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A New Application of Miller and Mocanu Lemma

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ABSTRACT: In this paper, we discuss a new application of Miller and Mocanu lemma (Miller and Mocanu, 1978) for $f(z)$ concerned with the classes $R(\alpha,\rho)$ and $P(\alpha,r)$.

Keywords: Analytic function, Jack lemma, Miller and Mocanu lemma

Miller ve Mocanu Lemmasının Yeni Bir Uygulaması

ÖZET: Bu çalışmada, $R(\alpha,\rho)$ ve $P(\alpha,r)$ sınıflarına ait $f(z)$ fonksiyonu için Miller ve Mocanu lemmasının (Miller and Mocanu, 1978) yeni bir uygulaması verilecektir.

Anahtar kelimeler: Analitik fonksiyon, Jack lemması, Miller ve Mocanu lemması

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INTRODUCTION

Let A be the class of normalized functions $f(z)$ $(f'(0)-1=f(0)=0)$, which are analytic in the open unit disk $\mathbb{U}=\{z\in\mathbb{C}:|z|<1\}$ given by the power series:

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

Let $R(\alpha,\rho)$ be the subclass of A consisting of functions $f(z)$ which satisfy

$$\left|\frac{f(z)}{z}-1\right|<(1-\alpha)\rho \qquad (z\in\mathbb{U}) \tag{2}$$

for some real $\alpha\,(0\leq\alpha<1)$ and $\rho>0$.

Further, let $P(\alpha,\rho)$ denote the subclass of A consisting of all functions $f(z)$ satisfying

$$\left|f'(z)-1\right|<(1-\alpha)\rho \qquad (z\in\mathbb{U}) \tag{3}$$

for some real $\alpha\,(0\leq\alpha<1)$ and $\rho>0$.

Singh and Singh (Singh and Singh, 1982) have considered some properties for the class $P(0,1)$. Further, Mocanu (Mocanu, 1988) and Nunokawa, Owa, Polatoglu, Caglar and Duman (Nunokawa et al., 2010) have discussed some problems concerned with the class $P(\alpha,\rho)$ with $(1-\alpha)\rho=\frac{2}{\sqrt{5}}$.

MATERIAL AND METHOD

In order to discuss our problem, we have to recall here the following lemma as a result of Miller and Mocanu (Miller and Mocanu, 1978) (also a result of Jack (Jack, 1971)).

Lemma 2.1 Let $w(z)$ be analytic in $\mathbb{U}$ with $w(0)=0$. If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|, \tag{4}$$

then we can write

$$z_0 w'(z_0)=kw(z_0), \tag{5}$$

where k is real and $k\geq 1$.

In the present paper, we discuss a new application of Lemma 2.1 for $f(z)$ concerned with the classes $R(\alpha,\rho)$ and $P(\alpha,\rho)$.

Let $f(z)\in A$. If there exists some convex function $g(z)\in A$ such that

$$Re\left(\frac{f'(z)}{g'(z)}\right)>\alpha \qquad (z\in\mathbb{U})$$

for some real $\alpha\,(0\leq\alpha<1)$, then we say that $f(z)$ is close-to-convex of order α in $\mathbb{U}$.

RESULTS AND DISCUSSION

First application of Lemma 2.1 is included in following theorem.

Theorem 3.1 If $f(z)\in A$ satisfies

$$Re\left(\frac{f(z)}{zf'(z)}\right)>\frac{1+(1-\alpha)\rho}{1+2(1-\alpha)\rho} \qquad (z\in\mathbb{U}) \tag{6}$$

for some real $\alpha\,(0\leq\alpha<1)$ and $\rho\geq\frac{1}{2(1-\alpha)}$, then $f(z) \in R(\alpha, \rho)$.

