POLİTEKNİK DERGİSİ

JOURNAL of POLYTECHNIC

ISSN: 1302-0900 (PRINT), ISSN: 2147-9429 (ONLINE)

URL: http://dergipark.gov.tr/politeknik

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Altın Riemann manifoldlarının tamamen umbilik yarı-invariant altmanifoldları üzerine bir çalışma

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Bu makaleye şu şekilde atıfta bulunabilirsiniz(To cite to this article): Erdoğan F. E. ve Yıldırım C.,“Altın Riemann manifoldlarının tamamen umbilik yarı-invaryant altmanifoldları üzerine bir çalışma”, ***Politeknik Dergisi***, 21(4): 967-970, (2018).

Erişim linki (To link to this article): http://dergipark.gov.tr/politeknik/archive

DOI: 10.2339/politeknik.389629

Altın Riemann Manifoldlarının Tamamen Umbilik Yarı-Invariant Altmanifoldları Üzerine Bir Çalışma

Araştırma Makalesi / Research Article

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(Geliş/Received : 02.10.2016 ; Kabul/Accepted : 26.08.2017)

ÖZ

Altın oran, sürekli yeni fikirler üretilen büyüleyici bir konudur. Altın yapı ile donatılmış bir Riemann manifoldu altın Riemann manifoldu olarak isimlendirilecektir. Tam olarak söyleyebiliriz ki, m-boyutlu $(\bar{M},\bar{g})$ Riemann manifoldu üzerindeki (1,1) tipinde bir $\bar{P}$ tensör alanı, Id, $\bar{M}$ üzerinde birim dönüşüm olduğu yerde, eğer $\bar{P}^2=\bar{P}+Id$, denklemini sağlarsa bir Golden yapıdır. Ayrıca, Riemannian metrik $\bar{g}(\bar{P}X,Y)=\bar{g}(X,\bar{P}Y)$, denklemini sağladığında $\bar{P}$-uyumludur denir ve $(\bar{M},\bar{g},\bar{P})$ de Golden Riemannian manifold olarak isimlendirilir. Bu makalenin asıl amacı, Golden yapılarıyla donatılmış Riemannian manifoldlarının geometrisini incelemektir. Bu amaçla, biz Golden Riemannian manifoldlarının tamamen umbilik yarı-invaryant altmanifoldlarını çalıştık. Ayrıca yapraklanmaların geometrisini inceledik ve distribüsyonların integrallenebilirlik şartlarını elde ettik.

Anahtar Kelimeler: Yarı-invaryant altmanifoldlar, Altın Riemann manifoldlar, tamamen umbilik altmanifoldlar.

On a Study of the Totally Umbilical Semi-Invariant Submanifolds of Golden Riemannian Manifolds

ABSTRACT

The Golden Ratio is fascinating topic that continually generated news ideas. A Riemannian manifold endowed with a Golden Structure will be called a Golden Riemannian manifold. Precisely, we can say that an (1,1)-tensor field $\bar{P}$ on a m-dimensional Riemann manifold $(\bar{M},\bar{g})$ is a Golden structure if it satisfies the equation $\bar{P}^2=\bar{P}+Id$, where Id is identity map on $\bar{M}$. Furthermore, $\bar{g}(\bar{P}X,Y)=\bar{g}(X,\bar{P}Y)$, the Riemannian metric is called $\bar{P}$-compatible and $(\bar{M},\bar{g},\bar{P})$ is named a Golden Riemannian manifold. The main purpose of the present paper is to study the geometry of Riemannian manifolds endowed with Golden structures. For this purpose, we study totally umbilical semi-invariant submanifold of the Golden Riemannian manifolds. Also, we obtain integrability conditions of the distributions and investigate the geometry of foliations.

Keywords: Semi-invariant submanifolds, Golden Riemannian manifolds, totally umbilical submanifolds

1. INTRODUCTION

The Golden proportion, also called the Golden ratio, Divine ratio, Golden section or Golden mean, has been well known since the time of Euclid. Many objects alive in the natural world that possess pentagonal symmetry, such as inflorescence of many flowers and phyllotaxis objects have a numerical description given by the Fibonacci numbers which are themselves based on the Golden proportion. The Golden proportion has also been found in the structure of musical compositions, in the ratios of harmonious sound frequencies and in dimensions of the human body. From ancient times it has played an important role in architecture and visual arts. The Golden proportion and the Golden rectangle (which is spanned by two sides in the Golden proportion) have been found in the harmonious proportions of temples, churches, statues, paintings, pictures and fractals.

