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# A Different Solution Method for the Confluent Hypergeometric Equation

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**ABSTRACT:** Fractional calculus theory includes definition of the derivatives and integrals of arbitrary order. This theory is used to solve some classes of singular differential equations and fractional order differential equations. One of these equations is the confluent hypergeometric equation. In this paper, we intend to solve this equation by applying $N^{\mu}$ method as a different solution method.

**Keywords:** Fractional calculus theory, fractional solutions, Confluent hypergeometric equation, $N^{\mu}$ method, generalized Leibniz rule

# Konfluent Hipergeometrik Denklemi İçin Farklı Bir Çözüm Metodu

**ÖZET:** Kesirli hesap teorisi, keyfi mertebeden türev ve integral tanımını kapsamaktadır. Diferansiyel denklemlerin ve kesirli diferansiyel denklemlerin bazı sınıflarını çözmek için bu teori kullanılmaktadır. Bu denklemlerden birisi konfluent hipergeometrik denklemidir. Bu makalede, farklı bir çözüm metodu olarak $N^{\mu}$ metodunun uygulanmasıyla bu denklemi çözmeyi hedeflemekteyiz.

**Anahtar Kelimeler:** Kesirli hesap teorisi, kesirli çözümler, Konfluent hipergeometrik denklemi, $N^{\mu}$ metot, genelleştirilmiş Leibniz kuralı

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## INTRODUCTION

The fractional calculus theory enables a set of axioms and methods to generalize the coordinate and corresponding derivative notions from integer $k$ to arbitrary order $\mu$, $\{x^k, \partial^k/\partial x^k\} \to \{x^\mu, \partial^\mu/\partial x^\mu\}$ in a good light. Fractional differential equations are applied in a widespread manner in robot technology, PID control systems, Schrödinger equation, KdV equations, heat transfer, relativity theory, economy, filtration, controller design, mechanics, optics, modelling and so on (Akgül, 2014; Akgül et al., 2015).

Riemann-Liouville fractional differentiation and fractional integration that are two most important definitions of fractional calculus are, respectively,

$$ {}_aD_t^\mu f(t) = \frac{1}{\Gamma(k-\mu)} \frac{d^k}{dt^k} \int_a^t f(\tau)(t-\tau)^{k-\mu-1}\, d\tau \quad (k-1 \le \mu < k), \tag{1} $$

and,

$$ {}_aD_t^{-\mu} f(t) = \frac{1}{\Gamma(\mu)} \int_a^t f(\tau)(t-\tau)^{\mu-1}\, d\tau \quad (t > a, \mu > 0), \tag{2} $$

where $k \in \mathbb{N}$ and $\Gamma$ is Euler's function gamma (Oldham and Spanier, 1974; Miller and Ross, 1993; Podlubny, 1999; Yilmazer and Ozturk, 2013).

## MATERIAL AND METHOD

**Definition 2.1.** If the function $f(z)$ is analytic (*regular*) inside and on $C$, where $C = \{C^-, C^+\}$, $C^-$ is a contour along the cut joining the points $z$ and $-\infty + i\mathrm{Im}(z)$, which starts from the point at $-\infty$, encircles the point $z$ once counter-clockwise, and returns to the point at $-\infty$, and $C^+$ is a contour along the cut joining the points $z$ and $\infty + i\mathrm{Im}(z)$, which starts from the point at $\infty$, encircles the point $z$ once counter-clockwise, and returns to the point at $\infty$,

$$ \begin{aligned} f_\mu(z) = [f(z)]_\mu &= \frac{\Gamma(\mu+1)}{2\pi i} \int_C \frac{f(\tau)d\tau}{(\tau - z)^{\mu+1}} \quad (\mu \notin \mathbb{Z}^-), \\ f_{-k}(z) &= \lim_{\mu \to -k} f_\mu(z) \quad (k \in \mathbb{Z}^+), \end{aligned} \tag{3} $$

where $\tau \neq z$,

$$-\pi \leq \arg(\tau - z) \leq \pi \quad \text{for } C^-,$$
$$0 \leq \arg(\tau - z) \leq 2\pi \quad \text{for } C^+. \tag{4}$$

