

On subclasses of close-to-star functions of order μ and type (α, β)

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ABSTRACT. Let $\mathcal{P}_\mu$ represent the class of analytic functions $\wp(z)$ defined in the open unit disc $\Delta = \{z : |z| < 1\}$ with $\wp(0) = 1$ and

$$\left| \frac{\wp(z) - 1}{\wp(z) + 1} \right| < \mu.$$

In this paper, we introduce two new subclasses $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $\mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$ of the class of close-to-star functions that satisfy the conditions:

$$\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \in \mathcal{P}_\mu$$

and

$$\left(\alpha \frac{((\mathcal{L}_{u,v}f(z))')^\lambda}{(g'(z))^\lambda} + \beta \frac{(\mathcal{L}_{u,v}f(z))^\lambda}{(g(z))^\lambda} \right) \in \mathcal{P}_\mu,$$

respectively. Functions f in the new classes are normalized analytic functions defined in the unit disc Δ such that g is starlike and $\mathcal{L}_{u,v}$ is the Carlson-Shaffer operator. Some reported results for $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$ include the integral representation formula, some coefficient estimates, Fekete-Szegő estimates for real and complex parameters, and some inclusion properties. All the results are sharp. Again, some early coefficient estimates for functions $f \in \mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$ are investigated. Furthermore, a number of remarks to show the relationship between the new classes and some existing classes are clearly discussed.

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1. INTRODUCTORY STATEMENT

Let $\mathcal{A}$ denote the class of complex-valued functions of the infinite series type

$$f(z) = z + a_2z^2 + a_3z^3 + \cdots \quad (1)$$

which are *analytic* in the unit disc

$$\{z : z \in \mathbb{C} \text{ and } |z| < 1\} = \Delta$$

and normalized by the conditions: $f(0) = 0 = f'(0) - 1$.

The image domain $f(\Delta)$ of an analytic-univalent function f describes a number of nice geometries and geometric characterizations of the function f . A geometric function is a complex-valued function whose image domain is a description of some certain geometries such as star, close-to-star, convex, close-to-convex, spiral, etc. The study of such functions is known as the Theory of Geometric Functions; a branch of Complex Analysis that explains the characteristics of geometric properties of the image domain $f(\Delta)$.

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A function $g \in \mathcal{A}$ of the infinite series type

$$g(z) = z + b_2 z^2 + b_3 z^3 + \cdots \quad (z \in \Delta) \quad (2)$$

is called a *starlike function* if and only if

$$\Re \left(\frac{zg'(z)}{g(z)} \right) > 0 \quad (z \in \Delta).$$

We shall let $\mathcal{S}^*$ represent the class of such functions.

Another class of interest is the class $\mathcal{P}$ of functions analytic in Δ and having positive real parts in the *open-right-half-plane* of the complex number space. An analytic function $\wp$ of the infinite series type

$$\wp(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots \quad (3)$$

is in $\mathcal{P}$ if it satisfies the conditions: $\wp(0) = 1$, $\Re \wp(z) > 0$, and $z \in \Delta$. Functions in class $\mathcal{P}$ are well-known to play significant role in the establishment of many properties of analytic functions. Further, let $\mathcal{P}_\mu$ represent the class of functions $\wp$ of type (3) that satisfy the conditions:

$$\left| \frac{\wp(z) - 1}{\wp(z) + 1} \right| < \mu \quad (0 < \mu \leq 1, z \in \Delta).$$

From geometric viewpoint, $\wp \in \mathcal{P}_\mu$ if and only if, $\wp(0) = 1$ and

$$\wp(\Delta) = \left\{ \omega : \omega \in \mathbb{C} \text{ and } \left| \omega - \frac{1 + \mu^2 r^2}{1 - \mu^2 r^2} \right| < \frac{2\mu r}{1 - \mu^2 r^2}, 0 < r < 1 \right\}$$

such that the image domain $\wp(\Delta)$ is an open symmetric disc whose center is at a point where $x = \frac{1 + \mu^2 r^2}{1 - \mu^2 r^2} > 0$ on the real axis. This disc has diameter endpoints $d_1 = \wp(-r) = (1 - \mu r)/(1 + \mu r)$ and $d_2 = \wp(r) = (1 + \mu r)/(1 - \mu r)$ such that $0 < d_1 < 1 < d_2$. Obviously, $\mathcal{P}_\mu \subseteq \mathcal{P}$. See Goodman [9, p.111] and Thomas *et al.* [25, p.23] for more information.

1.1. The Class of Close-to-star Functions. A function $f \in \mathcal{A}$ is said to be *close-to-star* if and only if it satisfies the conditions:

$$\Re \left(\frac{f(z)}{g(z)} \right) > 0 \quad (g \in \mathcal{S}^*, z \in \Delta).$$

The class of close-to-star functions was introduced by Reade [20] and notably, it is not necessary that close-to-star functions are univalent in Δ . The class of close-to-star functions and some of its subclasses have been investigated by many authors such as in [7, 13, 15, 21]. One may refer to the standard text of Goodman [9, pp.97-99] for more information.

In 1991, Sudharsan *et al.* [23] introduced the subclass $\mathcal{CS}(\alpha, \beta, \mu)$ of close-to-star functions. A function $f \in \mathcal{A}$ belongs to the class $\mathcal{CS}(\alpha, \beta, \mu)$ if it satisfies the conditions:

$$\left(\alpha \frac{f'(z)}{g'(z)} + \beta \frac{f(z)}{g(z)} \right) \in \mathcal{P}_\mu \quad (z \in \Delta),$$

$0 < \mu \leq 1$, $\alpha \geq 0$, $\beta > 0$ and $g \in \mathcal{S}^*$. Likewise, Babalola *et al.* [3] studied the class $\mathcal{C}_\lambda^*$ which contains functions $f \in \mathcal{A}$ that satisfy the conditions:

$$\Re \left(\frac{(f(z))^\lambda}{(g(z))^\lambda} \right) > 0 \quad (z \in \Delta),$$

where λ is a positive real number and $g \in \mathcal{S}^*$. Some generalized subclasses of the class of close-to-star functions involving operators such as Carlson-Shaffer linear operator, Opoola differential operator, and Srivastava-Attiya operator, were respectively studied in [1, 14, 22].