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Golden Riemannian manifolds were introduced by Crasmereanu and Hretcanu [3] by using Golden ratio. The authors also studied invariant submanifolds of a Golden Riemannian manifold and obtained interesting results in [4], [6]. The integrability of such Golden structures was also investigated by Gezer, Cengiz and Salimov in [5]. Moreover, the harmonicity of maps between Golden Riemannian manifolds was studied in [7].

Submanifolds of Riemannian manifolds endowed with some structure (complex, contact, product etc.) have rich geometric properties. In this way, CR-submanifolds have been studied by many authors see: [2], [3], [7] and they found many intersting results.

The structure of this article is following: In Preliminaries, we give some fundamental concepts and definitions needed for this paper. In Section3, we investigate totally umbilical semi-invariant submanifold of the Golden Riemannian manifolds, furthermore, we find integrability conditions of the distributions and analyze the geometry of foliations.

2. PRELIMINARIES

Let $(\bar{M}, \bar{g})$ be a Riemannian manifold. Then $\bar{M}$ is called Golden Riemannian manifold if there exists an $(1,1)$ tensor field $\bar{P}$ on $\bar{M}$ such that

$$\bar{P}^2 = \bar{P} + Id, \tag{1}$$

where Id is identity map on $\bar{M}$. Also,

$$\bar{g}(\bar{P}\,X, Y) = \bar{g}(X, \bar{P}\,Y). \tag{2}$$

The Riemannian metric (2) is called $\bar{P}$ -compatible and $(\bar{M}, \bar{g}, \bar{P})$ is named a Golden Rimannian manifold [3]. It is known [3] that a Golden structure φ is integrable if the Nijenhuis tensor N_φ vanishes. In [5], the authors show that a Golden structure is integrable if and only if $\bar{\nabla}_\varphi = 0$, where $\bar{\nabla}$ is Levi-Civita connection of $\bar{g}$.
Let M be a Golden Riemannian manifold isometrically immersed in $\bar{M}$ and denote by the same symbol g the Riemannian metric induced on M Let TM be the Lie algebra of vector fields in M and $TM^\perp$ the set of all vector fields normal to M. Then the Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y), \forall X, Y \in \Gamma(TM) \tag{3}$$

for all $X \in \Gamma(TM), N \in \Gamma(TM^\perp)$

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp Y, \tag{4}$$

where $\{\nabla_X Y, A_N X\}$ and $\{h(X, Y), \nabla_X^\perp Y\}$ belong to $\Gamma(TM)$ and $\Gamma(TM^\perp)$, respectively. ∇ and $\nabla^\perp$ are linear connections on M and $(TM^\perp)$, respectively. The second fundamental form h is a symmetric $\mathcal{F}(M)$-bilinear form on $\Gamma(TM)$ with values in $\Gamma(TM^\perp)$ and the shape operatör A_N is linear endomorphisim of $\Gamma(TM)$. Also we have, for $N \in \Gamma(TM^\perp)$

$$g(h(X, Y), N) = g(A_N X, Y). \tag{5}$$

Let M be a submanifold of Riemannian manifold $(\bar{M}, \bar{P})$ if there is a smooth vector field $\mathcal{H} \in \Gamma(TM^\perp)$ on M, called the curvature vector field of M, such that, for all $X, Y \in \Gamma(TM)$,

$$h(X, Y) = \mathcal{H} g(X, Y). \tag{6}$$

Hence, M is totally umbilical submanifold.

Definition 2.1 Let $(\bar{M}, \bar{P})$ be a Golden Riemannian manifold and M a real submanifold of $\bar{M}$. Then M is called a semi-invariant submanifold of $\bar{M}$, if it is endowed with the pair of orthogonal distributions $(D, D^\perp)$ satisfying the following conditions:

(a) $TM = D \oplus D^\perp$ (7)

