha)_{\alpha\in D}$ is called an orthonormal directed generalized basis, if for each $x \in A$, there is a unique net $(\lambda_\alpha)_{\alpha\in D}$ of scalars such that the net $\left(\sum\limits_{\alpha\leq\beta} \lambda_\alpha e_\alpha\right)_{\beta\in D}$ converges to x in A. Let us say in this case that the directed series $\sum\limits_{\alpha\in D} \lambda_\alpha e_\alpha$ converges to x.*

One can follow the earlier arguments to derive the following results.

Theorem 5.1. *Let $T : A \to B$ be a homomorphism from a topological algebra A into a topological algebra B. Let $(e_\alpha)_{\alpha\in D}$ be an orthnormal directed generalized basis in A such that $(T(e_\alpha))_{\alpha\in D}$ is an orthonormal directed generalized basis in B. Suppose $T\left(\sum\limits_{\alpha\in D} \lambda_\alpha e_\alpha\right) = \sum\limits_{\alpha\in D} \lambda_\alpha T(e_\alpha)$, for every $\sum\limits_{\alpha\in D} \lambda_\alpha e_\alpha \in A$. Then T has a closed graph in $A \times B$.*

Theorem 5.2. *Let A be a topological algebra with an orthonormal directed generalized basis $(e_\alpha)_{\alpha\in D}$. Let $*$ be an involution in A. Then there is a bijection $P : I \to I$ such that* $\left(\sum_{\alpha\in D}\lambda_\alpha e_\alpha\right)^* = \sum_{\alpha\in D}\overline{\lambda_\alpha}e_{P(\alpha)} = \sum_{\alpha\in D}\overline{\lambda_{P(\alpha)}}e_\alpha, \forall \sum_{\alpha\in D}\lambda_\alpha e_\alpha \in A$.

Theorem 5.3. *Let us now further assume that the directed set $(D, \leq)$ satisfies the condition: $\{\alpha \in D : \alpha \ngeq \beta\}$ is finite, for every $\beta \in D$. Let $T : A \to B$ be a homomorphism from a topological algebra A into a topological algebra B. Let $(e_\alpha)_{\alpha\in D}$ be an orthnormal directed generalized basis in A. Suppose* $T\left(\sum_{\alpha\in D}\lambda_\alpha e_\alpha\right) = \sum_{\alpha\in D}\lambda_\alpha T(e_\alpha)$, *for every $\sum_{\alpha\in D}\lambda_\alpha e_\alpha \in A$. Then T is continuous, if A is an $F-$algebra. T is bounded, if A is a sequentially complete locally convex topological algebra.*

Corollary 3.3 can be now restated without changing the words as a corollary to the previous theorem.

Corollary 5.4. *Let us now further assume that the directed set $(D, \leq)$ satisfies the condition: $\{\alpha \in D : \alpha \ngeq \beta\}$ is finite, for every $\beta \in D$. If A in Theorem 5.2 is an $F-$algebra, then the involution is continuous. If A in Theorem 5.2 is a sequentially complete locally convex topological algebra, then the involution is bounded.*

6. Conclusion

This article has chosen some known techniques and has applied them to new concepts of orthonormal generalized bases and orthonormal directed generalized bases. It is possible to extend some techniques used in this article to some more algebras. But, there is always a need for new techniques to tackle open problems in the theory of automatic continuity.

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