In that case, $f_\mu(z)\ (\mu > 0)$ is the fractional derivative of $f(z)$ of order $\mu$ and $f_\mu(z)\ (\mu < 0)$ is the fractional integral of $f(z)$ of order $-\mu$, confirmed (*in each case*) that

$$\left|f_\mu(z)\right| < \infty \quad (\mu \in \mathbb{R}). \tag{5}$$

(Yilmazer and Ozturk, 2013).

**Lemma 2.1.** (Linearity) When fractional order derivatives $f_\mu$ and $g_\mu$ exist, then

$$\textbf{(i)} \quad [c_1 f(z)]_\mu = c_1 [f(z)]_\mu,$$
$$\textbf{(ii)} \quad [c_1 f(z) + c_2 g(z)]_\mu = c_1 [f(z)]_\mu + c_2 [g(z)]_\mu, \tag{6}$$

where $f(z)$, $g(z)$ are analytic and single-valued functions, $c_1$ and $c_2$ are constants and, $\mu \in \mathbb{R}, z \in \mathbb{C}$.

**Lemma 2.2.** (Index law) If fractional order derivatives $(f_\nu)_\mu$ and $(f_\mu)_\nu$ exist, then

$$\{[f(z)]_\nu\}_\mu = [f(z)]_{\nu+\mu} = \{[f(z)]_\mu\}_\nu, \tag{7}$$

where $f(z)$ is an analytic and single-valued function, $\nu$ and $\mu \in \mathbb{R}$, $z \in \mathbb{C}$ , and $\left|\frac{\Gamma(\nu+\mu+1)}{\Gamma(\nu+1)\Gamma(\mu+1)}\right| < \infty$.'

**Property 2.1.** Let $\lambda$ be a constant. So,

$$\left(\mathrm{e}^{\lambda z}\right)_\nu = \lambda^\nu \mathrm{e}^{\lambda z} \quad (\lambda \neq 0, \nu \in \mathbb{R}, z \in \mathbb{C}). \tag{8}$$

**Property 2.2.** Let $\lambda$ be a constant. So,

$$\left(\mathrm{e}^{-\lambda z}\right)_\nu = \mathrm{e}^{-i\pi\nu} \lambda^\nu \mathrm{e}^{-\lambda z} \quad (\lambda \neq 0, \nu \in \mathbb{R}, z \in \mathbb{C}). \tag{9}$$

**Property 2.3.** Let $\lambda$ be a constant. So,

$$\left(z^\lambda\right)_\nu = \mathrm{e}^{-i\pi\nu} z^{\lambda-\nu} \frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)} \quad \left(\nu \in \mathbb{R}, z \in \mathbb{C}, \left|\frac{\Gamma(\nu-\lambda)}{\Gamma(-\lambda)}\right| < \infty\right). \tag{10}$$

**Property 2.4.**

$$\Gamma(z+1) = z\Gamma(z+1) = z!, \tag{11}$$

and,

$$\Gamma(\nu - k) = (-1)^k \frac{\Gamma(\nu)\Gamma(1-\nu)}{\Gamma(k+1-\nu)}, \tag{12}$$

where $k \in \mathbb{Z}_0^+$ and $\nu \in \mathbb{R}$.

**Lemma 2.3.** ($N^\mu$ method) If fractional order derivatives $f_\mu$ and $g_\mu$ exist, then generalized Leibniz rule is

$$N^\mu(f\ g) = (f.g)_\mu = \sum_{k=0}^{\infty} \frac{\Gamma(\mu+1)}{\Gamma(\mu+1-k)\Gamma(k+1)} f_{\mu-k}\ g_k, \tag{13}$$

where $f(z)$ and $g(z)$ are single-valued and analytic functions, $\mu \in \mathbb{R},\ z \in \mathbb{C}$ and $\left|\frac{\Gamma(\mu+1)}{\Gamma(\mu-k+1)\Gamma(k+1)}\right| < \infty$.

