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# Versatile Extension of the Unit Gompertz: Efficient Estimation and Application

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**Highlights**

- A novel extension of the unit Gompertz distribution is presented.
- Certain statistical characteristics of the new distribution are obtained.
- Six practical estimation techniques are considered for the estimation problem.
- Through simulation, several estimates are compared, and an actual data analysis is investigated.

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**Abstract**

Despite the availability of numerous statistical models for describing real-world data, the need remains for flexible distributions capable of accurately capturing diverse spread patterns, particularly within the unit interval. This study introduces the Kavya-Manoharan (KM)-unit Gompertz (KM-UGo) distribution, a novel model tailored for data confined to the unit interval. By combining the unit Gompertz distribution and the KM transformation, the KM-UGo distribution is an improved version of the existing unit-Gompertz distribution, offering more adaptability and the possibility of better model fit in a a wider range of data with diverse spread patterns. This enhances its ability to model various hazard rate shapes, including J-shaped, bathtub, increasing, inverted bathtub, and decreasing. The paper delves into the mathematical properties of the KM-UGo distribution, deriving key characteristics such as moments, probability-weighted moments, incomplete moments, residual and reversed residual life, quantile function, and entropy measures. Classical estimation techniques, including maximum likelihood, least squares, maximum product spacing, Cramér-von Mises, Anderson-Darling, and weighted least squares are employed to determine the distribution's parameters and the results are assessed using a Monte Carlo method. The study's findings showed that the maximum likelihood and maximum product spacing estimation methods offer more accurate and reliable parameter estimates. Furthermore, as demonstrated in simulation studies, larger sample sizes produce better parameter estimates, which are characterized by lower bias and higher accuracy. To illustrate its practical application, the KM-UGo distribution is applied to two real-world datasets residing within the unit interval.

## 1. INTRODUCTION

Many researchers have all through the last years conducted studies and offered different strategies for developing new distributions that start with the baseline distributions. Novel transformations have been applied to the continuous distribution to create new lifetime models. A major development in the field of data analysis and distribution transformations is the Kavya-Manoharan (KM) statistical transformation introduced by [1]. Its significance lies in its capacity to efficiently reshape data distributions, thus empowering researchers and analysts to derive deeper insights. The KM transformation is a valuable tool for creating new lifetime distributions without increasing the number of parameters. This is significant because additional parameters not only increases the variance but also increases the complexity and number of potential issues in estimating parameters. Also, this transformation can improve data fitting to match

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data more accurately than conventional models. This can lead to more accurate statistical inferences and improved decision-making.

The following are the definitions of the KM transformation's probability density function (PDF), denoted by $f(x;\Theta)$, and cumulative distribution function (CDF), denoted by $F(x;\Theta)$, as follows:

$$f(x;\Theta)=\frac{e}{e-1}g(x;\Theta)e^{-G(x;\Theta)};\quad x\in R, \tag{1}$$

and

$$F(x;\Theta)=\frac{e}{e-1}\left[1-e^{-G(x;\Theta)}\right];\ x\in R, \tag{2}$$

where $\Theta$ is the set of parameters. The $G(x;\Theta)$ is the CDF of the base-line distribution and $g(x;\Theta)$ is the PDF of the the base-line.

In several domains, modeling data sets constrained within the range (0, 1) has gained importance recently as a means of addressing product failure and survival rates. Therefore, due to its versatility when dealing with such probabilistic models, various unit distributions limited in the interval (0, 1) appear. Furthermore, there is an urgent demand for these sorts of distributions in several domains, including the medical, actuarial, and financial sciences. Previous research has shown the importance of flexible distributions. Most modern unit distributions are created by transforming existing distributions through suitably modified schemes. For example, if a random variable $X$ has a lifetime distribution, the transformations $U_1=\frac{1}{1+X}$, $U_2=\frac{X}{1+X}$, $U_3=\frac{X}{1-X}$, and $U_4=e^{-X}$, are popularly used to derive a distribution with support (0,1). Reference [2] proposed a transformation called Dinesh-Umesh-Sanjay (DUS) transformation. An exponentiated generalization of the DUS transformation called the power generalized DUS transformation was introduced by [3]. Then a new transformation called KM transformation was introduced. This study suggests a unit distribution based on the KM transformation as an alternate transformation to the well known transformations $U_1, U_2, U_3,$ and $U_4$.

In the literature, numerous scholars have therefore suggested unit distributions using the well known transformation mentioned earlier. Notable examples, the unit-Gompertz distribution and unit inverse power Lomax distribution.presented respectively by [4, 5] using the transformation $U_4=e^{-X}$. The one-parameter unit Lindley and unit power Lindely distributions were proposed, respectively, by [6, 7] using the tranformation $U_2=\frac{X}{1+X}$. The unit-Weibull (UW) distribution and the unit Burr XII distribution were proposed, respectively, using the transformation $U_4=e^{-X}$ by [8, 9]. Unit half-logistic geometric (UHLG) distribution and unit-exponentiated half-logistic distribution using the transformation $U_4=e^{-X}$, were presented respectively, by [10, 11]. Unit xgamma distribution using the transformation $U_2=\frac{X}{1+X}$ and unit Teissier distribution using the transformation $U_4=e^{-X}$ were provided, respectively, by [12, 13]. The unit exponentiated Lomax and the unit inverse exponentiated Weibull distributions, were suggested respectively, using the transformation $U_4=e^{-X}$, by [14, 15]. The power unit Burr XII and the unit power Burr X distributions, were introduced, respectively by [16, 17]. The unit exponential Pareto and Unit Gamma-Lindley distributions were proposed respectively by [18, 19], using the transformation $U_3=\frac{X}{1-X}$. Reference [20] introduced the unit power Lomax (UPL) distribution using the transformation $U_4=e^{-X}$. Other notable unit distributions have been proposed in the literature (see [21-27]).

Some newly modified distributions have been constructed based on the KM transformation.. Reference [28] introduced an improved Burr X distribution based on the KM transformation, with parameters estimated using ranked set sampling. The KM transformation was utilized to give an enhanced version of the log-logistic distribution by [29]. The KM generalized inverted Kumaraswamy distribution presented by [30]. The KM unit exponentiated half logistic distribution provided by [31].

We consider here the unit-Gompertz (UGo) distribution with (0, 1) as support. It can be viewed as an alternate model for reliability studies where distributions with finite support are owing to physical constraints like the design life of the system or a constrained power supply. The PDF and CDF, denoted by $g(y;\Theta)$, and $G(y;\Theta)$, respectively, $\Theta=(\mu,\eta)$, of the UGo distribution are as follows:

$$g(y;\Theta)=\mu\eta y^{-(\eta+1)}e^{-\mu(y^{-\eta}-1)};\;\; 0\le y\le 1;\;\; \mu,\eta>0, \tag{3}$$

and,

$$G(y;\Theta)=e^{-\mu(y^{-\eta}-1)}, \tag{4}$$

where $\mu$ and $\eta$ are two shape parameters. When modelling skewed data that is not adequately characterized by other widely used distributions, the UGo distribution is quite helpful.

This study aims to provide the Kavya-Manoharan unit-Gompertz (KM-UGo) distribution, a novel two-parameter distribution from the UGo distribution. The KM-UGo distribution offers potential improvements over the UGo distribution with the same number of parameters. This model is derived by merging the KM transformation with the UGo distribution, which builds on the advantages of both parent distributions. The KM-UGo distribution exhibits remarkable flexibility, characterized by a hazard rate function (HF) that can assume various shapes, including increasing, decreasing, bathtub, and upside-down bathtub. This versatility makes it adaptable to a wide range of real-world data scenarios. Due to its inherent flexibility, the KM-UGo distribution holds the potential to provide superior data fitting capabilities compared to the UGo distribution and other well-known distributions. Furthermore, the KM-UGo distribution possesses a tractable closed-form quantile function (QF). This valuable property facilitates the straightforward calculation of various statistical characteristics, such as quantiles, percentiles, and moments, and enables efficient random number generation. We are motivated to present the following:

a) To challenge the existing bounded distributions, a brand-new two-parameter distribution termed the KM-UGo distribution is proposed, which is specified on (0,1).
b) The density function exhibits several possible forms, such as symmetric, unimodal, reversed J-shaped, and left- and right-skewed. Furthermore, J-shaped, bathtub, up-side-down, decreasing, and increasing HF plots of the KM-UGo distribution are possible.
c) The statistical features that are derived include incomplete moments (IMs), QF, moments, probability-weighted moments (PWMs), residual and inverted residual lifetimes, and entropy measures.
d) The values of parameter estimates for the KM-UGo distribution are evaluated and compared using six standard estimation techniques. These techniques include the least squares (LS), the maximum product spacing (MPS), the Cramér-von Mises (CvM), the weighted LS (WLS), the Anderson-Darling (AD), and the maximum likelihood (ML).
e) To assess the accuracy of the various estimates, a simulation study is conducted. The utility of the KM-UGo distribution is assessed against several other models using two actual data sets.

This is the structure of the article: section 2 provides an overview of the formation of the KM-UGo distribution. Section 3 discusses the KM-UGo distribution's statistical properties. In section 4, the model parameter estimators utilizing different techniques of estimation are generated. To illustrate the findings in section 5, a simulation study is conducted. In section 6, two real data sets are used to demonstrate the importance of the KM-UGo distribution model. Section 7 provides the conclusion.