(b) The distribution D is invariant

$$\bar{P} D_x = D_x, \tag{8}$$

for each $x \in M$.
(c) The distribution $D^\perp$ is anti-invariant,

$$\bar{P} D^\perp \subset T_x M^\perp, \tag{9}$$

for each $x \in M$.
The distributions D and $D^\perp$ are called the horizontal distribution and vertical distribution on M, respectively. A semi-invariant submanifold M is said to be invariant and anti-invariant submanifold if we have $D_x^\perp = 0$ and $D_x = 0$, respectively, for each $x \in M$ We say that, M is proper semi-invariant submanifold if it is a semi-invariant submanifold, which is neither an invariant nor an anti-invariant submanifold. The projection morphisims of TM to D and $D^\perp$ are denoted by T and R respectively. Then, we have

$$X = TX + RX, \tag{10}$$

$$\bar{P} N = BN + CN, \tag{11}$$

for $X \in \Gamma(TM)$ and $N \in \Gamma(TM^\perp)$, where BN and CN denote the tangential and normal components of $\bar{P} N$, respectively. Applying $\bar{P}$ to (10), we obtain

$$\bar{P}\,X = \bar{P}\,TX + \bar{P}\,RX \tag{12}$$

If we put $\bar{P}\,TX = SX$ and $\bar{P}\,RX = LX$, we rewrite (12)

$$\bar{P}\,X = SX + LX, \tag{13}$$

where $SX \in \Gamma(TD)$ and $LX \in \Gamma(TM^\perp)$. We obtain complementary distribution to $\bar{P} D^\perp$ in $\Gamma(TM^\perp)$ by μ, then, we have

$$TM^\perp = \bar{P}(D^\perp) \oplus \mu. \tag{14}$$

PROPOSITION 2.1 Let M be a semi -invariant submanifold of the Golden Riemanniann manifold$(\bar{M}, \bar{P})$. Then, the distribution μ is invariant with respect to $\bar{P}$, [9].

COROLLARY 2.1 Let M be a semi -invariant submanifold of the Golden Riemanniann manifold $(\bar{M}, \bar{P})$. Then the projection S given by (13) is a Golden structure on M [9].
However, there is no guarantee for the projection L.

THEOREM 2.1 Let M be a semi-invariant submanifold of Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, D distribution is integrable if and only if

$$h(X, PY) = h(Y, PX)$$

for all $X, Y \in \Gamma(D)$ [9].

LEMMA 2.1 Let M be a semi-invariant submanifold of Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, we have

$$A_{PX}Y = -A_{PY}X \tag{15}$$

for all $X, Y \in \bar{P}(D^{\perp})$[9].

THEOREM 2.2 Let M be a semi-invariant submanifold of Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, $D^{\perp}$ distribution is integrable if and only if

$$PA_{PX}Y = A_{PX}Y \tag{16}$$

has no components in D for all $X, Y \in \Gamma(D^{\perp})$, [9].

THEOREM 2.3 Let M be a semi-invariant submanifold of Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, D distribution defines a totally geodesic foliation if and only if

$$PA_{PZ}X - A_{PZ}X \tag{17}$$

has no components in $D^{\perp}$ for all $\in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$ [9].

3. TOTALLY UMBILICAL SEMI-INVARIANT SUBMANIFOLDS (TOTAL UMBİLİK YARI-İNVARYANT ALT MANİFOLDLAR)

In this section, totally umbilical semi-invariant submanifolds of Golden Riemannian manifolds are studied.
Totally umbilical M implies the following result which shows that the submanifold is foliated by D.

THEOREM 3.1 Let M be a totally umbilical submanifold of a Golden Riemannian manifold $\bar{M}$. Then, the distribution D is always integrable.

PROOF: Let M be a totally umbilical submanifold of a Golden Riemannian manifold $\bar{M}$. Then, for all $X, Y \in \Gamma(D)$, from (1), (2) and (6), we have

$$h(PX, Y) = g(PX, Y)\mathcal{H}$$

$$h(PX, Y) = h(PX, PY) - h(X, Y). \tag{18}$$

In a similiar way, we obtain

$$h(X, PY) = h(PX, PY) - h(X, Y). \tag{19}$$

Thus, from (18) and (19), the proof is completed.

LEMMA 3.1 Let M be a totally umbilical submanifold of a Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, we have

$$g(Y, Z)g(\mathcal{H}, PX) + g(X, Z)g(\mathcal{H}, PY) = 0$$

for all $X, Y, Z \in \Gamma(D^{\perp})$.

PROOF: From Lemma 2.1 , we get

$$g(A_{PX}Y, Z) = -g(A_{PY}X, Z),$$

$$g(Y, Z)g(\mathcal{H}, PX) + g(X, Z)g(\mathcal{H}, PY) = 0$$

THEOREM 3.2 Let M be a totally umbilical submanifold of a Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then, $D^{\perp}$ is integrable.