1.2. The Carlson-Shaffer Linear Operator. In [6], the Carlson-Shaffer linear operator $\mathcal{L}_{u,v} : \mathcal{A} \rightarrow \mathcal{A}$ is defined by the relation

$$\mathcal{L}_{u,v}f(z) = \delta(u,v;z) \star f(z) = z + \sum_{s=2}^{\infty} (k)_{s-1} a_s z^s, \quad (4)$$

where the notation ' $\star$ ' means Hadamard product (see [9, 24, 25] for definition),

$$(k)_s = \frac{(u)_s}{(v)_s}, \quad u \in \mathbb{C}, \quad v \in \mathbb{C} \setminus \{0, -1, -2, \dots\},$$

and the function

$$\delta(u,v;z) = z + \sum_{s=2}^{\infty} \frac{(u)_{s-1}}{(v)_{s-1}} z^s$$

is known as the *incomplete beta function*. More so, the term $(\gamma)_s$ is the Pochhammer symbol which can be expressed in terms of gamma function

$$(\gamma)_s = \frac{\Gamma(\gamma+s)}{\Gamma(\gamma)} = \begin{cases} 1 & (s=0), \\ \gamma(\gamma+1)(\gamma+2) \cdots (\gamma+s-1) & (s \in \mathbb{N}). \end{cases} \quad (5)$$

We however, recommend the work of Oyekan and Opoola [16, Lemma 2.5] for some simplifications and applications of the gamma function in (5).

2. NEW SUBCLASSES OF CLOSE-TO-STAR FUNCTIONS

Motivated by the aforementioned works, we therefore, introduce two new subclasses $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $\mathcal{L}_{u,v}^{\lambda}(\alpha, \beta, \mu)$ of the class of close-to-star functions using the Carlson-Shaffer operator introduced in [6] and applied in [1, 16].

Definition 1. A function $f \in \mathcal{A}$ is said to belong to the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ if

$$\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \in \mathcal{P}_{\mu} \quad (6)$$

and it is said to belong to the class $\mathcal{L}_{u,v}^{\lambda}(\alpha, \beta, \mu)$ if

$$\left(\alpha \frac{((\mathcal{L}_{u,v}f(z))')^{\lambda}}{(g'(z))^{\lambda}} + \beta \frac{(\mathcal{L}_{u,v}f(z))^{\lambda}}{(g(z))^{\lambda}} \right) \in \mathcal{P}_{\mu} \quad (7)$$

where $0 < \mu \leq 1$, $\alpha \geq 0$, $\beta > 0$, $\alpha + \beta = 1$, $g \in \mathcal{S}^*$, λ is a positive real number, and $\mathcal{L}_{u,v}f(z)$ is as defined by (4). In addition, the power λ is meant as principal determinations only.

Remark 1. Some earlier known subclasses of classes $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $\mathcal{L}_{u,v}^{\lambda}(\alpha, \beta, \mu)$ are enumerated as follows.

- (1) $\mathcal{L}_{1,1}(0, 1, 1)$ is the class of close-to-star functions introduced by Reade [20].
- (2) $\mathcal{L}_{1,1}(\alpha, 1 - \alpha, 1)$ is the class $\mathcal{CS}(\alpha)$ of α -close-to-star functions studied by Kasi [12].
- (3) $\mathcal{L}_{1,1}(\alpha, 1 - \alpha, \mu)$ is the class $\mathcal{CS}(\alpha, \mu)$ introduced by Kasi [11].
- (4) $\mathcal{L}_{1,1}^{\lambda}(0, 1, 1)$ is the class $\mathcal{C}_{\lambda}^*$ introduced by Babalola et al. [3].
- (5) $\mathcal{L}_{u,v}(0, 1, 1)$ is the class $\mathcal{CS}_{u,v}^*$ introduced by Akbarally et al. [1].
- (6) $\mathcal{L}_{1,1}(\alpha, \beta, \mu)$ is the class $\mathcal{CS}(\alpha, \beta, \mu)$ introduced by Sudharsan et al. [23].

3. A SET OF LEMMAS

Let g and $\wp$ be as defined in (2) and (3), respectively. More so, let

$$w(z) = w_1 z + w_2 z^2 + \cdots \in \mathcal{W}$$

be an analytic function such that $w(0) = 0$, $|w(z)| < 1$ and $z \in \Delta$. The class $\mathcal{W}$ is known as the class of Schwarz (or unit bounded) functions. We shall need the following lemmas in proving our results.

Lemma 1 ([9, 19, 25]). Let $g \in \mathcal{S}^*$, then

$$|b_s| \leq s, \quad \forall s \geq 2.$$

Lemma 2 (Jack's Lemma, [10]). *Let w be an analytic and a non-constant function in Δ such that $w(0) = 0$. If $|w(z)|$ attains its maximum value at the point z_0 with $|z_0| = r$, then*

$$z_0 w'(z_0) = k w(z_0), \quad \forall k \geq 1.$$

Lemma 3 ([5]). *Let $w \in \mathcal{W}$, then*

$$|w_2| \leq 1 - |w_1|^2.$$

Lemma 4. ([18, Lemma 2]). *Let $\wp \in \mathcal{P}_\mu$, then*

$$|p_s| \leq 2\mu, \quad \forall s \geq 1, \quad \text{and } 0 < \mu \leq 1.$$

The result is sharp for the function $\wp_0(z) = \frac{1 - \mu z^s}{1 + \mu z^s}$.