## 2. THE KM-UGO DISTRIBUTION

This section introduces the KM-UGo distribution, a new two-parameter model developed by applying the KM transformation to the UGo distribution. The CDF of the KM-UGo distribution is obtained by inserting Equations (3) and (4) in Equations (1) and (2), as below:

$$F(y;\Theta)=\frac{e}{e-1}\left[1-e^{-\delta(y;\Theta)}\right];\ 0<y<1;\ \mu,\eta>0, \tag{5}$$

where $\delta(y;\Theta)=e^{-\mu(y^{-\eta}-1)}$, $\Theta=(\mu,\eta)$ is the set of parameters, $\mu$ and $\eta$ are shape parameters. The PDF of the KM-UGo distribution is given by:

$$f(y;\Theta)=\frac{e}{e-1}\left[\mu\eta y^{-(\eta+1)}\delta(y;\Theta)e^{-\delta(y;\Theta)}\right];\ \ 0<y<1. \tag{6}$$

The survival function and the HF of the KM-UGo distribution, for $0<y<1$, are given, in that order, by

$$S(y;\Theta)=1-\frac{e}{e-1}\left[1-e^{-\delta(y;\Theta)}\right],$$

and,

$$h(y;\Theta)=\frac{e\mu\eta y^{-(\eta+1)}\delta(y;\Theta)\left(e^{-\delta(y;\Theta)}\right)}{e-1-e\left(1-e^{-\delta(y;\Theta)}\right)}.$$

Figure 1 represents the PDF plots of the KM-UGo distribution for selected parameter values (PVs). It demonstrates the symmetric, unimodal, reversed J-shaped, left-and right-skewed characteristics of the KM-UGo distribution. Also, for certain PVs have increasing, J-shaped, bathtub, declining, and inverted bathtub HF graphs of the KM-UGo distribution. The key gap between the UGo and the KM-UGo distribution lies in its flexibility and applicability. The UGo distribution has limitations in capturing diverse shapes of data distributions, particularly those with complex skewness patterns. Also, it might not be suitable for modeling data with certain hazard rate shapes, such as bathtub-shaped or increasing-decreasing patterns. By incorporating the KM transformation, the KM-UGo gains significant flexibility. This transformation allows it to model a wider range of data shapes more effectively**.**

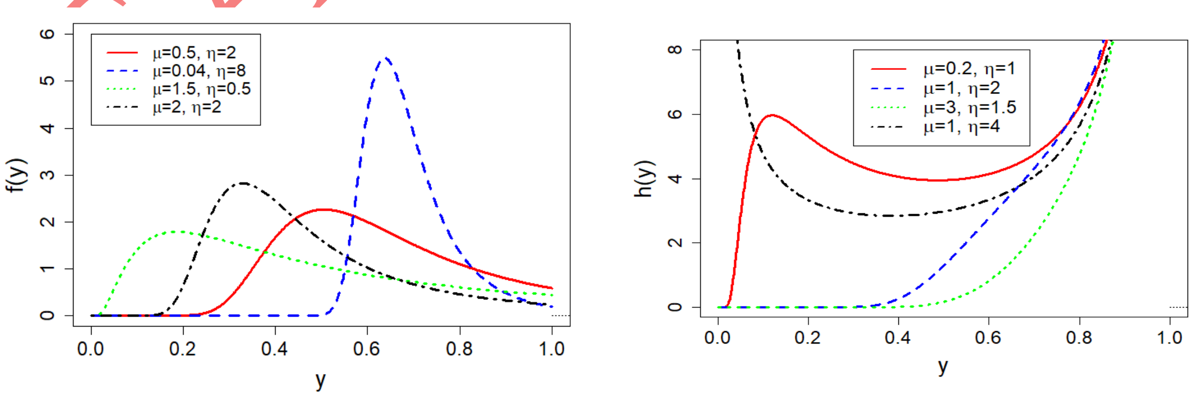


***Figure 1.*** *The KM-UGo distribution's PDF and HF*

The QF of a random variable $Y$ say $y = Q(q) = F^{-1}(q)$, where $q\sim$ uniform (0,1), is obtained as follows:

$$q=\frac{e}{e-1}\left[1-exp\left(-e^{-\mu\left(y^{-\eta}-1\right)}\right)\right].$$

Then, the QF of the KM-UGo distribution takes the following form

$$Q(q)=(\mu)^{1/\eta}\left[-\ln\left\{-\ln\left(1-q\left(\frac{e-1}{e}\right)\right)\right\}+\mu\right]^{-1/\eta}. \tag{7}$$

Setting $q$ = 0.25, 0.5 and 0.75 in Equation (7), we obtain, respectively, the first quartile ($Q_1$), the median ($Q_2$), and the third quartile ($Q_3$).

## 3. SOME STATISTICAL PROPERTIES

This section determines a number of statistical characteristics of the KM-UGo distribution, including moments, PWM, IMs, and moments of residual.

### 3.1. Moments Measures

The rth moment of the KM-UGo distribution is easily obtained from PDF (6) as follows:

$$\begin{aligned}E(Y^r)&=\int_0^1\frac{e}{e-1}\left[\mu\eta y^r y^{-(\eta+1)}\,\delta(y;\Theta)\,e^{-\delta(y;\Theta)}\right]dy\\&=\sum_{j,i=0}^{\infty}A_{i,j}(\mu)B\left(i+1,\frac{r}{\eta}-i-1\right),\end{aligned}$$

where B(.,.) is the beta function and $A_{i,j}(\mu)=\left(\frac{e}{e-1}\right)\frac{(-1)^{j+i}(j+1)^i(\mu)^{i+1}}{j!\;i!}$.

For some specified PVs, Table 1 lists numerical values for the mean $(\mu_1')$, variance $(\sigma^2)$, skewness $(\alpha_3)$, kurtosis $(\alpha_4)$, and the coefficient of variation (CV).

***Table 1.*** *Values of the KM-UGo distribution's moments*

| $\mu$ | $\eta$ | $\mu_1'$ | $\sigma^2$ | CV | $\alpha_3$ | $\alpha_4$ |
|---|---|---|---|---|---|---|
| 1 | | 0.646 | 0.031 | 0.048 | 0.092 | 2.13 |
| 1.5 | 1.5 | 0.715 | 0.025 | 0.035 | -0.144 | 2.191 |
| 2 | | 0.759 | 0.020 | 0.027 | -0.307 | 2.328 |
| 1 | | 0.716 | 0.022 | 0.031 | -0.032 | 2.169 |
| 1.5 | 2 | 0.774 | 0.017 | 0.022 | -0.251 | 2.295 |
| 2 | | 0.811 | 0.013 | 0.016 | -0.403 | 2.467 |
| 1 | | 0.796 | 0.013 | 0.016 | -0.163 | 2.259 |
| 1.5 | 3 | 0.84 | 0.009 | 0.011 | -0.365 | 2.442 |
| 2 | | 0.868 | 0.007 | 0.008 | -0.506 | 2.645 |

As illustrated in Table 1, the value of $\mu$ increases while maintaining the value of $\eta$ fixed, then the mean and kurtosis values rise while the values of other measures decline. It can be concluded that, as the value of $\eta$ rise while the value of $\mu$ remain fixed, the values of the mean and kurtosis rise, while the values of other measures decline. Additionally, the values of skewness indicate that the distribution is skewed to the left and right. Finally, according to values of $\alpha_4$, the KM-UGo distribution is platykurtic.

Furthermore, the rth IM of the KM-UGo distribution is obtained by using PDF (6) as follows:

$$\phi_r(x) = \frac{e}{e-1}\int_0^x \mu\eta\,\delta(y;\Theta)\,y^{r-(\eta+1)}\,e^{-\delta(y;\Theta)}dy$$
$$= \sum_{j,i=0}^{\infty} A_{i,j}(\mu)\,\mathrm{B}\left(i+1,\frac{r}{\eta}-i-1,x^{\eta}\right),$$

where $\mathrm{B}(.,.,y)$ is an incomplete beta function.

### 3.2. The PWM of the KM-UGo Distribution

Reference [32] originally proposed the PWM for the generalized distributions expressible in inverse form. The PWM of a random variable $Y$ for $s$ and $r$ are positive integers is defined by

$$\nu_{s,r} = \int_{-\infty}^{\infty} y^s\left[F(y)\right]^r f(y)\,dy. \tag{8}$$

Using PDF (6) and CDF (5) in Equation (8), the PWM of the KM-UGo distribution is produced in the following form,

$$\nu_{s,r} = \left(\frac{e}{e-1}\right)^r \int_0^1 \mu\eta\delta(y;\Theta)y^{s-(\eta+1)}\; e^{-\delta(y;\Theta)}\left[1-e^{-\delta(y;\Theta)}\right]^r dy$$
$$= \sum_{i=0}^{r} M_{j,i,k}\,\mathrm{B}\left(k+1,\frac{s}{\eta}-k-1\right),$$

where $M_{j,i,k} = \sum_{j,k=0}^{\infty}\binom{r}{i}\left(\frac{e}{e-1}\right)^r \frac{(-1)^{j+i+k}(i+1)^j(j+1)^k\mu^{k+1}}{j!k!}.$

### 3.3. Residual and Reversed Residual Life's

Residual life and reversed residual life are often used terms in risk analysis. Thus, among other relevant statistical functions, [33] investigated the survival function, mean, and variance. The residual life is the amount of time that elapses between time ($t$) and the time of failure of the conditional random variable. The following defines the rth moment of the residual life, let's say $I_r(t)$:

$$I_r(t) = \frac{1}{S(t;\Theta)}\int_t^{\infty}(y-t)^r f(y;\Theta)\,dy = \frac{1}{S(t;\Theta)}\sum_{n=0}^{r}\binom{r}{n}(-t)^{r-n}\int_t^{\infty} y^n f(y;\Theta)\,dy. \tag{9}$$

Additionally, by combining PDF (6) into Equation (9), the rth moment of residual life of the KM-UGo distribution can be obtained as follows:

$$I_r(t) = \frac{1}{S(t;\Theta)}\sum_{n=0}^{r}(-1)^{r-n}\binom{r}{n}t^{r-n}\,\frac{e}{e-1}\int_t^1 \mu\eta\,\delta(y;\Theta)y^{n-(\eta+1)}\,e^{-\delta(y;\Theta)}dy,$$

where, $S(t;\Theta)$ is the survival function. After some manipulation, the $I_r(t)$ takes the following form

$$I_r(t)=\frac{1}{S(t;\Theta)}\sum_{n=0}^{r} l_{i,j,n}\,\mathrm{B}\left(i+1,\ \frac{n}{\eta}-i-1,\ \left(1-t^{\eta}\right)\right),$$

where $l_{i,j,n}=(-1)^{r-n}\binom{r}{n}t^{r-n}\sum_{j,i=0}^{\infty}A_{i,j}(\mu).$

Further, the rth moment of reversed residual life of the KM-UGo distribution is derived as follows

$$\begin{aligned}\varepsilon_r(t)&=\frac{1}{F(t;\Theta)}\int_0^t (t-y)^r f(y;\Theta)\ dy\\&=\frac{1}{F(t;\Theta)}\sum_{n=0}^{r}(-1)^n\binom{r}{n}t^{r-n}\frac{e}{e-1}\int_0^t \mu\eta\delta(y;\Theta)y^{n-(\eta+1)}\ e^{-\delta(y;\Theta)}dy,\end{aligned}$$

which is the incomplete beta function, and takes the following form

$$\varepsilon_r(t)=\frac{1}{F(t;\Theta)}\sum_{n=0}^{r}U^{*}_{i,j,n}\,\mathrm{B}\left(\frac{n}{\eta}-i-1, i+1,\ t^{\eta}\right),$$

where $U^{*}_{i,j,n}=(-1)^{n}\binom{r}{n}t^{r-n}\sum_{j,i=0}^{\infty}A_{i,j}(\mu),$ and $\mathrm{B}(.,.,x)$ is the incomplete beta function.