PROOF: From Theorem 2.2, it is known that [9], $D^{\perp}$ distribution integrable if and only if,

$$PA_{PX}Y = A_{PX}Y$$

has no components in D. Then for all $X, Y \in \Gamma(D^{\perp})$ and $Z \in \Gamma(D)$, we have

$$g(PA_{PX}Y - A_{PX}Y, Z) = \begin{Bmatrix} g(PA_{PX}Y, Z) \\ -g(A_{PX}Y, Z) \end{Bmatrix}$$

$$g(PA_{PX}Y - A_{PX}Y, Z) = \begin{Bmatrix} g(h(Y, PZ), PX) \\ -g(h(Y, Z), PX) \end{Bmatrix}$$

$$= \begin{Bmatrix} g(Y, PZ)g(\mathcal{H}, PX) \\ -g(Y, Z)g(\mathcal{H}, PX) \end{Bmatrix}$$

$$g(PA_{PX}Y - A_{PX}Y, Z) = 0,$$

which completes proof.

THEOREM 3.3 Let M be a totally umbilical submanifold of a Golden Riemannian manifold $(\bar{M}, \bar{P})$. Then D is integrable for all $X, Y \in \Gamma(D)$.

PROOF: From $(\bar{\nabla}_X P)Y = 0$, we have

$$\begin{Bmatrix} \nabla_X PY + h(X, PY) \\ -P\nabla_X Y - Ph(X, Y) \end{Bmatrix} = 0$$

$$\begin{Bmatrix} \nabla_X PY + h(X, PY) \\ -S\nabla_X Y - L\nabla_X Y \\ -Bh(X, Y) - Ch(X, Y) \end{Bmatrix} = 0$$

$$\begin{Bmatrix} \nabla_X PY + g(X, PY)\mathcal{H} \\ -S\nabla_X Y - L\nabla_X Y \\ -B\mathcal{H}g(X, Y) - C\mathcal{H}g(X, Y) \end{Bmatrix} = 0,$$

taking the normal parts of the equation, we get

$$g(X, PY)\mathcal{H} - L\nabla_X Y - C\mathcal{H}g(X, Y) = 0.$$

Interchanging X and Y above equation, we can be obtained

$$g(Y,PX)\mathcal{H}-L\nabla_Y X-C\mathcal{H}g(Y,X)=0.$$

Thus, we find

$$\{g(Y,PX)-g(X,PY)\}\mathcal{H}+L[X,Y]=0,$$

which implies that

$$L[X,Y]=0\ ,$$

then $[X,Y]\in\Gamma(D)$, which completes the proof.

THEOREM 3.4 Let M be a totally umbilical submanifold of a Golden Riemannian manifold $(\bar{M},\bar{P})$ and $D^{\perp}$ be integrable. Then $\mathcal{H}\in\Gamma(\mu)$ for all $X,Y,Z\in\Gamma(D^{\perp})$.

PROOF: From (5), (6) and (16), we have

$$g(h(X,Y),PZ)=g(g(X,Y),PZ)$$

$$g(PA_{PZ}X,Y)=g(X,Y)g(\mathcal{H},PZ).$$

Thus, we obtain

$$g(A_{PZ}X,PY)=g(X,Y)g(\mathcal{H},PZ),$$

$$0=\ g(X,Y)g(\mathcal{H},PZ).$$

By virtue of $g(X,Y)\neq 0$,

$$0=\ g(\mathcal{H},PZ).$$

Hence, we get

$$g(\mathcal{H},PZ)=g(P\mathcal{H},PZ)-g(\mathcal{H},Z),$$

$$g(\mathcal{H},PZ)=g(P\mathcal{H},PZ)=0,$$

namely, we obtain $\mathcal{H}\in\Gamma(\mu)$.
Thus, the proof is completed.

4. CONCLUSION

In the last few years, the Golden proportion has played an increasing role in modern physical research and it has a unique significant role in atomic physics. The Golden proportion is found to govern the transition from Newtons physics to relativistic mechanics and the Golden rectangle has been used to derive the dilation of time intervals and the Lorentz contraction of lengths in special relativity. The Golden proportion has also interesting properties in topology of fourmanifolds, in conformal field theory, in mathematical probability theory and in Cantorian spacetime as well as in the El Naschie's field theory.

The classification of the submanifolds is an important part of the application, while the geometry of the submanifolds is examined. By examining the geometry of the submanifolds, a lot of information about the main manifold can be obtained.

In this paper, we study totally umbilical semi-invariant submanifold of the Golden Riemannian manifolds. Also, we obtain integrability conditions of the distributions and investigate the geometry of foliations.

We hope that the current work contributes to motivate this research in both mathematics and physics.

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