## RESULTS AND DISCUSSION

The hypergeometric equation

$$z(1-z)\frac{d^2y(z)}{dz^2} + [c-(a+b+1)z]\frac{dy(z)}{dz} - aby(z) = 0, \tag{14}$$

has three regular singular points at $z = 0,1$ and $\infty$ $(a, b$ *and* $c$ *are parameters)*. The singularities can be merged at $b$ and infinity, where $z = x/b$ and $b \to \infty$. Thus, confluent equation is

$$x\frac{d^2y}{dx^2}+(c-x)\frac{dy}{dx}-ay=0, \tag{15}$$

solutions of which are the confluent hypergeometric functions, which are defined as $M(a,c;x)$. The confluent hypergeometric equation has a regular singular point at $x=0$ and an essential singularity at infinity. $J_n(x)$ (*Bessel functions*) and $L_n(x)$ (*Laguerre polynomials*), can be formed in terms of the solutions of the confluent hypergeometric equation as

$$J_n(x)=\frac{\mathrm{e}^{-ix}}{n!}\left(\frac{x}{2}\right)^n M\left(n+\frac{1}{2},2n+1;2ix\right),$$

$$L_n(x)=M(-n,1;x).$$

Linearly independent solutions of "Eq. 15." are defined as

$$y_1(x)=M(a,c;x)=1+\frac{a}{c}\frac{x}{1!}+\frac{a(a+1)}{c(c+1)}\frac{x^2}{2!}+\frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{x^3}{3!}+\cdots,$$

$$(c\neq 0,-1,-2,\dots),$$

and,

$$y_2(x)=x^{1-c}M(a+1-c,2-c;x)\quad (c\neq 2,3,4,\dots),$$

Integral representation of the confluent hypergeometric functions ${}_1F_1(a,b;x)$ can be defined as

$$M(a,c;x)=\frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}\int_0^1 \mathrm{e}^{xt}t^{a-1}(1-t)^{c-a-1}dt \quad (a,c\in\mathbb{R},c>a>0).$$

(Bayın, 2006).

**Remark 3.1.** The familiar Bessel differential equation of general order $l^2$:

$$z^2\frac{d^2f}{dz^2}+z\frac{df}{dz}+(z^2-l^2)f=0,$$

which is named after F. Wilheim Bessel. More precisely, just as in the earlier works (Lin et al., 2005; Wang et al., 2006), we aim here at demonstrating how the underlying simple fractional-calculus approach to the solutions of the classical differential equation, which were considered in the earlier works (Lin et al., 2005; Wang et al., 2006; Akgül et al., 2013), would lead us analogously to several interesting consequences including *(for example)* an alternative investigation of solutions of the confluent hypergeometric equation.

**Theorem 3.1.** Let $y \in \{y: 0 \neq |\, y_\mu| < \infty, \mu \in \mathbb{R}\}$. "Eq. 15." can be written as

$$xy_2 + (c - x)y_1 - ay = 0 \quad (x \neq 0). \tag{16}$$

And "Eq. 16." has particular solutions as follows

$$y^{(\mathrm{I})} = A[x^{a-c}\mathrm{e}^x]_{a-1}, \tag{17}$$

$$y^{(\mathrm{II})} = Bx^{-(c+1)}[x^{1+a}\mathrm{e}^x]_{-(2+c-a)}, \tag{18}$$

where $y_k = d^k y/dx^k \quad (k = 0,1,2, \dots), y_0 = y = y(x), A, B$ are constants.