Lemma 5 ([12, 17]). *Let $\wp \in \mathcal{P}_\mu$, then there exists a function $w \in \mathcal{W}$ such that*

$$\wp(z) = \frac{1 - \mu w(z)}{1 + \mu w(z)}, \quad 0 < \mu \leq 1, \quad \text{and } z \in \Delta.$$

Lemma 6. *Let $\wp \in \mathcal{P}_\mu$ and $0 < \mu \leq 1$.*

(1) *Then,*

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2\mu - \frac{|p_1|^2}{2\mu}. \quad (8)$$

(2) *If $\eta \in \mathbb{R}$, then*

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq \begin{cases} 2\mu^2(1 - \eta) & \text{for } \eta \leq \frac{\mu-1}{\mu}; \\ 2\mu & \text{for } \frac{\mu-1}{\mu} \leq \eta \leq \frac{\mu+1}{\mu}; \\ 2\mu^2(\eta - 1) & \text{for } \eta \geq \frac{\mu+1}{\mu}. \end{cases} \quad (9)$$

(3) *And if $\eta \in \mathbb{C}$, then*

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq \begin{cases} 2\mu & \text{for } |1 - \eta| \leq \frac{1}{\mu}; \\ 2\mu^2|1 - \eta| & \text{for } |1 - \eta| \geq \frac{1}{\mu}. \end{cases} \quad (10)$$

In another way,

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq 2\mu \max \left\{ 1, \mu|1 - \eta| \right\}. \quad (11)$$

Proof. Since $\wp \in \mathcal{P}_\mu$, then by Lemma 5 we can write

$$\wp(z) = \frac{1 - \mu w(z)}{1 + \mu w(z)} = (1 - \mu w(z))(1 + \mu w(z))^{-1} \quad (12)$$

or

$$1 + p_1 z + p_2 z^2 + \cdots = 1 - 2\mu w_1 z + 2\mu(\mu w_1^2 - w_2)z^2 + \cdots.$$

Comparing corresponding coefficients means

$$p_1 = -2\mu w_1 \implies \begin{cases} \frac{p_1^2}{2} = 2\mu^2 w_1^2 \\ w_1 = \frac{p_1}{-2\mu} \\ |w_1|^2 = \frac{|p_1|^2}{4\mu^2} \end{cases} \quad (13)$$

and

$$p_2 = 2\mu(\mu w_1^2 - w_2) = 2\mu^2 w_1^2 - 2\mu w_2. \quad (14)$$

Considering (13) and (14) implies that

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2\mu|w_2|$$

and the application of Lemma 3 means

$$\left| p_2 - \frac{p_1^2}{2} \right| \leq 2\mu(1 - |w_1|^2). \quad (15)$$

Some simplifications after putting the third result in (13) into (15) gives inequality (8).

Also, considering a real number η shows that

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| = \left| p_2 - \frac{p_1^2}{2} + \frac{p_1^2}{2} - \eta \frac{p_1^2}{2} \right| \leq \left| p_2 - \frac{p_1^2}{2} \right| + \frac{|p_1|^2}{2} |1 - \eta| \quad (16)$$

and using (8) in (16) means that

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq \left(2\mu - \frac{|p_1|^2}{2\mu} \right) + \frac{|p_1|^2}{2} |1 - \eta|$$

or

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq 2\mu - \frac{|p_1|^2}{2\mu} (1 - \mu|1 - \eta|). \quad (17)$$

So, setting $(1 - \mu|1 - \eta|) \geq 0 \implies \frac{\mu-1}{\mu} \leq \eta \leq \frac{\mu+1}{\mu}$ which shows that inequality (17) has the least upper bound of 2μ . This is the second result in (9). On the other hand, setting $(1 - \mu|1 - \eta|) \leq 0 \implies \eta \leq \frac{\mu-1}{\mu}$ or $\eta \geq \frac{\mu+1}{\mu}$ which shows that the least upper bound of inequality (17) can be expressed as $2\mu^2|1 - \eta|$. This is the first and third results in (9).

To prove inequality (10), observe that for $\eta \in \mathbb{C}$ in (17), setting $(1 - \mu|1 - \eta|) \geq 0 \implies |1 - \eta| \leq \frac{1}{\mu}$ or $|1 - \eta| \geq \frac{1}{\mu}$. For $|1 - \eta| \leq \frac{1}{\mu}$ the least upper bound is 2μ and for $|1 - \eta| \geq \frac{1}{\mu}$ the least upper bound can be expressed as $2\mu^2|1 - \eta|$ and the proof is complete. $\square$

Remark 2. Setting $\mu = 1$ in (8) gives the result of Pommerenke [19, p.165] and setting $\mu = 1$ in (9) and (10) gives the results of Babalola and Opoola [4, Corollary 2.5].

4. THE MAIN RESULTS

In this paper, we obtain the integral representation formula, some sharp early coefficient estimates, the Fekete-Szegő estimates (for real and complex parameters) and some inclusion relations for functions $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$. Further, we feature the early coefficient estimates for functions $f \in \mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$.

4.1. Integral Representation of $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$.

Theorem 1. A function f in (1) is in the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ if and only if there exists a function $g \in \mathcal{S}^*$ and a function $\wp \in \mathcal{P}_\mu$ such that

$$\mathcal{L}_{u,v}f(z) = \frac{1}{\alpha(g(z))^{\frac{\beta}{\alpha}}} \int_0^z (g(t))^{\frac{\beta}{\alpha}} g'(t) \wp(t) dt \quad (18)$$

where $(\alpha + \beta) = 1$ and the powers appearing in the formula are meant as principal values only.

Proof. Let $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$ so that for $g \in \mathcal{S}^*$,

$$\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) = \wp(z). \quad (19)$$

Multiplying both sides of (19) by $\frac{1}{\alpha} g'(z) (g(z))^{\frac{\beta}{\alpha}}$ yields

$$(\mathcal{L}_{u,v}f(z))' (g(z))^{\frac{\beta}{\alpha}} + \frac{\beta}{\alpha} (g(z))^{\frac{\beta}{\alpha}-1} g'(z) \mathcal{L}_{u,v}f(z) = \frac{1}{\alpha} (g(z))^{\frac{\beta}{\alpha}} g'(z) \wp(z)$$

where an equivalence relation gives

$$\frac{d}{dz} (\mathcal{L}_{u,v}f(z) (g(z))^{\frac{\beta}{\alpha}}) = \frac{1}{\alpha} (g(z))^{\frac{\beta}{\alpha}} g'(z) \wp(z).$$

The integration of both sides with respect to z from 0 to z yields the representation formula (18). We obtain the converse from the fact that $\mathcal{L}_{u,v}f(z)$ given by (4) has a Taylor's series type

$$\mathcal{L}_{u,v}f(z) = z + (k)_1 a_2 z^2 + \cdots$$

$\square$

Remark 3. It is known that the extremal functions in $\mathcal{S}^*$ and $\mathcal{P}_\mu$ are $g_0(z) = \frac{z}{(1-z)^2}$ and $\wp_0(z) = \frac{1+\mu z}{1-\mu z}$, respectively. Hence, following from (18), we get

$$\mathcal{L}_{u,v}f(z) = \frac{1}{\alpha \left(\frac{\sqrt{z}}{(1-z)}\right)^{\frac{2\beta}{\alpha}}} \int_0^z \left(\frac{\sqrt{t}}{(1-t)}\right)^{\frac{2\beta}{\alpha}} \frac{(1+t)(1+\mu t)}{(1-t)^3(1-\mu t)} dt \quad (20)$$

as the extremal function in the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$. Some other examples of functions in $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ are reported as follows.