Proof Let us consider $w(z)$ defined by

$$\frac{f(z)}{z} = 1+(1-\alpha)w(z) \qquad (z \in \mathbb{U}) \tag{7}$$

for $f(z) \in A$ satisfying the condition (6). Then we see that

$$\frac{f(z)}{zf'(z)} = \frac{1+(1-\alpha)w(z)}{1+(1-\alpha)w(z)\left(1+\frac{zw'(z)}{w(z)}\right)}, \tag{8}$$

that is, that

$$Re\left(\frac{f(z)}{zf'(z)}\right) = Re\left(\frac{1+(1-\alpha)w(z)}{1+(1-\alpha)w(z)\left(1+\frac{zw'(z)}{w(z)}\right)}\right) > \frac{1+(1-\alpha)\rho}{1+2(1-\alpha)\rho} \tag{9}$$

for (z∈$\mathbb{U}$). Because $w(z)$ is analytic in $\mathbb{U}$ with $w(0)=0$, we suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)| = |w(z_0)| = \rho \quad \text{with} \quad \rho \geq \frac{1}{2(1-\alpha)}.$$

Then Lemma 2.1 gives us that $w(z_0)=\rho e^{i\theta}$

and $z_0 w'(z_0) = k w(z_0)$ $(k \geq 1)$.

Therefore, we have that

$$Re\left(\frac{f(z_0)}{z_0 f'(z_0)}\right) = Re\left(\frac{1+(1-\alpha)\rho e^{i\theta}}{1+(1-\alpha)\rho e^{i\theta}(1+k)}\right) \tag{10}$$

$$= \frac{1+(1-\alpha)^2\rho^2(1+k)+(1-\alpha)\rho(2+k)\cos\theta}{1+(1-\alpha)^2\rho^2(1+k)^2+2(1-\alpha)\rho(1+k)\cos\theta}$$

$$\leq \frac{1+(1-\alpha)\rho}{1+(1-\alpha)\rho(1+k)}$$

$$\leq \frac{1+(1-\alpha)\rho}{1+2(1-\alpha)\rho} \qquad \left(\rho \geq \frac{1}{2(1-\alpha)}\right).$$

This contradicts our condition (6) of the theorem.

Therefore, we see that there isn't $z_0 \in \mathbb{U}$ such that $|w(z_0)| = \rho$. This means that $|w(z)| < \rho$ (z∈$\mathbb{U}$)

Thus we conclude that

$$\left|\frac{f(z)}{z}-1\right| = (1-a)|w(z)| < (1-\alpha)\rho \quad (z\in\mathbb{U}),$$

that is, that $f(z) \in R(\alpha, \rho)$.

Theorem 3.2 If $f(z) \in A$ satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{1+2(1-\alpha)\rho}{1+(1-\alpha)\rho} \quad (z\in\mathbb{U}) \qquad (11)$$

for some $\alpha\,(0 \le \alpha < 1)$ and $\rho \ge \frac{1}{1-\alpha}$, then $f(z) \in P(\alpha, \rho)$.

Proof Defining the function $w(z)$ by (7), we have that

$$Re\left(\frac{zf'(z)}{f(z)}\right) = Re\left(\frac{1+(1-\alpha)\rho w(z)\left(1+\frac{zw'(z)}{w(z)}\right)}{1+(1-\alpha)\rho w(z)}\right) < \frac{1+2(1-\alpha)\rho}{1+(1-\alpha)\rho} \qquad (12)$$

for $z\in\mathbb{U}$. If we suppose that there exists a point $z_0 \in U$ such that

$$\max_{|z|\le|z_0|}|w(z)| = |w(z_0)| = \rho \qquad \left(\rho \ge \frac{1}{1-\alpha}\right)$$

then we can write that $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0) \quad (k \ge 1)$. This shows that

$$Re\left(\frac{z_0 f'(z_0)}{f(z_0)}\right) = Re\left(\frac{1+(1-\alpha)\rho e^{i\theta}(1+k)}{1+(1-\alpha)\rho e^{i\theta}}\right) \qquad (13)$$

$$= \frac{1+(1-\alpha)^2\rho^2(1+k)+(1-\alpha)\rho(2+k)\cos\theta}{1+(1-\alpha)^2\rho^2+2(1-\alpha)\rho\cos\theta}$$

$$\ge \frac{1+(1-\alpha)\rho(1+k)}{1+(1-\alpha)\rho}$$

$$\ge \frac{1+2(1-\alpha)\rho}{1+(1-\alpha)\rho} \qquad \left(\rho \ge \frac{1}{1-\alpha}\right),$$

which contradicts the condition (11). Therefore, there isn't $z_0\in\mathbb{U}$ such that $|w(z_0)| = \rho \ge \frac{1}{1-\alpha}$. Thus we see that

$$\left|\frac{f(z)}{z}-1\right| = (1-\alpha)|w(z)| < (1-\alpha)\rho \quad (z\in\mathbb{U}),$$

that is, that $f(z) \in R(\alpha, \rho)$.