### 3.4. Some Entropy Measures

For studies on reliability and risk assessment, entropy measures are crucial. It has been applied in a variety of biological applications in addition to those in the physical and medicinal fields. Entropy quantifies the variance of the uncertainty associated with the random variable $Y$ distribution. The Tsallis, Arimoto, Havrda and Charvát (HC), and Rényi entropies of the KM-UGo distribution are presented here. The following defines the Rényi entropy of the KM-UGo distribution

$$R_\gamma=\frac{1}{1-\gamma}\log\left[\int_{-\infty}^{\infty} f(y;\Theta)^{\gamma}dy\right];\qquad \gamma>0 \text{ and } \gamma\neq 1. \tag{10}$$

The Rényi entropy of the KM-UGo distribution is obtained by using PDF (6) in Equation (10) as follows:

$$R_\gamma=\frac{1}{1-\gamma}\log\left[\int_0^1\left(\frac{e}{e-1}\right)^{\gamma}\ \mu^{\gamma}\eta^{\gamma}y^{-\gamma(\eta+1)}e^{-\mu\gamma(y^{-\eta}-1)}\,dy\right].$$

Thus, the KM-UGo distribution's Rényi entropy has the following structure

$$R_\gamma=\frac{1}{1-\gamma}\log\left[\sum_{j,i=0}^{\infty}D_{j,i}(\Theta,\gamma)\ \mathrm{B}\left(i+1,\frac{\gamma(\eta+1)}{\eta}-i-1\right)\right],$$

where $D_{j,i}(\Theta,\gamma)=\left(\frac{e}{e-1}\right)^{\gamma}\frac{(-1)^{j+i}\mu^{\gamma+i}\eta^{\gamma-1}\gamma^{j}(\gamma+j)^{i}}{j!i!}.$

As stated by [34], the HC measure is a helpful expanded measure for Shannon's entropy. The HC of the KM-UGo distribution is given from PDF (6) as shown below.

$$HC_{\gamma}=\frac{1}{2^{1-\gamma}-1}\left[\sum_{j,i=0}^{\infty}D_{j,i}(\Theta,\gamma)\,\mathrm{B}\left(i+1,\frac{\gamma(\eta+1)}{\eta}-i-1\right)-1\right].$$

Reference [35] proposed an extension of Shannon's entropy. The following formula can be used for calculating the KM-UGo distribution's Tsallis entropy from PDF (6) utilizing the technique that was previously mentioned

$$T_{\gamma}=\frac{1}{\gamma-1}\left[1-\sum_{j,i=0}^{\infty}D_{j,i}(\Theta,\gamma)\,\mathrm{B}\left(i+1,\frac{\gamma(\eta+1)}{\eta}-i-1\right)\right].$$

An alternative for the Shannon entropy measure, the Arimoto's entropy was introduced by [36] and has comparable characteristics. Using PDF (6), which follows is a method to get Arimoto's entropy of the KM-UGo distribution.

$$A_{\gamma}=\frac{\gamma}{1-\gamma}\left[\left[\sum_{j,i=0}^{\infty}D_{j,i}(\Theta,\gamma)\,\mathrm{B}\left(i+1,\frac{\gamma(\eta+1)}{\eta}-i-1\right)\right]^{\frac{1}{\gamma}}-1\right].$$

Table 2 shows certain predefined PVs for different entropy measurements of the KM-UGo distribution.

***Table 2.*** *Some of the KM-UGo distribution's entropy measurements*

| $\gamma$ | $\eta$ | $\mu$ | $R_{\gamma}$ | $T_{\gamma}$ | $HC_{\gamma}$ | $A_{\gamma}$ |
|---|---|---|---|---|---|---|
| 0.1 | 0.5 | 1 | -0.0277 | -0.0273 | 0.0529 | 0.0245 |
| | 2 | 2 | -0.4896 | -0.3959 | 0.7679 | 0.1098 |
| | 5 | 3 | -1.3062 | -0.7681 | 1.4896 | 0.1111 |
| | 7 | 4 | -1.7315 | -0.8772 | 1.7011 | 0.1111 |
| | 9 | 5 | -2.0751 | -0.9394 | 1.8218 | 0.1111 |
| | 11 | 6 | -2.3632 | -0.9787 | 1.8978 | 0.1111 |
| 0.5 | 0.5 | 1 | -0.1239 | -0.1202 | 0.2052 | 0.1166 |
| | 2 | 2 | -0.7355 | -0.6154 | 1.0505 | 0.5207 |
| | 5 | 3 | -1.7244 | -1.1555 | 1.9726 | 0.8217 |
| | 7 | 4 | -2.2266 | -1.3431 | 2.2927 | 0.8921 |
| | 9 | 5 | -2.6259 | -1.4619 | 2.4957 | 0.9276 |
| | 11 | 6 | -2.9572 | -1.5441 | 2.6359 | 0.948 |
| 0.9 | 0.5 | 1 | -0.2119 | -0.2097 | 0.3131 | 0.2094 |
| | 2 | 2 | -0.8144 | -0.7821 | 1.1679 | 0.7786 |
| | 5 | 3 | -1.8538 | -1.6921 | 2.5268 | 1.6754 |
| | 7 | 4 | -2.379 | -2.1172 | 3.1616 | 2.0906 |
| | 9 | 5 | -2.7944 | -2.4379 | 3.6405 | 2.4022 |
| | 11 | 6 | -3.1377 | -2.6931 | 4.0216 | 2.6492 |

The values of Table 2 indicate that $T_{\gamma}$ and $R_{\gamma}$ fall while the values of the other measures rise when the values of $\gamma$ while maintaining $\eta$ and $\mu$ fixed. Thus, it follows that when $\eta$ and $\mu$ increase for fixed values of $\gamma$, $T_{\gamma}$ and $R_{\gamma}$ values decrease and the values of the other measures increase.

## 4. PARAMETER ESTIMATION OF KM-UGO DISTRIBUTION

This section investigates six different parameter estimation techniques that are utilized for the KM-UGo distribution model. The examined techniques are the ML, LS, CvM, WLS, AD, and MPS. For estimating the model parameters of the KM-UGo distribution, each of these techniques offers a unique strategy. The analysis will include an evaluation and comparison of various strategies' performances.

### 4.1. Maximum Likelihood Estimator

Using the ML approach, the unknown parameters of the KM-UGo distribution are estimated here. Assuming a random sample $y_1, y_2 \ldots y_m$ of size $m$ from the KM-UGo distribution, the log-likelihood function, indicated by $\ln L$, is given by

$$\ln L = m\ln(\frac{e}{e-1}) + m\ln(\mu) + m\ln(\eta) - (\eta+1)\sum_{r=1}^{m}\ln y_r - \mu\sum_{r=1}^{m}(y_r^{-\eta}-1) - \sum_{r=1}^{m}\delta(y_r;\Theta).$$

Equating the following non-linear equations with zero and solving them via package optim using R program yields the ML estimators of the parameters $\mu$ and $\eta$

$$\frac{\partial \ln L}{\partial \mu} = \frac{m}{\mu} - \sum_{r=1}^{m}(y_r^{-\eta}-1)(1-\delta(y_r;\Theta)) = 0,$$

and

$$\frac{\partial \ln L}{\partial \eta} = \frac{m}{\eta} - \sum_{r=1}^{m}\ln(y_r) + \sum_{r=1}^{m}\mu y_r^{-\eta}\ln(y_r)(1+\delta(y_r;\Theta)) = 0,$$

where, $\delta(y_r;\Theta) = e^{-\mu(y_r{}^{-\eta}-1)}$.

### 4.2. Least Squares and Weighted Least Squares

Let $y_1, y_2, \ldots, y_m$ be a random sample of size $m$ from the KM-UGo distribution. Suppose that $y_{(1)} < y_{(2)} < \ldots < y_{(m)}$ denotes the corresponding ordered sample. The LS and WLS estimators of unknown parameters of the KM-UGo distribution are obtained by minimizing the error of sum squares.

$$L^*(\Theta) = \sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y(r);\Theta)}\right] - \frac{r}{m+1}\right]^2. \tag{11}$$

Equating the following non-linear equations with zero and solving them via package optim using R program yields the LS estimators of the parameters $\mu$, and $\eta$

$$\frac{\partial L^*(\Theta)}{\partial \mu} = \sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y(r);\Theta)}\right] - \frac{r}{m+1}\right]\Xi_1(y_{(r)};\Theta) = 0,$$

and,

$$\frac{\partial L^*(\Theta)}{\partial \eta} = \sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y(r);\Theta)}\right] - \frac{r}{m+1}\right]\Xi_2(y_{(r)};\Theta) = 0,$$

where

$$\left.\begin{aligned} \Xi_1(y_{(r)};\Theta) &= \left(\frac{e}{e-1}\right)\delta(y_{(r)};\Theta)(y_{(r)}^{-\eta}-1), \\ \Xi_2(y_{(r)};\Theta) &= -\left(\frac{e}{e-1}\right)\mu y_{(r)}^{-\eta}(\ln y_{(r)})\,\delta(y_{(r)};\Theta) \end{aligned}\right\}. \tag{12}$$

Similar to the LS estimators, the following function is minimized to yield the WLS estimators of $\mu$ and $\eta$

$$WL(\Theta)=\sum_{r=1}^{m}\frac{(m+2)(m+1)^2}{r(m-r+1)}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{r}{m+1}\right]^2.$$

The following non-linear equations are solved by employing an iterative method, yielding the WLS estimators of $\mu$, and $\eta$

$$\frac{\partial WL(\Theta)}{\partial\mu}=\sum_{r=1}^{m}\frac{(m+2)(m+1)^2}{r(m-r+1)}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{r}{m+1}\right]\Xi_1(y_{(r)};\Theta)=0,$$

and

$$\frac{\partial WL(\Theta)}{\partial\eta}=\sum_{r=1}^{m}\frac{(m+2)(m+1)^2}{r(m-r+1)}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{r}{m+1}\right]\Xi_2(y_{(r)};\Theta)=0,$$

where $\Xi_1(y_{(r)};\Theta)$ and $\Xi_2(y_{(r)};\Theta)$ are defined in Equation (12).