Example 1. Take $g_s(z) = z + sz^s \in \mathcal{S}^*$ and $\wp(z) = 1 \in \mathcal{P}_\mu$, then from (18) we get

$$\begin{aligned} \mathcal{L}_{u,v}f(z) &= \frac{1}{\alpha(g_s(z))^{\frac{\beta}{\alpha}}} \int_0^z (g_s(t))^{\frac{\beta}{\alpha}} g'_s(t) \wp(t) dt \\ &= \frac{1}{\alpha(z + sz^s)^{\frac{\beta}{\alpha}}} \int_0^z (t + st^s)^{\frac{\beta}{\alpha}} (1 + s^2 t^{s-1}) dt \\ &= z + \frac{s}{\alpha} (1 - \beta) z^s \quad (s \geq 2, \alpha > 0, \alpha + \beta = 1). \end{aligned} \quad (21)$$

Observe that we can harvest infinitely many functions from (21) by varying values for s , α and β . Clearly, the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ is non-void of functions.

4.2. Coefficient Estimates for $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$.

Theorem 2. Let the function f in (1) belong to the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$, then for $(\alpha + \beta) = 1$,

$$|a_2| \leq \frac{2[3 - (\alpha + 2\beta) + \mu]}{(2\alpha + \beta)|(k)_1|} \quad (22)$$

and

$$|a_3| \leq \frac{(2\alpha + \beta)\{12 + 14\mu - 3(\alpha + 3\beta)\} - 8\mu}{(2\alpha + \beta)(3\alpha + \beta)|(k)_2|}. \quad (23)$$

The results are sharp for the extremal function in (20).

Proof. Since $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $(\alpha + \beta) = 1$, then we can write (6) as

$$\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) = \wp(z)$$

or

$$\alpha(\mathcal{L}_{u,v}f(z))'g(z) + \beta\mathcal{L}_{u,v}f(z)g'(z) = \wp(z)g(z)g'(z). \quad (24)$$

Putting (2), (3), and (4) into (24) simplifies to

$$\begin{aligned} &z + \{(\alpha + 2\beta)b_2 + (2\alpha + \beta)(k)_1 a_2\}z^2 \\ &\quad + \{(\alpha + 3\beta)b_3 + 2(k)_1 b_2 a_2 + (3\alpha + \beta)(k)_2 a_3\}z^3 + \cdots \\ &= z + \{3b_2 + p_1\}z^2 + \{4b_3 + 2b_2^2 + 3b_2 p_1 + p_2\}z^3 + \cdots \end{aligned} \quad (25)$$

The comparison of both sides of (25) means

$$a_2 = \frac{[3 - (\alpha + 2\beta)]b_2 + p_1}{(2\alpha + \beta)(k)_1} \quad (26)$$

and

$$a_3 = \frac{4 - (\alpha + 3\beta)}{(3\alpha + \beta)(k)_2} b_3 + \frac{3(2\alpha + \beta) - 2}{(2\alpha + \beta)(3\alpha + \beta)(k)_2} b_2 p_1 + \frac{1}{(3\alpha + \beta)(k)_2} p_2. \quad (27)$$

From (26),

$$|a_2| \leq \frac{|[3 - (\alpha + 2\beta)]||b_2| + |p_1|}{(2\alpha + \beta)|(k)_1|}$$

so that the use of inequalities $|p_s| \leq 2\mu$ (Lemma 4) and $|b_s| \leq s$ (Lemma 1) yields (22). More so, from (27) we get

$$|a_3| \leq \frac{|4 - (\alpha + 3\beta)|}{(3\alpha + \beta)|(k)_2|} |b_3| + \frac{|3(2\alpha + \beta) - 2|}{(2\alpha + \beta)(3\alpha + \beta)|(k)_2|} |b_2| |p_1| + \frac{1}{(3\alpha + \beta)|(k)_2|} |p_2|$$

and using the inequalities $|p_s| \leq 2\mu$ (Lemma 4) and $|b_s| \leq s$ (Lemma 1) yields (23). $\square$

Remark 4. We observe the following existing results.

- (1) When $\alpha = 0$ and $u = v = \mu = \beta = 1$, then (22) and (23) will reduce to the estimates:

$$|a_2| \leq 4 \quad \text{and} \quad |a_3| \leq 9$$

which are the results of Reade [20, Theorem 4].

- (2) When $u = 1 = v$ and $\beta = 1 - \alpha$, then (22) and (23) will reduce to the estimates:

$$|a_2| \leq 2 \left(\frac{1 + \mu + \alpha}{(1 + \alpha)} \right) \quad \text{and} \quad |a_3| \leq \frac{6\alpha^2 + \alpha(14\mu + 9) + 6\mu + 3}{(1 + \alpha)(1 + 2\alpha)}$$

which are the results of Kasi [11, Theorem 4].

- (3) When $u = v = \mu = 1$ and $\beta = 1 - \alpha$, then (22) and (23) will reduce to the estimates:

$$|a_2| \leq 2 + \frac{2}{(1 + \alpha)} \quad \text{and} \quad |a_3| \leq \frac{\alpha(6\alpha + 23) + 9}{(1 + \alpha)(1 + 2\alpha)}$$

which are the results of Kasi [12, Theorem 3.1.4].

- (4) When $\alpha = 0$ and $\beta = 1 = \mu$, then (22) and (23) will reduce to the estimates:

$$|a_2| \leq \frac{4}{|(k)_1|} \quad \text{and} \quad |a_3| \leq \frac{9}{|(k)_2|}$$

which are the results of Akbarally et al. [1, Theorem 2.1].