Example 3.1 Let us consider the function $f(z)$ given by

$$f(z) = z+(1-\alpha)\rho\, z^2 \quad (z\in\mathbb{U}) \qquad (14)$$

for some $\alpha\,(0 \le \alpha < 1)$ and $\rho \ge \frac{1}{1-\alpha}$. Then we have that, for $z = e^{i\theta}$,

$$Re\left(\frac{zf'(z)}{f(z)}\right)=Re\left(\frac{1+2(1-\alpha)\rho e^{i\theta}}{1+(1-a)re^{i\theta}}\right)\geq\frac{1+2(1-\alpha)\rho}{1+(1-a)\rho}. \qquad (15)$$

Thus $f(z)$ satisfies the condition (11) and that

$$\left|\frac{f(z)}{z}-1\right|=(1-\alpha)\rho|z|<(1-\alpha)\rho \quad (z\in\mathbb{U}) \qquad (16)$$

Therefore, $f(z)$ given by (14) belongs to the class $R(\alpha,\rho)$.

We give a condition for starlikeness of functions $f(z)\in A$.

Theorem 3.3 If $f(z)\in A$ satisfies

$$Re\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\}<1+\frac{\rho}{2(1+\rho)} \quad (z\in\mathbb{U}) \qquad (17)$$

for some $\rho>0,$ then

$$\left|\frac{zf'(z)}{f(z)}-1\right|<\rho \qquad (z\in Y) \qquad (18)$$

Proof Let define the function $w(z)$ by

$$\frac{zf'(z)}{f(z)}=1+w(z). \qquad (19)$$

Then $w(z)$ is analytic in Y with $w(0)=0.$ It follows from (19) that

$$1+\frac{zf''(z)}{f'(z)}=1+w(z)+\frac{zw'(z)}{1+w(z)} \qquad (20)$$

so that

$$\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}=1+\frac{zw'(z)}{(1+w(z))^2}. \qquad (21)$$

Thus we see that

$$Re\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\}=Re\left\{1+\frac{zw'(z)}{(1+w(z))^2}\right\}<1+\frac{\rho}{2(1+\rho)} \tag{22}$$

for $z\in\mathbb{U}$.

Suppose that there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\leq|z_0|}|w(z)|=|w(z_0)|=\rho \qquad (\rho>0).$$

Then, Lemma 2.1 shows that $w(z_0)=\rho e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)$ $(k\geq 1)$. This implies that

$$Re\left\{\frac{1+\frac{z_0f''(z_0)}{f'(z_0)}}{\frac{z_0f'(z_0)}{f(z_0)}}\right\}=Re\left\{1+\frac{k\rho e^{i\theta}}{(1+\rho e^{i\theta})^2}\right\}=1+\frac{k\rho}{2(\rho+\cos\theta)}$$

$$\geq 1+\frac{k\rho}{2(1+\rho)}$$

$$\geq 1+\frac{\rho}{2(1+\rho)} \qquad (\rho>0)$$

This contradicts the condition (17) of the theorem.

Thus $|w(z)|<\rho$ for $z\in\mathbb{U}$. This means that

$$\left|\frac{zf'(z)}{f(z)}-1\right|=|w(z)|<\rho \qquad (z\in\mathbb{U}).$$

Theorem 3.4 Let $f(z)\in A$. If there exists some convex function $g(z)\in A$ such that

$$Re\left(\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right)<\frac{(1-\alpha)\rho}{1+(1-\alpha)\rho} \qquad (z\in\mathbb{U}) \tag{23}$$

for some real $\alpha\,(0\le\alpha<1)$ and $\rho\ge\frac{1}{\sqrt{2}(1-\alpha)}$, then

$$\left|\frac{f'(z)}{g'(z)}-1\right|<(1-\alpha)\rho \qquad (z\in\mathbb{U}) \tag{24}$$

Proof Let the function $w(z)$ be defined by

$$\frac{f'(z)}{g'(z)}=1+(1-\alpha)w(z) \qquad (z\in\mathbb{U}) \tag{25}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0)=0$. This implies that

$$Re\left(\frac{zf''(z)}{f'(z)}-\frac{zg''(z)}{g'(z)}\right)=Re\left(\frac{(1-\alpha)zw'(z)}{1+(1-\alpha)w(z)}\right)<\frac{(1-\alpha)\rho}{1+(1-\alpha)\rho} \tag{26}$$

for $z\in$U with $0\le\alpha<1$ and $\rho\ge\frac{1}{\sqrt{2}(1-\alpha)}$.