### 4.3. Anderson-Dalring Estimators

Minimizing the following function yields the CvM estimators of the given parameters $\Theta=(\mu,\eta)^T$

$$C(\Theta)=\frac{1}{12m}+\sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{2r-1}{2m}\right]^2.$$

The following non-linear equations are solved for zero by employing an iterative method, yielding the CvM estimators of the parameters $\mu,\eta$, using R program.

$$\frac{\partial C(\Theta)}{\partial\mu}=\sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{2r-1}{2m}\right]\Xi_1(y_{(r)};\Theta)=0,$$

and,

$$\frac{\partial C(\Theta)}{\partial\eta}=\sum_{r=1}^{m}\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]-\frac{2r-1}{2m}\right]\Xi_2(y_{(r)};\Theta)=0,$$

where $\Xi_1(y_{(r)};\Theta)$ and $\Xi_2(y_{(r)};\Theta)$ are defined in equation (12).

Similarly, minimizing the following function provides the AD estimators of the given parameters $\Theta=(\mu,\eta)^T$,

$$A^{\bullet}(\Theta)=-m-\sum_{r=1}^{m}\frac{2r-1}{m}\log\left[\left[\frac{e}{e-1}\left[1-e^{-\delta(y_{(r)};\Theta)}\right]\right]+\log\left[1-\frac{e}{e-1}\left[1-e^{-\delta(y_{(m+1-r)};\Theta)}\right]\right]\right].$$

The following non-linear equations are solved by employing an iterative method, yielding the ADEs of the parameters $\mu$ and $\eta$

$$\frac{\partial A^{\bullet}(\Theta)}{\partial\mu}=\sum_{r=1}^{m}\frac{2r-1}{m}\left[\frac{\Xi_1\left(y_{(r)};\Theta\right)}{F\left(y_{(r)};\Theta\right)}-\frac{\Xi_1\left(y_{(r+m-1)};\Theta\right)}{S\left(y_{(r+m-1)};\Theta\right)}\right]=0,$$

and,

$$\frac{\partial A^{\bullet}(\Theta)}{\partial\eta}=\sum_{r=1}^{m}\frac{2r-1}{m}\left[\frac{\Xi_2\left(y_{(r)};\Theta\right)}{F\left(y_{(r)};\Theta\right)}-\frac{\Xi_2\left(y_{(r+m-1)};\Theta\right)}{S\left(y_{(r+m-1)};\Theta\right)}\right]=0,$$

where $\Xi_1(y_{(m+1-r)};\Theta)$ and $\Xi_2(y_{(m+1-r)};\Theta)$ are defined in Equation (12) by replacing ($r$) with $(m+1-r)$
.

### 4.4. Maximum Product of Spacings

The ML approach can be replaced by the MPS method, which approaches the Kullback-Leibler information metric. Although ML estimation is the most popular and extensively used approach, it does not work well in some situations involving big samples and complex continuous distributions. Reference [37] introduced the MPS approach as a substitute for the ML method. Numerous disciplines have used MPS, including econometrics, hydrology, statistics, pure mathematics, and magnetic resonance imaging. The distance between the CDF values at subsequent data points serves as the foundation for the MPS approach. Let $Y_{(1)} < Y_{(2)} < \ldots < Y_{(m)}$ be the ordered statistics from the distribution with sample size $m$, and $y_{(1)} < y_{(2)} < \ldots < y_{(m)}$ be the ordered observed values. The uniform spacings can be defined as follows, based on a random sample of size $m$ from the KM-UGo distribution

$$D_r(\Theta) = F\left(y_{(r)}\middle|\Theta\right) - F\left(y_{(r-1)}\middle|\Theta\right),\ \ r = 1,2,\ldots,m+1,$$

where $F\left(y_{(0)}\middle|\Theta\right) = 0$, $F\left(y_{(m+1)}\middle|\Theta\right) = 1$ and $\sum_{r=1}^{m+1} D_r(\Theta) = 1$.

The MPS estimator for the KM-UGo distribution is given by maximizing the geometric mean of the spacings

$$S^*(\Theta) = \frac{1}{1+m}\sum_{r=1}^{m+1} \ln D_r(\Theta) = \frac{1}{1+m}\sum_{r=1}^{m+1} \ln\left[\frac{e}{e-1}\left[\left(1-e^{-\delta(y_{(r)};\Theta)}\right)-\left(1-e^{-\delta(y_{(r-1)};\Theta)}\right)\right]\right].$$

The following non-linear equations are solved for zero by employing an iterative method, yielding the MPS estimator of the parameters $\mu$ and $\eta$

$$\frac{\partial S^*(\Theta)}{\partial \mu} = \frac{1}{1+m}\sum_{r=1}^{m+1}\left[\frac{\left(\Xi_1(y_{(r)};\Theta)+\Xi_1(y_{(r-1)};\Theta)\right)}{D_r(\Theta)}\right] = 0,$$

and

$$\frac{\partial S^*(\Theta)}{\partial \eta} = \frac{1}{1+m}\sum_{r=1}^{m+1}\left[\frac{\left(\Xi_1(y_{(r)};\Theta)+\Xi_1(y_{(r-1)};\Theta)\right)}{D_r(\Theta)}\right] = 0,$$

where $\Xi_1(y_{(r-1)};\Theta)$ and $\Xi_2(y_{(r-1)};\Theta)$ are defined in Equation (12) by replacing ($r$) with ($r$–1).

## 5. NUMERICAL STUDY

The performance of various estimates was evaluated and compared in this section, using a numerical analysis concerning their relative absolute biases (RAB), chosen PVs, and mean squared errors (MSEs) for various sample sizes. The numerical procedures are described through the following steps.

**Step 1:** Generate a random sample from the KM-UGo distribution by using the inverse transformation (7) with sample sizes $m$ = (50, 75, 100, 125 and 150).

**Step 2:** Some PVs are selected as below.

**Set 1=** $(\eta = 1, \mu = 0.1)$ **Set 2=** $(\eta = 1, \mu = 0.5)$

**Set 3 =** $(\eta = 1.5, \mu = 0.3)$ **Set 4=** $(\eta = 1.5, \mu = 0.7)$

**Set 5**= $(\eta = 2, \mu = 0.8)$ **Set 6**= $(\eta = 2, \mu = 1)$
**Set 7**= $(\eta = 2.5, \mu = 0.6)$ **Set 8**= $(\eta = 2.5, \mu = 0.9)$

**Step 3:** Obtain the parameter estimates of $\eta$ and $\mu$ using the provided estimation methods for the selected sample sizes.

**Step 4:** The first three steps are 1000 times repeated for every sample size and chosen PV. Afterwards, the MSEs and RABs of various estimates of of $\eta$ and $\mu$ are calculated. The MSEs and RABs have the following formulas

$$RAB(\Theta) = \frac{1}{1000}\sum_{k=1}^{1000}\left|\frac{\hat{\Theta}_k - \Theta_k}{\Theta_k}\right|, \quad MSE(\Theta) = \frac{1}{1000}\sum_{k=1}^{1000}\left(\hat{\Theta}_k - \Theta_k\right)^2.$$

**Step 5:** The numerical results of the simulation study are listed in Table 3.
The findings obtained regarding the behavior of the estimated parameters from the KM-UGo distribution are as follows:

- **i.** The RABs of all estimates decrease with increasing sample sizes based on different estimation techniques (see Table (3)).
- **ii.** The MSEs for the $\mu$ and $\eta$ estimate increase as the value of the parameter the $\mu$ increases for all estimation methods.
- **iii.** The MSEs for the $\mu$ and $\eta$ estimate rise as the PVs of $\eta$ rise for all methods of estimation.
- **iv.** The MSEs of all estimates based on different methods decline as the sample size increases for different selected PVs (see Figures 2 and 3).
- **v.** The maximum likelihood estimate (MLE) and maximum product estimate (MPE) of $\mu$ have the smallest MSE compared to other estimates at values of set 5 and set 8 (Figure 4).
- **vi.** The MLE and ADE of $\eta$ are the best method among all other methods at values of set 6 and set 7 (Figure 5).
- **vii.** When compared to other approaches assessed, the MLE and MPSE procedures typically yield more precise and reliable parameter estimates. Moreover, regardless of the estimation technique employed, simulation studies consistently demonstrate that larger sample sizes result in improved parameter estimates, characterized by reduced bias and increased precision.
- **viii.** The study discovered that, regardless of the estimation method employed, the RABs of all parameter estimates reduced by increasing sample size, which is in line with statistical theory. This illustrates the anticipated pattern, which states that larger datasets typically result in parameter estimates that are more accurate and less RABs.

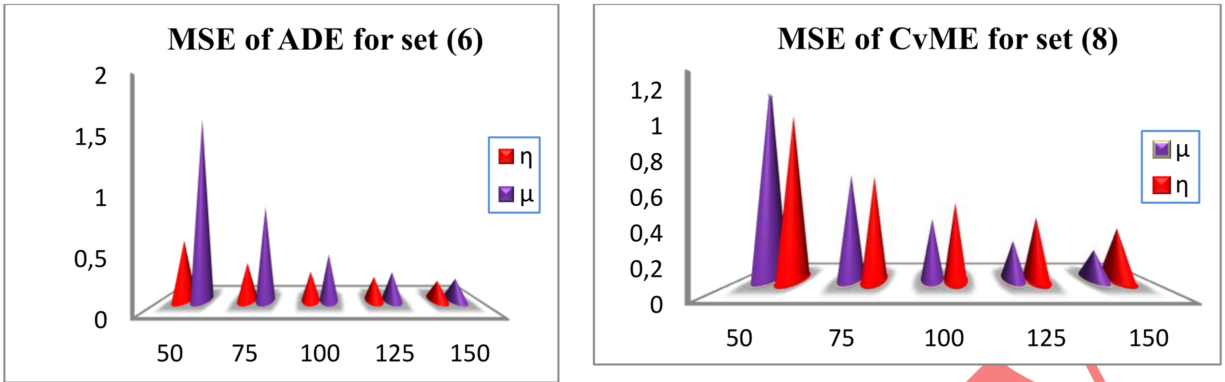


***Figure 2.*** *The MSEs of the AD estimates and Cramér-von Mises estimates of the KM-UGo distribution for all m values*