- (5) When $u = 1 = v$, then (22) and (23) will reduce to the estimates:

$$|a_2| \leq \frac{2[\mu + 3 - (\alpha + 2\beta)]}{(2\alpha + \beta)}$$

and

$$|a_3| \leq \frac{2\mu(10\alpha + 3\beta) + 4(4\alpha - \beta) + 2\alpha^2 + 7\beta^2 + 3\alpha\beta}{(3\alpha + \beta)(2\alpha + \beta)}$$

which are the results of Sudharsan et al. [23, Theorem 2] although with an erratum for the bound on $|a_3|$.

4.3. Fekete-Szegő Estimtates. Another well investigated type of coefficient problems of functions f in (1) is the Fekete-Szegő functional defined by

$$\mathcal{F}(\xi, f) = |a_3 - \xi a_2^2| \quad (28)$$

where a_2 and a_3 are coefficients of f in (1), and ξ is a real (or complex) parameter. The functional (28) was initiated by Fekete and Szegő [8] and it has been considered for several subclasses of $\mathcal{A}$. One may explore the works in [2, 13, 14] for more information.

Theorem 3. Let the function f in (1) belong to the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and let $(\alpha + \beta) = 1$. If ξ and $(k)_s$ are real parameters, then

$$|a_3 - \xi a_2^2| \leq \begin{cases} A_\beta^\alpha(\mu, k_1, k_2, \xi) - \frac{U_\beta^\alpha(\mu, k_1, k_2, \xi)}{(3\alpha + \beta)|(k)_2|} & \text{for } \xi \leq (\mu - 1)\Omega_\beta^\alpha(\mu, k_1, k_2); \\ A_\beta^\alpha(\mu, k_1, k_2, \xi) + \frac{2\mu}{(3\alpha + \beta)|(k)_2|} & \text{for } (\mu - 1)\Omega_\beta^\alpha(\mu, k_1, k_2) \leq \xi \leq (\mu + 1)\Omega_\beta^\alpha(\mu, k_1, k_2); \\ A_\beta^\alpha(\mu, k_1, k_2, \xi) + \frac{U_\beta^\alpha(\mu, k_1, k_2, \xi)}{(3\alpha + \beta)|(k)_2|} & \text{for } \xi \geq (\mu + 1)\Omega_\beta^\alpha(\mu, k_1, k_2); \end{cases} \quad (29)$$

where

$$A_{\beta}^{\alpha}(\mu, k_1, k_2, \xi) = \frac{3(2\alpha + \beta)\{4 - (\alpha + 3\beta)\} + 4\mu\{3(2\alpha + \beta) - 2\}}{(2\alpha + \beta)(3\alpha + \beta)|(k)_2|} + \frac{4|\xi|\{3 - (\alpha + 2\beta)\}\{2\mu + [3 - (\alpha + 2\beta)]\}}{(2\alpha + \beta)^2|(k)_1|^2}, \quad (30)$$

$$U_{\beta}^{\alpha}(\mu, k_1, k_2, \xi) = \frac{2\mu^2\{2(3\alpha + \beta)(k)_2\xi - (2\alpha + \beta)^2(k)_1^2\}}{(2\alpha + \beta)^2(k)_1^2}, \quad (31)$$

and

$$\Omega_{\beta}^{\alpha}(\mu, k_1, k_2) = \frac{(2\alpha + \beta)^2(k)_1^2}{2\mu(3\alpha + \beta)(k)_2}. \quad (32)$$

Proof. Let $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu)$, then putting (26) and (27) into (28), and using the fact that $(\alpha + \beta) = 1$ with some groupings and simplifications, we get

$$\begin{aligned} |a_3 - \xi a_2^2| &= \left| \frac{4 - (\alpha + 3\beta)}{(3\alpha + \beta)(k)_2} b_3 - \frac{[3 - (\alpha + 2\beta)]^2}{(2\alpha + \beta)^2(k)_1^2} \xi b_2^2 \right. \\ &\quad + \left\{ \frac{3(2\alpha + \beta) - 2}{(2\alpha + \beta)(3\alpha + \beta)(k)_2} - \frac{2[3 - (\alpha + 2\beta)]}{(2\alpha + \beta)^2(k)_1^2} \xi \right\} b_2 p_1 \\ &\quad \left. + \frac{1}{(3\alpha + \beta)(k)_2} \left\{ p_2 - \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \times \frac{p_1^2}{2} \right\} \right| \end{aligned}$$

or

$$\begin{aligned} |a_3 - \xi a_2^2| &\leq \frac{4 - (\alpha + 3\beta)}{(3\alpha + \beta)|(k)_2|} |b_3| + \frac{[3 - (\alpha + 2\beta)]^2}{(2\alpha + \beta)^2|(k)_1|^2} |\xi| |b_2|^2 \\ &\quad + \left\{ \frac{3(2\alpha + \beta) - 2}{(2\alpha + \beta)(3\alpha + \beta)|(k)_2|} + \frac{2[3 - (\alpha + 2\beta)]}{(2\alpha + \beta)^2|(k)_1|^2} |\xi| \right\} |b_2| |p_1| \\ &\quad + \frac{1}{(3\alpha + \beta)|(k)_2|} \left| p_2 - \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \times \frac{p_1^2}{2} \right|. \end{aligned} \quad (33)$$