If there exists a point $z_0\in\mathbb{U}$ such that

$$\max_{|z|\le|z_0|}|w(z)|=|w(z_0)|=\rho \qquad \left(\rho\ge\frac{1}{\sqrt{2}(1-\alpha)}\right),$$

then we can write that $w(z_0)=\rho e^{i\theta}$ and $z_0w'(z_0)=kw(z_0)\quad(k\ge1)$. Thus we see that

$$Re\left(\frac{z_0f''(z_0)}{f'(z_0)}-\frac{z_0g''(z_0)}{g'(z_0)}\right)=Re\left(\frac{(1-\alpha)k\rho e^{i\theta}}{1+(1-\alpha)\rho e^{i\theta}}\right) \tag{27}$$

$$\geq \frac{(1-\alpha)\rho k}{1+(1-\alpha)\rho}$$

$$\geq \frac{(1-\alpha)\rho}{1+(1-\alpha)\rho},$$

which contradicts our condition (23). This implies that there isn't such a point $z_0 \in \mathbb{U}$ and that $|w(z)| < \rho$ ($z \in \mathbb{U}$). Therefore, we obtain that

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < (1-\alpha)|w(z)| < (1-\alpha)\rho \qquad (z \in \mathbb{U}) \tag{28}$$

Thus, the proof of the theorem is completed.

Example 3.2 Let us consider $f(z) \in A$ and some convex function $g(z) \in A$ such that

$$\frac{f'(z)}{g'(z)} = 1 + \frac{1}{\sqrt{2}}\, z.$$

Then we have that

$$Re\left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) = Re\left(\frac{z}{\sqrt{2}+z}\right) < \sqrt{2} - 1 \quad (z \in \mathbb{U}).$$

Further, we see that

$$\left|\frac{f'(z)}{g'(z)} - 1\right| < \frac{1}{\sqrt{2}} \qquad (z \in \mathbb{U}),$$

that is, that $f(z)$ is close-to-convex of order $1 - \frac{1}{\sqrt{2}}$ in $\mathbb{U}$.

For close-to-convex functions, Singh and Singh (Singh and Singh, 1982) have shown that if $f(z) \in A$ satisfies $|zf''(z)| < 1$ ($z \in \mathbb{U}$), then

$$|f'(z) - 1| < 1 \qquad (z \in \mathbb{U}),$$

which means that $f(z)$ is close-to-convex in Y. Further, Nunokawa (Nunokawa, 1993) has proved that if $f(z) \in A$ satisfies $|f''(z)| < 1$ ($z \in \mathbb{U}$), then $f(z)$ is univalent in .

We apply our new application for such problems.

Theorem 3.5 If $f(z) \in A$ satisfies

$$|zf''(z)| < (1-\alpha)\rho \quad (z \in \mathbb{U}) \tag{29}$$

for some real α $(0 \leq \alpha < 1)$ and $\rho > 0$, then

$$|f'(z) - 1| < (1-\alpha)\rho \quad (z \in \mathbb{U}). \tag{30}$$

Proof Define the function $w(z)$ by

$$f'(z) = 1 + (1-\alpha)w(z) \quad (z \in \mathbb{U}). \tag{31}$$

Then $w(z)$ is analytic in $\mathbb{U}$ and $w(0) = 0$. Let us suppose that there exists a point $z_0 \in \mathbb{U}$ such that

$$\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = \rho,$$

then $w(z)$ satisfies $w(z_0) = \rho e^{i\theta}$ and $z_0 w'(z_0) = kw(z_0)$ $(k \geq 1)$ by Lemma 2.1. This gives us that

$$|z_0 f''(z_0)| = (1-\alpha)|z_0 w'(z_0)| = (1-\alpha)k\rho \geq (1-\alpha)\rho, \tag{32}$$

which contradicts the inequality (29).