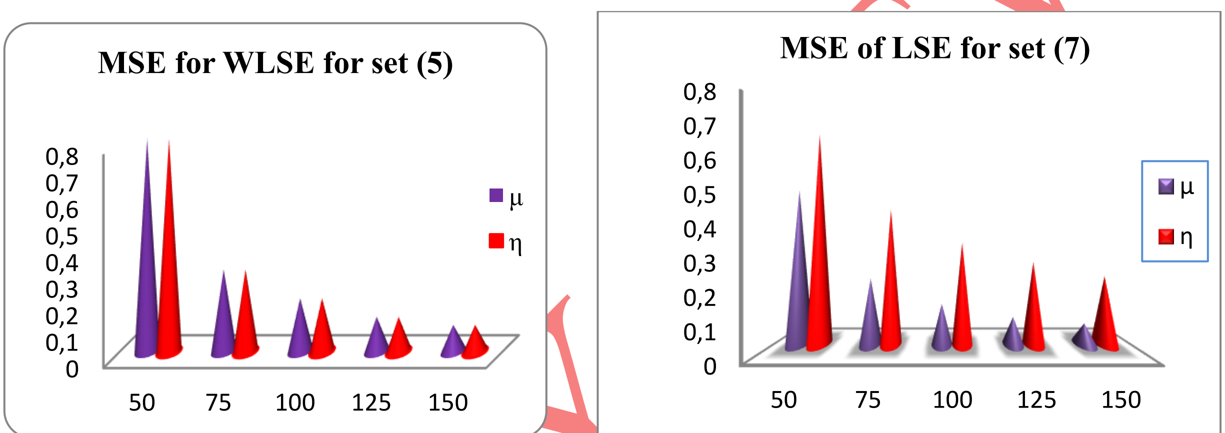


***Figure 3.*** *The MSEs of the LS estimates and WLS estimates of KM-UGo distribution for all m values*

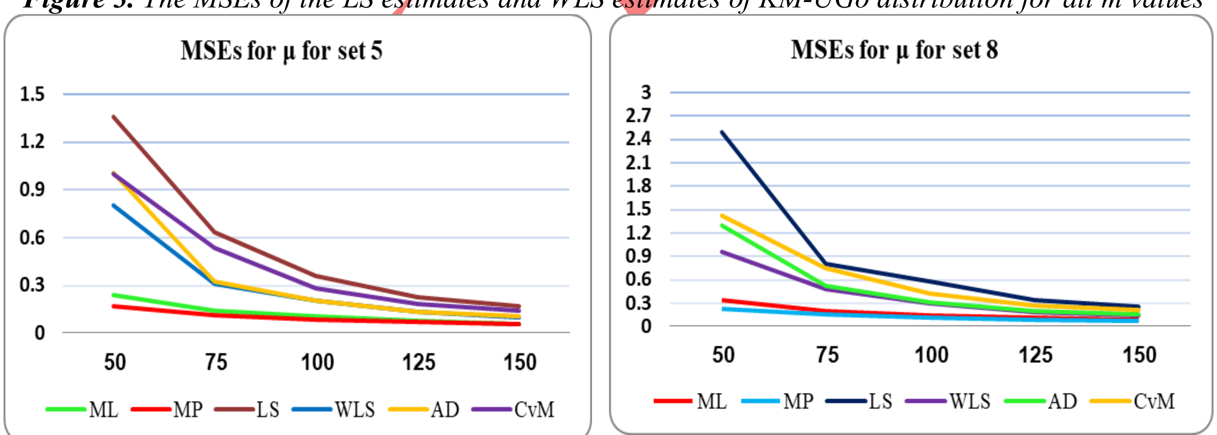


***Figure 4.*** *The MSEs of different $\mu$ estimates of KM-UGo distribution for all m values*

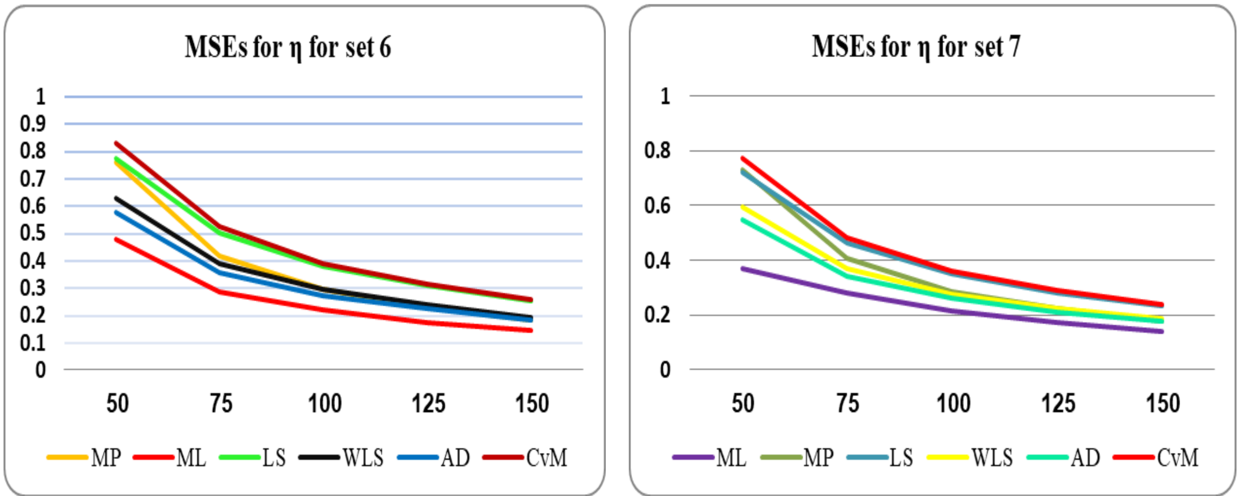


***Figure 5.*** *The MSEs for different* $\eta$ *estimates of KM-UGo distribution for all m values*

***Table 3.*** *MSEs and RABs of the KM-UGo estimates*

| PVs | | | *m* | ML | | MPS | | LS | | WLS | | AD | | CvM | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu$ | $\eta$ | | | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB |
| 0.1 | 1 | $\mu$ | 50 | 0.003 | 0.005 | 0.002 | 0.165 | 0.006 | 0.209 | 0.004 | 0.137 | 0.004 | 0.123 | 0.005 | 0.089 |
| | | $\eta$ | | 0.028 | 0.045 | 0.043 | 0.17 | 0.038 | 0.006 | 0.032 | 0.009 | 0.03 | 0.013 | 0.041 | 0.0345 |
| | | $\mu$ | 75 | 0.002 | 0.007 | 0.002 | 0.119 | 0.003 | 0.138 | 0.002 | 0.086 | 0.002 | 0.082 | 0.003 | 0.061 |
| | | $\eta$ | | 0.017 | 0.028 | 0.024 | 0.079 | 0.024 | 0.004 | 0.019 | 0.007 | 0.019 | 0.007 | 0.025 | 0.022 |
| | | $\mu$ | 100 | 0.001 | 0.007 | 0.001 | 0.097 | 0.002 | 0.107 | 0.0018 | 0.066 | 0.0018 | 0.064 | 0.002 | 0.05 |
| | | $\eta$ | | 0.013 | 0.022 | 0.017 | 0.062 | 0.018 | 0.004 | 0.015 | 0.005 | 0.014 | 0.005 | 0.019 | 0.016 |
| | | $\mu$ | 125 | 0.001 | 0.002 | 0.001 | 0.086 | 0.0017 | 0.081 | 0.001 | 0.046 | 0.001 | 0.047 | 0.0015 | 0.036 |
| | | $\eta$ | | 0.01 | 0.018 | 0.013 | 0.052 | 0.014 | 0.002 | 0.012 | 0.006 | 0.012 | 0.005 | 0.015 | 0.014 |
| | | $\mu$ | 150 | 0.0008 | 0.0005 | 0.0008 | 0.076 | 0.0014 | 0.066 | 0.001 | 0.039 | 0.001 | 0.039 | 0.001 | 0.029 |
| | | $\eta$ | | 0.008 | 0.015 | 0.011 | 0.044 | 0.012 | 0.002 | 0.01 | 0.005 | 0.009 | 0.004 | 0.012 | 0.011 |
| 0.5 | | $\mu$ | 50 | 0.072 | 0.006 | 0.056 | 0.161 | 0.291 | 0.31 | 0.171 | 0.193 | 0.171 | 0.182 | 0.261 | 0.171 |
| | | $\eta$ | | 0.065 | 0.076 | 0.101 | 0.185 | 0.098 | 0.012 | 0.081 | 0.013 | 0.076 | 0.018 | 0.105 | 0.053 |
| | | $\mu$ | 75 | 0.045 | 0.009 | 0.038 | 0.115 | 0.124 | 0.192 | 0.077 | 0.115 | 0.078 | 0.112 | 0.096 | 0.102 |
| | | $\eta$ | | 0.039 | 0.047 | 0.056 | 0.125 | 0.063 | 0.01 | 0.051 | 0.009 | 0.047 | 0.009 | 0.066 | 0.033 |
| | | $\mu$ | 100 | 0.035 | 0.009 | 0.03 | 0.093 | 0.082 | 0.146 | 0.056 | 0.086 | 0.056 | 0.087 | 0.068 | 0.082 |
| | | $\eta$ | | 0.029 | 0.036 | 0.039 | 0.098 | 0.048 | 0.009 | 0.038 | 0.007 | 0.036 | 0.006 | 0.049 | 0.023 |
| | | $\mu$ | 125 | 0.026 | 0.004 | 0.024 | 0.083 | 0.057 | 0.108 | 0.039 | 0.061 | 0.039 | 0.064 | 0.049 | 0.059 |
| | | $\eta$ | | 0.024 | 0.03 | 0.031 | 0.083 | 0.038 | 0.005 | 0.031 | 0.009 | 0.029 | 0.007 | 0.039 | 0.021 |
| | | $\mu$ | 150 | 0.02 | 0.0009 | 0.019 | 0.075 | 0.044 | 0.087 | 0.03 | 0.048 | 0.031 | 0.052 | 0.039 | 0.047 |
| | | $\eta$ | | 0.019 | 0.025 | 0.025 | 0.071 | 0.032 | 0.004 | 0.025 | 0.008 | 0.024 | 0.005 | 0.033 | 0.017 |
| 0.3 | 1.5 | $\mu$ | 50 | 0.023 | 0.002 | 0.019 | 0.159 | 0.068 | 0.243 | 0.043 | 0.152 | 0.043 | 0.141 | 0.049 | 0.114 |
| | | $\eta$ | | 0.103 | 0.061 | 0.159 | 0.153 | 0.149 | 0.009 | 0.125 | 0.011 | 0.118 | 0.015 | 0.159 | 0.044 |
| | | $\mu$ | 75 | 0.015 | 0.006 | 0.013 | 0.114 | 0.032 | 0.154 | 0.023 | 0.094 | 0.023 | 0.091 | 0.026 | 0.075 |
| | | $\eta$ | | 0.063 | 0.038 | 0.089 | 0.104 | 0.096 | 0.008 | 0.078 | 0.008 | 0.073 | 0.008 | 0.1 | 0.028 |
| | | $\mu$ | 100 | 0.012 | 0.006 | 0.01 | 0.092 | 0.023 | 0.119 | 0.017 | 0.072 | 0.017 | 0.071 | 0.019 | 0.06 |
| | | $\eta$ | | 0.048 | 0.029 | 0.063 | 0.081 | 0.072 | 0.007 | 0.059 | 0.006 | 0.056 | 0.005 | 0.074 | 0.019 |
| | | $\mu$ | 125 | 0.009 | 0.002 | 0.008 | 0.082 | 0.016 | 0.089 | 0.012 | 0.051 | 0.012 | 0.052 | 0.015 | 0.044 |
| | | $\eta$ | | 0.038 | 0.024 | 0.049 | 0.068 | 0.058 | 0.004 | 0.047 | 0.007 | 0.045 | 0.006 | 0.059 | 0.017 |
| | | $\mu$ | 150 | 0.007 | 0.0004 | 0.006 | 0.073 | 0.013 | 0.072 | 0.009 | 0.039 | 0.009 | 0.043 | 0.012 | 0.036 |
| | | $\eta$ | | 0.031 | 0.02 | 0.039 | 0.058 | 0.048 | 0.003 | 0.039 | 0.006 | 0.037 | 0.005 | 0.049 | 0.014 |
| 0.7 | | $\mu$ | 50 | 0.165 | 0.015 | 0.119 | 0.163 | 1.02 | 0.407 | 0.669 | 0.259 | 0.659 | 0.251 | 0.591 | 0.216 |