From (33), consider the expression

$$\left| p_2 - \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \times \frac{p_1^2}{2} \right| = \left| p_2 - \eta \frac{p_1^2}{2} \right|$$

for

$$\eta = \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2}.$$

Now, taking the first inequality in (9) (Lemma 6) implies

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq 2\mu^2(1 - \eta) = 2\mu^2 \left(1 - \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \right) = -U(\alpha, \beta, \mu, k, \xi) \quad (34)$$

in (31) and

$$\eta \leq \frac{\mu - 1}{\mu} \implies \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \leq \frac{\mu - 1}{\mu} \implies \xi \leq (\mu - 1)\Omega(\alpha, \beta, \mu, k) \quad (35)$$

in (32). Also, taking the second inequality in (9) (Lemma 6) implies

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq 2\mu \quad (36)$$

and

$$\frac{\mu - 1}{\mu} \leq \eta \leq \frac{\mu + 1}{\mu} \implies \frac{\mu - 1}{\mu} \leq \frac{2(3\alpha + \beta)(k)_2\xi}{(2\alpha + \beta)^2(k)_1^2} \leq \frac{\mu + 1}{\mu}$$

so that

$$(\mu - 1)\Omega(\alpha, \beta, \mu, k) \leq \xi \leq (\mu + 1)\Omega(\alpha, \beta, \mu, k) \quad (37)$$

in (32). Lastly, taking the third inequality in (9) (Lemma 6) implies

$$\left| p_2 - \eta \frac{p_1^2}{2} \right| \leq 2\mu^2(\eta - 1) = 2\mu^2 \left(\frac{2(3\alpha + \beta)(k)_2 \xi}{(2\alpha + \beta)^2(k)_1^2} - 1 \right) = \mathcal{U}(\alpha, \beta, \mu, k, \xi) \quad (38)$$

in (31) and

$$\eta \geq \frac{\mu + 1}{\mu} \implies \frac{2(3\alpha + \beta)(k)_2 \xi}{(2\alpha + \beta)^2(k)_1^2} \geq \frac{\mu + 1}{\mu} \implies \xi \geq (\mu + 1)\Omega(\alpha, \beta, \mu, k) \quad (39)$$

in (32). Putting (34 – 39) into (33), using Lemmas 1 and 4, and further simplification give the result in (29). $\square$

Theorem 4. Let the function f in (1) belong to the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and let $(\alpha + \beta) = 1$. If ξ or (and) $(k)_s$ are complex parameters, then

$$|a_3 - \xi a_2^2| \leq \Lambda_\beta^\alpha(\mu, k_1, k_2, \xi) + \frac{2\mu}{(3\alpha + \beta)|(k)_2|} \max\{1, \varphi\} \quad (40)$$

where

$$\varphi = \mu \left| 1 - \frac{2(3\alpha + \beta)(k)_2 \xi}{(2\alpha + \beta)^2(k)_1^2} \right| = \mu \frac{|2(3\alpha + \beta)(k)_2 \xi - (2\alpha + \beta)^2(k)_1^2|}{(2\alpha + \beta)^2|(k)_1|^2}$$

and $\Lambda_\beta^\alpha(\mu, k_1, k_2, \xi)$ is as defined in (30).

Proof. Using Lemmas 1 and 4; and inequality (11) (Lemma 6) in (33) yields the result in (40). $\square$

4.4. Inclusion Relations for the Class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$. For convenience, let $\mathcal{L}_{u,v}(\alpha, \beta, 1) = \mathcal{L}_{u,v}(\alpha, \beta)$. Hence we establish the following theorems.

Theorem 5. Let $f \in \mathcal{A}$ and $g \in \mathcal{S}^*$ with $f(0) = f'(0) - 1 = g(0) = g'(0) - 1 = 0$. If

$$\Re \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0 \quad (z \in \Delta),$$

$\alpha \geq 0$, $\beta > 0$, and $(\alpha + \beta) = 1$, then

$$\Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0 \quad (z \in \Delta).$$

Proof. Firstly, let $\alpha > 0$, $\beta > 0$ and consider the function

$$\phi(z) = \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \quad (z \in \Delta).$$

It can easily be observed that $\Re \phi(0) = \alpha + \beta > 0$ and $\Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) = 1 > 0$. Secondly, if we let $\alpha = 0$, $\beta > 0$ and $z \in \Delta \setminus \{0\}$, then $\Re \phi(z) > 0$ implies $\Re \left(\beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0$. Now to prove that $\Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0$, consider the analytic function

$$\frac{\mathcal{L}_{u,v}f(z)}{g(z)} = \frac{1 - w(z)}{1 + w(z)} \quad (z \in \Delta)$$

for $w(z) \neq -1$ and $w(0) = 0$. So, it suffices to prove that $|w(z)| < 1$ ($z \in \Delta$), otherwise, by Jack's lemma (Lemma 2) there exists $z_0 \in \Delta$ such that $|w(z_0)| = 1$ and

$$z_0 w'(z_0) = k w(z_0), \quad k \geq 1.$$

That is

$$\phi(z_0) = \frac{1 - w(z_0)}{1 + w(z_0)} - \frac{2\alpha k w(z_0)}{(1 + w(z_0))^2} \times \frac{g(z_0)}{z_0 g'(z_0)}$$

with $|w(z_0)| = 1$. Since $\Re\left(\frac{1-w(z_0)}{1+w(z_0)}\right) = 0$, $\Re\left(\frac{g(z_0)}{z_0 g'(z_0)}\right) > 0$, and $\frac{w(z_0)}{(1+w(z_0))^2}$ is real and positive, then $\Re\phi(z_0) \leq 0$ which is a contradiction to the fact that $\Re\phi(z) > 0$, $\forall z \in \Delta$. Therefore,

$$\Re\left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0 \quad (z \in \Delta).$$

□

Remark 5. From Theorem 5, it is easily seen that

$$\mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(0, 1).$$

Theorem 6. Let $\alpha \geq 0$ and $\beta > 0$, then

$$\mathcal{CS}(\alpha) \subset \mathcal{L}_{u,v}(\alpha, \beta)$$

if $\alpha + \beta > 1$, and

$$\mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{CS}(\alpha)$$

if $0 < \alpha + \beta < 1$.

Proof. Firstly, observe that

$$\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) = \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + (1-\alpha) \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) + (\alpha + \beta - 1) \frac{\mathcal{L}_{u,v}f(z)}{g(z)}. \quad (41)$$

Case 1. Let $\alpha + \beta = 1 \implies \beta = (1 - \alpha)$ and if $f \in \mathcal{CS}(\alpha)$, then we have

$$\Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + (1-\alpha) \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0 \quad (z \in \Delta).$$

Since $\mathcal{CS}(\alpha)$ consists of close-to-star functions (see [12, Theorem 3.1.1]), then

$$\Re\left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0.$$

So, $\alpha + \beta = 1$ in (41) implies

$$\Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0.$$

Therefore, $f \in \mathcal{L}_{u,v}(\alpha, \beta, \mu) \implies \mathcal{CS}(\alpha) \subset \mathcal{L}_{u,v}(\alpha, \beta)$ when $\alpha + \beta = 1$.