**Proof. (I)** At first, by applying $N^\mu$ method to the both sides of "Eq. 16.", we obtain

$$xy_{2+\mu} + (\mu + c - x)y_{1+\mu} - (\mu + a)y_\mu = 0. \tag{19}$$

If we suppose that

$$\mu + a = 0, \tag{20}$$

then, we have

$$\mu = -a. \tag{21}$$

By substituting "Eq. 21." into "Eq. 19.", we obtain

$$(y_{1-a})_1 + [(c - a)x^{-1} - 1]y_{1-a} = 0. \tag{22}$$

Let

$$y_{1-a} = u = u(x) \quad [y(x) = u_{a-1}]. \tag{23}$$

So, we have differential equation as

$$u_1 + [(c - a)x^{-1} - 1]u = 0. \tag{24}$$

Solution of the "Eq. 24." is

$$u = Ax^{a-c}\mathrm{e}^x, \tag{25}$$

and we obtain a fractional solution of the "Eq. 16." as

$$y^{(\mathrm{I})} = A[x^{a-c}\mathrm{e}^{x}]_{a-1}. \tag{26}$$

**(II)** We suppose that

$$y = x^{t}h \quad (x \neq 0), \tag{27}$$

where $h = h(x)$. Then,

$$y_1 = tx^{t-1}h + x^{t}h_1, \tag{28}$$

and,

$$y_2 = t(t-1)x^{t-2}h + 2tx^{t-1}h_1 + x^{t}h_2. \tag{29}$$

By substituting "Eq. 27.", "Eq. 28." and "Eq. 29." into "Eq. 16.", we have

$$xh_2 + (2t + c - x)h_1 + [t(t + c + 1)x^{-1} - (t + a)]h = 0. \tag{30}$$

We suppose that

$$t(t + c + 1) = 0. \tag{31}$$

So,

$$t = 0 \quad \text{or} \quad t = -(c + 1). \tag{32}$$

**(II-i)** We have "Eq. 16." from "Eq. 30.", where $t = 0$ .

**(II-ii)** We write equation as

$$xh_2 - (c + 2 + x)h_1 + (c + 1 - a)h = 0. \tag{33}$$

where $t = -(c + 1)$. Now, by applying N method to the both sides of "Eq. 33.", we obtain

$$xh_{2+\mu} + [\mu - (c + 2 + x)]h_{1+\mu} + (-\mu + c + 1 - a)h_{\mu} = 0. \tag{34}$$

If we choose that

$$-\mu + c + 1 - a = 0, \tag{35}$$

then, we find

$$\mu = c + 1 - a. \tag{36}$$

By substituting “Eq. 36.” into “Eq. 34.”, we have

$$(h_{2+c-a})_1 - [(1 + a)x^{-1} + 1]h_{2+c-a} = 0. \tag{37}$$

Let

$$h_{2+c-a} = w = w(x), \quad [h(x) = w_{-(2+c-a)}]. \tag{38}$$

Then, we have another differential equation as

$$w_1 - [(1 + a)x^{-1} + 1]w = 0. \tag{39}$$

The solution of “Eq. 39” is

$$w = Bx^{1+a}\mathrm{e}^x, \tag{40}$$

and,

$$h = B(x^{1+a}\mathrm{e}^x)_{-(2+c-a)}, \tag{41}$$

and finally, we find another fractional solution of the “Eq. 16.” as

$$y^{(\mathrm{II})} = Bx^{-(c+1)}(x^{1+a}\mathrm{e}^x)_{-(2+c-a)}. \tag{42}$$

**Example 3.1.** Let $a = 2$ and $c = 1$ for “Eq. 16.”. So, we have equation as

$$xy_2 + (1 - x)y_1 - 2y = 0 \quad (x \neq 0). \tag{43}$$

Therefore, we can write solutions of the “Eq. 43.” as follows

$$y^{(\mathrm{I})} = A(x\mathrm{e}^x)_1,$$

$$y^{(\mathrm{II})} = Bx^{-2}(x^3\mathrm{e}^x)_{-1}.$$

So, we obtain solutions by calculating as

$$y^{(\mathrm{I})} = A(x\mathrm{e}^x)_1 = A\frac{d(x\mathrm{e}^x)}{dx},$$

$$= A\mathrm{e}^x(x + 1),$$

and,

$$y^{(\mathrm{II})} = Bx^{-2}(x^3\mathrm{e}^x)_{-1} = Bx^{-2}\int x^3\mathrm{e}^x\,dx,$$