| PVs | | | *m* | ML | | MPS | | LS | | WLS | | AD | | CvM | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| ***μ*** | ***η*** | | | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB |
| | | *η* | | 0.193 | 0.089 | 0.302 | 0.216 | 0.301 | 0.014 | 0.247 | 0.016 | 0.228 | 0.019 | 0.322 | 0.062 |
| | | *μ* | 75 | 0.101 | 0.015 | 0.081 | 0.118 | 0.421 | 0.249 | 0.199 | 0.142 | 0.205 | 0.141 | 0.289 | 0.141 |
| | | *η* | | 0.116 | 0.055 | 0.168 | 0.146 | 0.193 | 0.012 | 0.153 | 0.011 | 0.141 | 0.009 | 0.202 | 0.038 |
| | | *μ* | 100 | 0.077 | 0.013 | 0.063 | 0.096 | 0.225 | 0.182 | 0.136 | 0.106 | 0.138 | 0.107 | 0.179 | 0.109 |
| | | *η* | | 0.089 | 0.043 | 0.119 | 0.114 | 0.146 | 0.011 | 0.115 | 0.008 | 0.108 | 0.006 | 0.15 | 0.026 |
| | | *μ* | 125 | 0.057 | 0.007 | 0.049 | 0.086 | 0.147 | 0.134 | 0.091 | 0.075 | 0.094 | 0.078 | 0.123 | 0.079 |
| | | *η* | | 0.071 | 0.036 | 0.092 | 0.097 | 0.118 | 0.006 | 0.093 | 0.01 | 0.088 | 0.008 | 0.122 | 0.024 |
| | | *μ* | 150 | 0.044 | 0.003 | 0.039 | 0.078 | 0.11 | 0.107 | 0.07 | 0.058 | 0.073 | 0.063 | 0.096 | 0.062 |
| | | *η* | | 0.058 | 0.03 | 0.074 | 0.083 | 0.098 | 0.005 | 0.077 | 0.009 | 0.073 | 0.006 | 0.099 | 0.019 |
| 0.8 | 2 | *μ* | 50 | 0.238 | 0.021 | 0.165 | 0.165 | 1.358 | 0.435 | 0.8 | 0.266 | 1.005 | 0.278 | 0.997 | 0.252 |
| | | *η* | | 0.386 | 0.096 | 0.607 | 0.231 | 0.611 | 0.015 | 0.499 | 0.017 | 0.459 | 0.021 | 0.655 | 0.066 |
| | | *μ* | 75 | 0.142 | 0.019 | 0.111 | 0.119 | 0.632 | 0.273 | 0.309 | 0.158 | 0.324 | 0.158 | 0.533 | 0.168 |
| | | *η* | | 0.232 | 0.059 | 0.337 | 0.156 | 0.394 | 0.013 | 0.31 | 0.012 | 0.285 | 0.009 | 0.411 | 0.041 |
| | | *μ* | 100 | 0.107 | 0.015 | 0.086 | 0.097 | 0.359 | 0.205 | 0.202 | 0.117 | 0.206 | 0.119 | 0.278 | 0.125 |
| | | *η* | | 0.177 | 0.046 | 0.238 | 0.122 | 0.298 | 0.012 | 0.233 | 0.009 | 0.218 | 0.007 | 0.306 | 0.028 |
| | | *μ* | 125 | 0.078 | 0.009 | 0.067 | 0.087 | 0.224 | 0.149 | 0.131 | 0.082 | 0.136 | 0.086 | 0.185 | 0.089 |
| | | *η* | | 0.142 | 0.038 | 0.186 | 0.103 | 0.241 | 0.007 | 0.189 | 0.011 | 0.177 | 0.008 | 0.248 | 0.025 |
| | | *μ* | 150 | 0.059 | 0.004 | 0.054 | 0.079 | 0.167 | 0.119 | 0.101 | 0.064 | 0.105 | 0.069 | 0.143 | 0.071 |
| | | *η* | | 0.115 | 0.032 | 0.148 | 0.089 | 0.199 | 0.006 | 0.155 | 0.009 | 0.167 | 0.006 | 0.204 | 0.021 |
| 1 | | *μ* | 50 | 0.467 | 0.035 | 0.287 | 0.166 | 3.717 | 0.558 | 1.69 | 0.332 | 1.759 | 0.313 | 1.862 | 0.304 |
| | | *η* | | 0.479 | 0.109 | 0.759 | 0.26 | 0.771 | 0.017 | 0.629 | 0.02 | 0.575 | 0.023 | 0.831 | 0.076 |
| | | *μ* | 75 | 0.262 | 0.027 | 0.192 | 0.121 | 1.367 | 0.339 | 0.779 | 0.202 | 0.904 | 0.206 | 0.841 | 0.196 |
| | | *η* | | 0.288 | 0.068 | 0.419 | 0.175 | 0.5 | 0.015 | 0.391 | 0.013 | 0.357 | 0.011 | 0.523 | 0.046 |
| | | *μ* | 100 | 0.192 | 0.022 | 0.147 | 0.099 | 0.955 | 0.265 | 0.431 | 0.144 | 0.442 | 0.148 | 0.665 | 0.167 |
| | | *η* | | 0.219 | 0.053 | 0.296 | 0.138 | 0.379 | 0.014 | 0.294 | 0.009 | 0.273 | 0.007 | 0.39 | 0.032 |
| | | *μ* | 125 | 0.138 | 0.014 | 0.115 | 0.089 | 0.506 | 0.186 | 0.252 | 0.099 | 0.268 | 0.105 | 0.399 | 0.117 |
| | | *η* | | 0.175 | 0.044 | 0.231 | 0.117 | 0.308 | 0.008 | 0.238 | 0.012 | 0.223 | 0.007 | 0.316 | 0.029 |
| | | *μ* | 150 | 0.104 | 0.007 | 0.092 | 0.082 | 0.364 | 0.146 | 0.19 | 0.077 | 0.204 | 0.084 | 0.3 | 0.092 |
| | | *η* | | 0.143 | 0.037 | 0.184 | 0.1004 | 0.254 | 0.006 | 0.194 | 0.011 | 0.184 | 0.006 | 0.259 | 0.023 |
| 0.6 | 2.5 | *μ* | 50 | 0.111 | 0.01 | 0.084 | 0.163 | 0.53 | 0.351 | 0.323 | 0.221 | 0.323 | 0.211 | 0.363 | 0.188 |
| | | *η* | | 0.369 | 0.083 | 0.732 | 0.201 | 0.722 | 0.013 | 0.594 | 0.015 | 0.55 | 0.019 | 0.773 | 0.058 |
| | | *μ* | 75 | 0.069 | 0.012 | 0.067 | 0.117 | 0.225 | 0.218 | 0.127 | 0.127 | 0.128 | 0.126 | 0.167 | 0.119 |