Thus we say that there is no such a point $z_0 \in \mathbb{U}$. This means that $|w(z)| < \rho$ ($z \in \mathbb{U}$), that is

$$|f'(z) - 1| = (1-\alpha)|w(z)| < (1-\alpha)\rho \quad (z \in \mathbb{U}) \tag{33}$$

If we take $\rho = 1$ in Theorem 3.5, then we have

Remark 3.1 Corollary 4.5 is the generalization of the results by Singh and Singh (Singh and Singh, 1982) and Nunokawa (Nunokawa, 1993) .

Example 3.3 Let define the function $f(z)$ by

$$f(z) = z + \frac{(1-\alpha)\rho}{2} z^2.$$

It follows that

$$|zf''(z)| = (1-\alpha)\rho|z| < (1-\alpha)\rho \quad (z \in \mathbb{U}).$$

Thus $f(z)$ satisfies the inequality (29) of Theorem 3.5.

Further, $f(z)$ satisfies

$$\left|f'(z)-1\right| = (1-\alpha)\rho|z| < (1-\alpha)\rho \quad (z\in\mathbb{U}),$$

which is the inequality (30).

Remark 3.2 If $0 < (1-\alpha)\rho \leq 1$ in Theorem 3.5, then the inequality (30) shows that

$$\left|\arg f'(z)\right| < \sin^{-1}(1-\alpha)\rho \quad (z\in\mathbb{U}).$$

CONCLUSIONS

Corollary 4.1 If $f(z) \in A$ satisfies

$$Re\left(\frac{f(z)}{zf'(z)}\right) > \frac{1+\rho}{1+2\rho} \quad (z\in\mathbb{U}) \quad (34)$$

for some $\rho \geq \frac{1}{2}$, then $f(z) \in R(\alpha, \rho)$.

Corollary 4.2 If $f(z) \in A$. satisfies

$$Re\left(\frac{zf'(z)}{f(z)}\right) < \frac{1+2\rho}{1+\rho} \quad (z\in\mathbb{U}) \quad (35)$$

for some $\rho \geq 1$ then $f(z) \in P(\theta, \rho)$.

Corollary 4.3 If $f(z) \in A$ satisfies

$$Re\left\{\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right\} < \frac{5-3\alpha}{2(2-\alpha)} \quad (z\in\mathbb{U}) \quad (36)$$

for some real $\alpha\ (0 \leq \alpha < 1)$, then

$$\left|\frac{zf'(z)}{f(z)}-1\right| < 1-\alpha \quad (z\in\mathbb{U}) \quad (37)$$

This implies that $f(z)$ is starlike of order α in $\mathbb{U}$ so that $f(z) \in S^*(\alpha)$.

Corollary 4.4 Let $f(z) \in A$. If there exists some convex function $g(z) \in A$ such that

$$Re\left(\frac{zf''(z)}{f'(z)} - \frac{zg''(z)}{g'(z)}\right) < \sqrt{2}-1 \quad (z\in\mathbb{U}), \quad (38)$$

then

$$\left|\frac{f'(z)}{g'(z)}-1\right| < \frac{1}{\sqrt{2}} \quad (z\in\mathbb{U}). \quad (39)$$

This implies that $f(z)$ is close-to-convex of order $1-\frac{1}{\sqrt{2}}$ and that

$$\left|\arg\left(\frac{f'(z)}{g'(z)}\right)\right| < \frac{\pi}{4} \quad (z\in\mathbb{U}). \quad (40)$$

Corollary 4.5 If $f(z) \in A$ satisfies

$$\left|zf''(z)\right| < 1-\alpha \quad (z\in\mathbb{U}). \quad (41)$$

for some real $\alpha\ (0 \leq \alpha < 1)$, then

$$\left|f'(z)-1\right| < 1-\alpha \quad (z\in\mathbb{U}) \quad (42)$$

which means that $f(z)$ is close-to-convex of order α in $\mathbb{U}$.

Corollary 4.6 If $f(z) \in A$ satisfies the inequality (29) with $0 \leq \alpha < 1$ and $0 < \rho \leq \frac{1}{1-\alpha}$, then we have

$$\left|f'(z)-1\right| < 1-\left(1-\left(1-\alpha\right)\rho\right) \quad (z\in\mathbb{U}). \qquad (43)$$

which shows that $f(z)$ is close-to-convex of order $1-(1-\alpha)\rho$ in $\mathbb{U}$.

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