$$= Bx^{-2}[\mathrm{e}^x(x^3 - 3x^2 + 6x - 6)].$$

**Theorem 3.2.** Let $|(x^{a-c})_k| < \infty \ (k \in \mathbb{Z}^+ \cup \{0\}),\ x \neq 0,$ and $\left|\frac{1}{x}\right| < 1.$ The solution of "Eq. 26." can be written as follows

$$y^{(\mathrm{I})} = Ax^{a-c}\mathrm{e}^x\ {}_2F_0\left[1 - a, c - a; \frac{1}{x}\right]. \tag{44}$$

**Proof.** By means of "Eq. 13.", we have

$$y^{(\mathrm{I})} = A\sum_{k=0}^{\infty}\frac{\Gamma(a)}{\Gamma(a-k)\Gamma(k+1)}(x^{a-c})_k(\mathrm{e}^x)_{a-1-k}. \tag{45}$$

By using "Eq. 8.", "Eq. 10.", "Eq. 11." and "Eq. 12.", we can rewrite the "Eq. 45." as follows

$$y^{(\mathrm{I})} = A\sum_{k=0}^{\infty}\frac{\Gamma(k+1-a)}{(-1)^k\Gamma(1-a)}\frac{1}{k!}(-1)^k x^{a-c-k}\frac{\Gamma(k+c-a)}{\Gamma(c-a)}\mathrm{e}^x,$$

$$= Ax^{a-c}\mathrm{e}^x\sum_{k=0}^{\infty}[1-a]_k\,[c-a]_k\,\frac{1}{k!}\left(\frac{1}{x}\right)^k,$$

$$= Ax^{a-c}\mathrm{e}^x\ {}_2F_0\left[1 - a, c - a; \frac{1}{x}\right]. \tag{46}$$

**Theorem 3.3.** Let $|(x^{1+a})_k| < \infty \ (k \in \mathbb{Z}^+ \cup \{0\}),\ x \neq 0,$ , and $\left|\frac{1}{x}\right| < 1.$ The solution of "Eq. 42." can be written as follows

$$y^{(\mathrm{II})} = Bx^{a-c}\mathrm{e}^x\ {}_2F_0\left[2 - a + c, -1 - a; \frac{1}{x}\right]. \tag{47}$$

**Proof.** By means of "Eq. 13.", we have

$$y^{(\mathrm{II})} = Bx^{-(c+1)} \sum_{k=0}^{\infty} \frac{\Gamma(a-c-1)}{\Gamma(a-c-1-k)} \frac{1}{k!} (x^{1+a})_k (\mathrm{e}^x)_{-(2+c-a+k)}. \tag{48}$$

By using "Eq. 8.", "Eq. 10.", "Eq. 11." and "Eq. 12.", we rewrite the "Eq. 48." as follows

$$y^{(\mathrm{II})} = Bx^{-(c+1)} \sum_{k=0}^{\infty} \frac{\Gamma(k+2-a+c)}{(-1)^k \Gamma(2-a+c)} \frac{1}{k!} (-1)^k x^{1+a-k} \frac{\Gamma(k-1-a)}{\Gamma(-1-a)} \mathrm{e}^x,$$

$$= Bx^{a-c} \mathrm{e}^x \sum_{k=0}^{\infty} [2-a+c]_k \, [-1-a]_k \frac{1}{k!} \left(\frac{1}{x}\right)^k,$$

$$= Bx^{a-c} \mathrm{e}^x \; {}_2F_0 \left[2-a+c, -1-a; \frac{1}{x}\right]. \tag{49}$$

## CONCLUSION

In this paper, we used N method for the confluent hypergeometric equation. We also obtained hypergeometric forms of the fractional solutions. The most important advantage of this method is that can be applied for singular equations.

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