| PVs | | | *m* | ML | | MPS | | LS | | WLS | | AD | | CvM | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\mu$ | $\eta$ | | | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB | MSE | RAB |
| | | $\eta$ | | 0.283 | 0.051 | 0.407 | 0.136 | 0.464 | 0.011 | 0.369 | 0.01 | 0.341 | 0.009 | 0.484 | 0.036 |
| | | $\mu$ | 100 | 0.053 | 0.011 | 0.045 | 0.095 | 0.138 | 0.163 | 0.089 | 0.096 | 0.089 | 0.096 | 0.113 | 0.094 |
| | | $\eta$ | | 0.216 | 0.039 | 0.288 | 0.106 | 0.35 | 0.105 | 0.278 | 0.008 | 0.261 | 0.006 | 0.36 | 0.025 |
| | | $\mu$ | 125 | 0.039 | 0.006 | 0.035 | 0.084 | 0.093 | 0.12 | 0.061 | 0.068 | 0.062 | 0.07 | 0.079 | 0.068 |
| | | $\eta$ | | 0.173 | 0.033 | 0.225 | 0.089 | 0.283 | 0.006 | 0.226 | 0.009 | 0.212 | 0.007 | 0.291 | 0.022 |
| | | $\mu$ | 150 | 0.031 | 0.002 | 0.028 | 0.076 | 0.071 | 0.096 | 0.047 | 0.053 | 0.049 | 0.057 | 0.062 | 0.055 |
| | | $\eta$ | | 0.141 | 0.028 | 0.179 | 0.077 | 0.234 | 0.005 | 0.185 | 0.008 | 0.176 | 0.006 | 0.239 | 0.018 |
| 0.9 | | $\mu$ | 50 | 0.333 | 0.027 | 0.221 | 0.165 | 2.485 | 0.519 | 0.958 | 0.285 | 1.293 | 0.293 | 1.417 | 0.278 |
| | | $\eta$ | | 0.675 | 0.103 | 1.065 | 0.46 | 1.078 | 0.016 | 0.88 | 0.019 | 0.806 | 0.022 | 1.158 | 0.071 |
| | | $\mu$ | 75 | 0.194 | 0.023 | 0.147 | 0.119 | 0.798 | 0.294 | 0.484 | 0.178 | 0.52 | 0.179 | 0.738 | 0.183 |
| | | $\eta$ | | 0.406 | 0.064 | 0.589 | 0.166 | 0.697 | 0.014 | 0.546 | 0.012 | 0.5 | 0.01 | 0.728 | 0.044 |
| | | $\mu$ | 100 | 0.145 | 0.018 | 0.114 | 0.098 | 0.576 | 0.232 | 0.296 | 0.129 | 0.302 | 0.132 | 0.428 | 0.144 |
| | | $\eta$ | | 0.309 | 0.049 | 0.417 | 0.13 | 0.527 | 0.013 | 0.411 | 0.009 | 0.383 | 0.007 | 0.543 | 0.029 |
| | | $\mu$ | 125 | 0.105 | 0.011 | 0.089 | 0.088 | 0.337 | 0.166 | 0.184 | 0.09 | 0.193 | 0.095 | 0.273 | 0.103 |
| | | $\eta$ | | 0.247 | 0.041 | 0.325 | 0.11 | 0.427 | 0.007 | 0.333 | 0.012 | 0.312 | 0.009 | 0.439 | 0.027 |
| | | $\mu$ | 150 | 0.079 | 0.006 | 0.071 | 0.08 | 0.247 | 0.132 | 0.139 | 0.07 | 0.148 | 0.077 | 0.208 | 0.081 |
| | | $\eta$ | | 0.201 | 0.035 | 0.259 | 0.095 | 0.353 | 0.006 | 0.273 | 0.011 | 0.258 | 0.006 | 0.361 | 0.023 |

## 6. APPLICATIONS TO REAL DATA

This section presents a data analysis aimed at evaluating the KM-UGo distribution's goodness-of-fit in comparison to five other models, including the Kumaraswamy (Kum) distribution, the Topp-Leone (TL) distribution, the unit Gamma/Gompertz (UG/Go) distribution [38], the UGo distribution, the unit log-logistic (ULL) distribution [39], and the new power function (NP) distribution [40].

### 6.1. First Data Set

The first real data set was previously used by Reference [41] recorded the data as

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| 0.68879 | 0.50813 | 0.66621 | 0.74526 | 0.86947 | 0.88076 | 0.84688 | 0.91463 | 0.75655 |
| 0.55329 | 0.79042 | 0.82429 | 0.92593 | 0.80172 | 0.79042 | 0.83559 | 0.68879 | 0.74526 |
| 0.80172 | 0.93722 | 0.85818 | 0.98238 | 0.29359 | 0.99368 | 0.67751 | 0.80172 | 0.93722 |
| 0.63234 | 0.64363 | 0.73397 | 0.89205 | 0.64363 | 0.77913 | 0.41779 | 0.58717 | 0.88076 |
| 0.91463 | 0.80172 | 0.68879 | 0.72267 | 0.90334 | 0.76784 | 0.93722 | 0.51454 | 0.38392 |

The summary of these datasets has been discussed as follows: $Q_1$ = 067751, $Q_2$ = 0.79042, $Q_3$ = 0.0.8808, mean= 0.7480, $\alpha_3 = -1.17406$ and $\alpha_4 = 4.185955$. The MLEs and standard errors (SEs) for all models are given in Table 4. The measures of fit statistic using the maximized log-likelihood (-2logL), Akaike information criterion ($E_1$), Bayesian information criterion ($E_2$), the correct Akaike information criterion ($E_3$), Hannan-Quinn information criterion ($E_4$), the Kolmogorov Smirnov test (KST) statistic values along with P-value, Cramér-von Mises test (CvMT) and Anderson-Dalring test (ADT) are calculated in Table 5. The best model to match the data can be determined by looking at the models with the lowest values for -2logL, $E_1$, $E_2$, $E_3$, $E_4$, KST, CvMT, ADT, and the biggest P-value.

***Table 4.*** *MLEs and SEs of all model parameters for the first data*

| **parameters** | **Distributions** | | | | |
|---|---|---|---|---|---|
| | **UG** | **UW** | **KM-UGo** | **NP** | **UHLG** |
| $\mu$ | 13.274 | 8.848 | 29.059 | — | — |
| SE | (29.15) | (1.319) | (64.485) | | |
| $\eta$ | 0.214 | — | 0.128 | 0.901 | — |
| SE | (0.428) | | (0.273) | (0.051) | |
| $\lambda$ | — | — | — | 2.246 | 7.471 |
| SE | | | | (0.863) | (1.743) |

***Table 5.*** *The statistics measures for the first data*

| **Measures** | **Distribution** | | | | |
|---|---|---|---|---|---|
| | **UG** | **TL** | **KM-UGo** | **NP** | **UHLG** |
| -2log L | -19.954 | -14.101 | -20.452 | -17.969 | -15.329 |
| $E_1$ | -35.908 | -26.202 | -36.903 | -31.938 | -28.658 |
| $E_2$ | -32.295 | -24.395 | -33.289 | -28.325 | -26.851 |
| $E_3$ | -35.623 | -26.109 | -36.617 | -31.653 | -28.565 |
| $E_4$ | -34.562 | -25.528 | -35.556 | -30.592 | -27.985 |
| KST | 0.125 | 0.183 | 0.095 | 0.142 | 0.159 |
| P-value | 0.484 | 0.1 | 0.807 | 0.321 | 0.201 |
| CvMT | 0.07 | 0.081 | 0.07 | 0.093 | 0.501 |
| ADT | 0.475 | 0.544 | 0.419 | 0.606 | 0.438 |

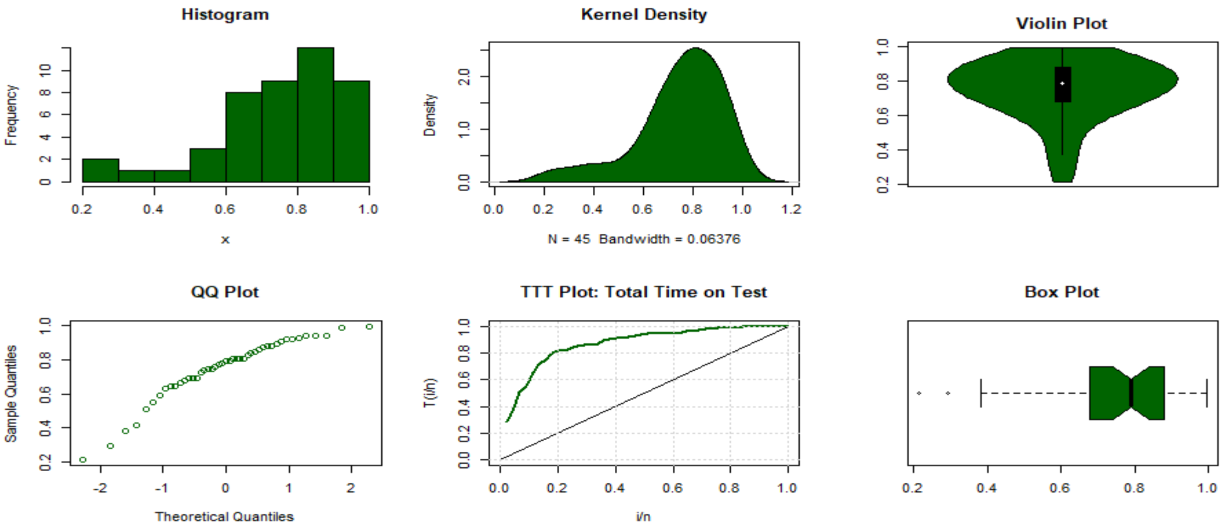


***Figure 6.*** *Nonparametric plots for the first data of the KM-UGo distribution*

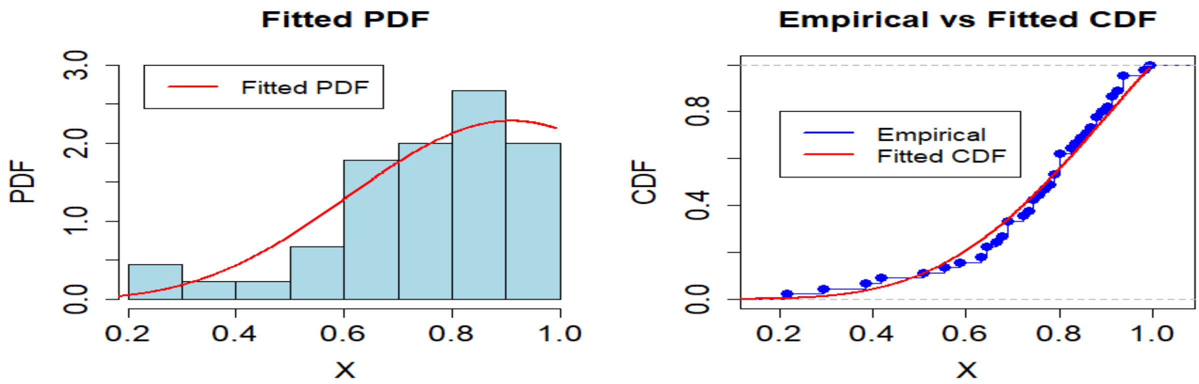


***Figure 7.*** *The CDF plot (right) with empirical line, fitted PDF plot (left) for the first data of the KM-UGo distribution*

KM-UGo distribution offers a noticeably better fit than the other four models, according to the data. The panel of Figure 6 shows that the box plot is left-skewed. Also, the total test on time (TTT) plot exhibits a concave shape initially and then transitions to a convex shape suggesting a unimodal HF. Figure 7 presents the empirical findings for the KM-UGo distribution.