Case 2. Let $0 < \alpha + \beta < 1$, then from (41), we get the relation

$$\begin{aligned} \Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + (1-\alpha) \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) \\ = \Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) + (1-\alpha-\beta) \Re\left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right). \end{aligned} \quad (42)$$

Let $f \in \mathcal{L}_{u,v}(\alpha, \beta)$, then

$$\Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0$$

and by Theorem 5, we have $\Re\left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0$. Thus, if $0 < \alpha + \beta < 1$, then (42) implies

$$\Re\left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + (1-\alpha) \frac{\mathcal{L}_{u,v}f(z)}{g(z)}\right) > 0.$$

Therefore, $f \in \mathcal{CS}(\alpha) \implies \mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{CS}(\alpha)$ when $0 < \alpha + \beta < 1$.

□

Remark 6. If $\alpha + \beta = 1$, then $\mathcal{L}_{v,v}(\alpha, \beta) = \mathcal{CS}(\alpha)$.

Theorem 7. Let $\alpha \geq 0$ and $\beta > 0$, then

$$(1) \quad \mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha', \beta') \text{ for } 0 \leq \alpha' < \alpha, \text{ and } 0 < \beta \leq \beta'.$$

- (2) $\mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha, \beta')$ for $\beta' > \beta$.
 (3) $\mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha', \beta)$ for $\beta > 0$ and $0 \leq \alpha' < \alpha$.

Proof. Let $f \in \mathcal{L}_{u,v}(\alpha, \beta)$.

- (1) Consider the relation

$$\begin{aligned} & \Re \left(\alpha' \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta' \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \\ &= \Re \left(\frac{\alpha'}{\alpha} \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \right) + \left(\beta' - \frac{\alpha'\beta}{\alpha} \right) \Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right). \end{aligned}$$

Now, if $0 \leq \alpha' < \alpha$ and $0 < \beta \leq \beta'$, then $0 \leq (\alpha'/\alpha) < 1$ and $(\beta' - \alpha'\beta/\alpha) = \beta(\beta'/\beta - \alpha'/\alpha)$. Therefore, if $f \in \mathcal{L}_{u,v}(\alpha, \beta)$, then using Theorem 5, implies

$$\Re \left(\alpha' \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta' \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0.$$

Therefore, $f \in \mathcal{L}_{u,v}(\alpha', \beta') \implies \mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha', \beta')$.

- (2) Consider the relation

$$\begin{aligned} & \Re \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta' \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \\ &= \Re \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) + (\beta' - \beta) \Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right). \end{aligned}$$

If $f \in \mathcal{L}_{u,v}(\alpha, \beta)$ and $\beta' > \beta$, then we have

$$\Re \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta' \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0.$$

Therefore, $f \in \mathcal{L}_{u,v}(\alpha, \beta') \implies \mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha, \beta')$.

- (3) Consider the relation

$$\begin{aligned} & \Re \left(\alpha' \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \\ &= \Re \left(\frac{\alpha'}{\alpha} \left(\alpha \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) \right) + \beta \left(1 - \frac{\alpha'}{\alpha} \right) \Re \left(\frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right). \end{aligned}$$

If $f \in \mathcal{L}_{u,v}(\alpha, \beta)$, $\beta > 0$, and $0 \leq \alpha' < \alpha$, then α'/α and $\beta(1 - \alpha'/\alpha)$ are positive values, hence,

$$\Re \left(\alpha' \frac{(\mathcal{L}_{u,v}f(z))'}{g'(z)} + \beta \frac{\mathcal{L}_{u,v}f(z)}{g(z)} \right) > 0 \quad (z \in \Delta).$$

Therefore, $f \in \mathcal{L}_{u,v}(\alpha', \beta) \implies \mathcal{L}_{u,v}(\alpha, \beta) \subset \mathcal{L}_{u,v}(\alpha', \beta)$.

□

4.5. Coefficient Estimates for $f \in \mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$.

Theorem 8. Let f in (1) belongs to $\mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$, then for $(\alpha + \beta) = 1$,

$$|a_2| \leq \frac{2}{\lambda(2\alpha + \beta)|(k)_1|} [\mu + \lambda|3 - (\alpha + 2\beta)|] \quad (43)$$

if $\lambda > 0$;

$$\begin{aligned} |a_3| \leq & \frac{1}{(3\alpha + \beta)\lambda|(k)_2|} \left\{ 2\mu + 2\mu^2 \frac{(4\alpha + \beta)(1 - \lambda)}{(2\alpha + \beta)^2 \lambda} + 3\lambda|4 - (\alpha + 3\beta)| \right. \\ & + 4\lambda\mu \left| 3 - \left[E - \frac{(1 - \lambda)}{\lambda} F \right] \right| + 4 \left| \frac{\lambda[(1 - \lambda)(\alpha + 4\beta) + (9\lambda - 5)]}{2} \right. \\ & \left. \left. - \left(E - \frac{(1 - \lambda)}{2\lambda} F \right) \lambda^2[3 - (\alpha + 2\beta)] \right| \right\} \quad (44) \end{aligned}$$

if $0 < \lambda \leq 1$; and

$$|a_3| \leq \frac{1}{(3\alpha + \beta)\lambda|(k)_2|} \left\{ 2\mu + 3\lambda|4 - (\alpha + 3\beta)| + 4\lambda\mu \left| 3 - \left[E + \frac{(\lambda - 1)}{\lambda} F \right] \right| \right. \\ \left. + 4 \left| \frac{\lambda[(9\lambda - 5) - (\lambda - 1)(\alpha + 4\beta)]}{2} - \left(E + \frac{(\lambda - 1)}{2\lambda} F \right) \lambda^2[3 - (\alpha + 2\beta)] \right| \right\} \quad (45)$$

if $\lambda \geq 1$. Note:

$$E = \frac{2}{2\alpha + \beta} \quad \text{and} \quad F = \frac{(4\alpha + \beta)[3 - (\alpha + 2\beta)]}{(2\alpha + \beta)^2}.$$