### 6.2. Second Data Set

The second dataset consists of 48 rock samples from a petroleum reservoir, as reported in [42]. These samples represent twelve core specimens taken from the reservoir, with each core being analyzed across four cross-sections. For each core sample, permeability was measured, and each cross-section was evaluated based on three variables: the total pore area, the total pore perimeter, and the pore shape. The dataset is recorded as follows:

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 0.090 | 0.149 | 0.183 | 0.117 | 0.122 | 0.167 | 0.190 | 0.164 |
| 0.204 | 0.162 | 0.151 | 0.148 | 0.229 | 0.232 | 0.173 | 0.153 |
| 0.204 | 0.263 | 0.200 | 0.145 | 0.114 | 0.219 | 0.240 | 0.162 |
| 0.281 | 0.179 | 0.192 | 0.133 | 0.225 | 0.341 | 0.312 | 0.276 |
| 0.198 | 0.327 | 0.154 | 0.276 | 0.177 | 0.439 | 0.164 | 0.254 |
| 0.329 | 0.230 | 0.464 | 0.420 | 0.201 | 0.263 | 0.128 | 0.200 |

The summary of these datasets has been listed: $Q_1$ =0.1600, $Q_2$ =0.1990, $Q_3$ =0.2562, mean=0.2155, $\alpha_3$ = 1.215241 and $\alpha_4$ =4.234513. The MLEs and SEs for all models are given in Table 6 The measures of fit statistic including -2logL, $E_1$, $E_2$, $E_3$, $E_4$, KST values along with P-value, CvMT and ADT are calculated in Table 7. The model with minimum values for -2logL, $E_1$, $E_2$, $E_3$, $E_4$, KS, CvMT, ADT, and largest P-value can be chosen as the best model to fit the data. The KM-UGo distribution offers a noticeably better fit than the other four models, according to the data.

***Table 6.*** *MLEs and SEs of all model parameters for the second data*

| **Parameters** | **Distributions** | | | | |
|---|---|---|---|---|---|
| | **UGo** | **Kum** | **KM-UGo** | **UG/Go** | **ULL** |
| $\mu$<br>SE | 0.008<br>(0.004) | — | 0.013<br>(0.007) | 2.783<br>(0.385) | 7.417<br>(0.901) |
| $\eta$<br>SE | 2.731<br>(0.263) | 42.93<br>(16.62) | 2.62<br>(0.281) | 1.678<br>(0.779) | 1.596<br>(0.054) |
| $\lambda$<br>SE | — | 2.67<br>(0.286) | — | 134.207<br>(52.05) | — |

***Table 7.*** *The statistical measures for the second data*

| **Measures** | **Models** | | | | |
|---|---|---|---|---|---|
| | **UGo** | **Kum** | **KM-UGo** | **UG/Go** | **ULL** |
| -2log L | -56.913 | -52.507 | -57.105 | -52.768 | -55.458 |
| $E_1$ | -109.82 | -101.01 | -110.21 | -99.537 | -106.92 |
| $E_2$ | -106.08 | -97.271 | -106.47 | -93.923 | -103.175 |
| $E_3$ | -109.56 | -100.75 | -109.94 | -98.992 | -106.651 |
| $E_4$ | -108.41 | -99.599 | -108.79 | -97.415 | -105.503 |
| KST | 0.082 | 0.142 | 0.067 | 0.164 | 0.097 |
| P-value | 0.906 | 0.287 | 0.982 | 0.1501 | 0.756 |
| CvMT | 0.028 | 0.205 | 0.017 | 0.028 | 0.112 |
| ADT | 0.218 | 1.293 | 0.209 | 0.289 | 0.708 |



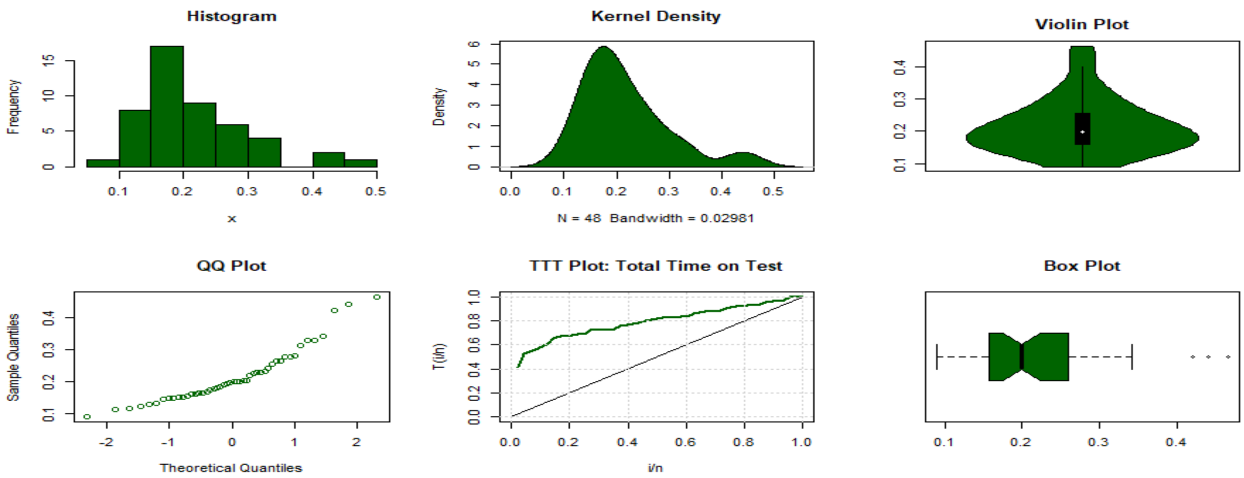


***Figure 8.*** *Nonparametric plots for the second data of the KM-UGo distribution*

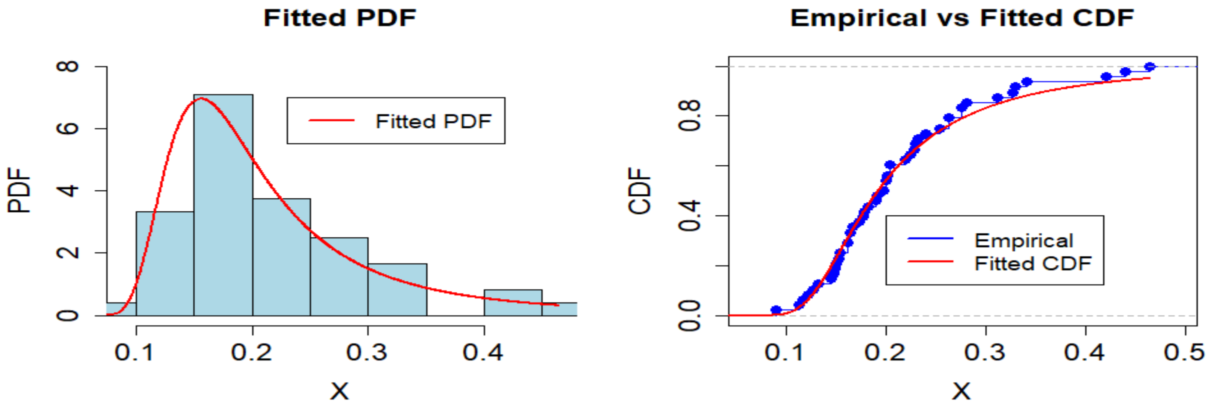


***Figure 9.*** *The CDF plot (right) with empirical line, fitted PDF plot (left) for the first data of the KM-UGo distribution*

Figure 8 illustrates the positive skewness of the box plot which indicate that most rock samples have lower permeability values, with a few samples having significantly higher permeability, while the panel displays the concave, or rising, TTT plot meaning that permeability have an increasing failure rate, with most samples exhibiting lower values and a few samples showing significantly higher values. This could reflect heterogeneity in the reservoir and the presence of rare but influential characteristics. Figure 9 presents the empirical findings for the KM-UGo distribution percentage of rock samples from a petroleum reservoir reported in Table 6. Real-data applications suggest that the KM-UGo distribution offers a more flexible alternative to the UGo distribution as a baseline distribution. Analyses performed using two distinct real-world datasets demonstrate the practical utility of the model, indicating that the KM-UGo model provides a better fit than other unit distributions for these datasets.

Subsequently, various estimates for the KM-UGo distribution were derived utilising the recommended estimation techniques for both real datasets. The Broyden–Fletcher–Goldfarb–Shanno (BFGS) optimization algorithm was employed to determine the optimal parameter estimates. This quasi-Newton method is widely recognized for its efficiency and robustness in solving unconstrained nonlinear optimization problems, making it particularly suitable for maximizing likelihood functions or minimizing objective functions in statistical estimation. We note that parameter estimation via the MPS method is impossible with these datasets, as they contain equal values. Table 8 presents the parameter estimates and their SEs for the KM-UGo distribution obtained using various estimation methods for both datasets.

***Table 8.*** *The parameter estimates using various estimation techniques of KM-UGo.*

| Parameter | Data | ML | LS | WLS | AD | CvM |
|---|---|---|---|---|---|---|
| $\mu$ | First data | 32.00215 | 2.711166 | 3.501802 | 12.0881 | 2.057336 |
| $\eta$ | | 0.116236 | 1.215248 | 0.963418 | 0.304979 | 1.528709 |
| $\mu$ | Second data | 0.013115 | 0.012899 | 0.011791 | 0.010484 | 0.011747 |
| $\eta$ | | 2.618305 | 2.643862 | 2.690002 | 2.757091 | 2.702258 |

## 7. CONCLUDING REMARKS

In this paper, a new heavy-tailed distribution which is called KM-UGo distribution is suggested and presented. Moments, incomplete moments, PWM, residual and reversed residual life's, quantile function, and entropy measures of the KM-UGo distribution are obtained. Six estimation methods are used in estimating the unknown parameters of the new KM-UGo distribution. A simulation study examined the asymptotic behavior of the KM-UGo distribution's parameter estimates. From the simulation study, it can be noted that the MSEs and RABs of the parameter estimate decrease with increasing sample size. Also, the MPS method is the best for $\mu$ estimate, and the ML method is the best for $\eta$ estimate compared to other methods. Finally, the applicability of the suggested KM-UGo distribution in lifetime data analysis was demonstrated using two real datasets. The outcomes made it obvious that the KM-UGo distribution provides a superior fit than the other compared distributions. The current model has some limitations. Notably, parameter estimation for the KM-UGo distribution currently relies on classical methods, which are primarily applicable to complete datasets. Future research directions include implementing Bayesian estimation methods for the KM-UGo distribution's parameters. Additionally, extending the model to handle censored data and conducting broader real-world applications are recommended to assess its practical utility.

## CONFLICTS OF INTEREST

No conflict of interest was declared by the authors

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