Proof. Since $f \in \mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$ and $(\alpha + \beta) = 1$, then we can write (7) as

$$\left(\alpha \frac{((\mathcal{L}_{u,v}f(z))')^\lambda}{(g'(z))^\lambda} + \beta \frac{(\mathcal{L}_{u,v}f(z))^\lambda}{(g(z))^\lambda} \right) = \wp(z)$$

or

$$\alpha((\mathcal{L}_{u,v}f(z))')^\lambda(g(z))^\lambda + \beta(\mathcal{L}_{u,v}f(z))^\lambda(g'(z))^\lambda = \wp(z)(g(z))^\lambda(g'(z))^\lambda. \quad (46)$$

Putting (2), (3), and (4) into (46) simplifies to

$$\alpha z^\lambda(1 + A_1z + A_2z^2 + \cdots) + \beta z^\lambda(1 + B_1z + B_2z^2 + \cdots) = z^\lambda(1 + C_1z + C_2z^2 + \cdots) \quad (47)$$

where

$$\begin{aligned} A_1 &= \lambda b_2 + 2\lambda(k)_1 a_2, \\ A_2 &= \lambda b_3 + \frac{\lambda(\lambda - 1)}{2} b_2^2 + 2\lambda^2(k)_1 a_2 b_2 + 3\lambda(k)_2 a_3 + 2\lambda(\lambda - 1)(k)_1^2 a_2^2, \\ B_1 &= 2\lambda b_2 + \lambda(k)_1 a_2, \\ B_2 &= 3\lambda b_3 + 2\lambda(\lambda - 1)b_2^2 + 2\lambda^2(k)_1 a_2 b_2 + \lambda(k)_2 a_3 + \frac{\lambda(\lambda - 1)}{2} (k)_1^2 a_2^2, \\ C_1 &= p_1 + 3\lambda b_2, \end{aligned}$$

and

$$C_2 = p_2 + 3\lambda b_2 p_1 + 4\lambda b_3 + \frac{(9\lambda - 5)\lambda b_2^2}{2}.$$

Equating the coefficients implies

$$(2\alpha + \beta)\lambda(k)_1 a_2 = [p_1 + 3\lambda b_2] - (\alpha + 2\beta)\lambda b_2 \quad (48)$$

and

$$(3\alpha + \beta)\lambda(k)_2 a_3 = p_2 + \frac{(4\alpha + \beta)(1 - \lambda)}{(2\alpha + \beta)^2 \lambda} \times \frac{p_1^2}{2} + [4 - (\alpha + 3\beta)]\lambda b_3 + D_1 b_2 p_1 + D_2 b_2^2, \quad (49)$$

where

$$D_1 = \lambda \left[3 - \left(E - \frac{(1 - \lambda)}{\lambda} F \right) \right]$$

and

$$D_2 = \frac{\lambda[(1 - \lambda)(\alpha + 4\beta) - (5 - 9\lambda)]}{2} - \left(E - \frac{(1 - \lambda)}{2\lambda} F \right) \lambda^2[3 - (\alpha + 2\beta)].$$

Using Lemmas 1 and 4 in (48) and (49) gives the inequalities (43) and (44). Now, if $\lambda \geq 1$, we write

$$(3\alpha + \beta)\lambda(k)_2 a_3 = p_2 - \frac{(4\alpha + \beta)(\lambda - 1)}{(2\alpha + \beta)^2 \lambda} \times \frac{p_1^2}{2} + [4 - (\alpha + 3\beta)]\lambda b_3 + D_1 b_2 p_1 + D_2 b_2^2$$

or

$$(3\alpha + \beta)\lambda|(k)_2||a_3| \leq \left| p_2 - \frac{(4\alpha + \beta)(\lambda - 1)}{(2\alpha + \beta)^2 \lambda} \times \frac{p_1^2}{2} \right| + |4 - (\alpha + 3\beta)|\lambda|b_3| + |D_1||b_2||p_1| + |D_2||b_2^2|.$$

Now, consider the expression

$$\left| p_2 - \frac{(4\alpha + \beta)(\lambda - 1)}{(2\alpha + \beta)^2\lambda} \times \frac{p_1^2}{2} \right| = \left| p_2 - \eta \times \frac{p_1^2}{2} \right|$$

where

$$\eta = \frac{(4\alpha + \beta)(\lambda - 1)}{(2\alpha + \beta)^2\lambda} \geq 0$$

with $\lambda \geq 1$. Using Lemmas 1, 4 and 6 yields the results in the Theorem. $\square$

Remark 7. Setting $u = v = \mu = \beta = 1$, and $\alpha = 0$ in (43), (44) and (45) gives

$$|a_2| \leq 2 \left(1 + \frac{1}{\lambda} \right) \quad \text{and} \quad |a_3| \leq \begin{cases} 3 + \frac{2(2\lambda + 1)}{\lambda^2} & (0 < \lambda \leq 1), \\ 3 \left(1 + \frac{2}{\lambda} \right) & (\lambda \geq 1). \end{cases}$$

which are the results of Babalola et al. [3, Theorem 1].

5. CONCLUSION

In this study, we introduced and studied some interesting properties of two subclasses of close-to-star functions. The classes were represented by the notations: $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $\mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$; and their definitions incorporated the idea of the use of the well-known Carlson-Shaffer linear operator, set theory, and the linear combination of two geometrical conditions, where one of the conditions is for a close-to-starlikeness.

Using various approaches, we were able to determine the integral representation formula which shows that class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ is non-empty, the extremal function for the class, some sharp upper coefficient estimates, the Fekete-Szegő estimates both for real and complex parameters and several inclusion relations for the functions f in the class $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$. Afterwards, some upper coefficient estimates for functions f in the class $\mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$ were established.

Astoundingly, several subclasses of the new classes $\mathcal{L}_{u,v}(\alpha, \beta, \mu)$ and $\mathcal{L}_{u,v}^\lambda(\alpha, \beta, \mu)$ were reported in many of the remarks. These show that the new classes generalized several existing and new subclasses of analytic functions.

We earnestly hope that these results will motivate researchers for further studies on close-to-star functions and many of its generalizations involving the use of other known (or new) (q-)operators in the theory of analytic-univalent functions.

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