Iğdır Üniversitesi Fen Bilimleri Enstitüsü Dergisi
Iğdır University Journal of the Institute of Science and Technology

Solution of a Nonlinear Schrödinger Equation with Galerkin's Method

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ABSTRACT: In this paper, we consider an initial boundary value problem for a two-dimensional nonlinear Schrödinger equation. We prove by using Galerkin's method that the solution of the initial boundary value problem exists and it has a unique solution. Also, we get an estimation for the solution of the initial boundary value problem.

Keywords: Galerkin method, initial boundary value problem, Schrödinger equation

Lineer Olmayan bir Schrödinger Denkleminin Galerkin Metoduyla Çözümü

ÖZET: Bu çalışmada iki boyutlu lineer olmayan bir Schrödinger denklemi için bir başlangıç sınır değer problemi göz önüne alırız. Galerkin metodunu kullanarak başlangıç sınır değer probleminin çözümünün var ve tek olduğunu ispatlarız. Ayrıca, başlangıç sınır değer probleminin çözümü için bir değerlendirme elde ederiz.

Anahtar kelimeler: Başlangıç sınır değer problemi, Galerkin metodu, Schrödinger denklemi

Cilt/*Volume*: 7, Sayı/*Issue*: 2, Sayfa/*pp*: 225-239, 2017
ISSN: 2146-0574, e-ISSN: 2536-4618 DOI: 10.21597/jist.2017.152

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Geliş tarihi / *Received:* 25.01.2017
Kabul tarihi / *Accepted:* 21.03.2017

INTRODUCTION

The nonlinear Schrödinger equation is a nonlinear mathematical equation that describes the evolution over time of a physical system. It arises in nonlinear optics (Kelley, 1965; Talanov, 1965), the evolution of water waves (Hashimoto and Ono, 1972), hydromagnetic and plasma waves (Schimizu and Ichikawa, 1972), nonlinear instability problems (Stewartson and Stuart, 1971).

In the present paper, we study a nonlinear Schrödinger equation that usually arises in the dispersion of light beams (waves) in a nonlinear medium. We investigate the existence and uniqueness of the solutions of nonlinear Schrödinger equation. For purpose, we use Galerkin's method and constitute the approximate solutions of the initial boundary value problem. By means of the approximate solutions, we prove that the solution of initial the boundary value problem exists and it has a unique solution.

The nonlinear Schrödinger equation and boundary value problems for Schrödinger equation were previously studied in (Tsutsumi, 1991; Bu, 1994; Strauss and Bu, 2001; Bu et al., 2005; Holmer, 2005; Iskenderov and Yagubov, 2007; Mahmudov, 2007; Kaikina, 2013; Yildirim Aksoy et al., 2016). The Schrödinger equation considered in the literature is usually one-dimensional. But, in the paper (Iskenderov and Yagubov, 2007), an initial boundary value problem for a multi-dimensional (except for two-dimensional) Schrödinger equation is examined.

MATERIAL AND METHODS

The basic of Galerkin's Method is based on finding an approximate solution in a finite-dimensional space spanned by a set of basis functions. To obtain the approximate solution, we project the partial differential equation onto a finite-dimensional subspace. This gives a system of ordinary differential equations for the approximate solutions, which has a solution with standart ordinary differential equations theory. Each approximate solution satisfies an estimation called as a priori estimation for solutions of the partial differential equation. These estimations allow to obtain a solution of the partial differential equation.

We formulate the initial boundary value problem as follows:

$$i\frac{\partial \psi}{\partial t} + a_0 \Delta \psi - a(x)\psi - v(x)\psi + a_1 |\psi|^2 \psi = f(x,t), \quad (x,t) \in \Omega \tag{1}$$

$$\psi(x,0) = \varphi(x), \quad x \in D \tag{2}$$

$$\psi(\xi,t)\big|_S = 0, \quad (\xi,t) \in S, \tag{3}$$

where, $D \subset R^2$ is a bounded domain, Γ is the sufficiently smooth boundary of domain D, T, $a_0 > 0$ are given number, $x = (x_1, x_2) \in D$ is an arbitrary point, $\Omega = D \times (0,T)$, $\Omega_t = D \times (0,t)$, $0 \le t \le T$, $S = \Gamma \quad (0,T)$ is the lateral surface of cylinder Ω, Δ is the Laplace operator such that $\Delta\psi = \frac{\partial^2 \psi}{\partial x_1^2} + \frac{\partial^2 \psi}{\partial x_2^2}$ and $\Delta\psi = \left(\frac{\partial \psi}{\partial x_1}, \frac{\partial \psi}{\partial x_2}\right)$, υ indicates the unit outward normal vector to Γ, a_1 is a complex number such that

$$\operatorname{Im} a_1 > 0, \quad \operatorname{Re} a_1 < 0, \quad \operatorname{Im} a_1 \geq 2\left|\operatorname{Re} a_1\right|, \tag{4}$$

$a(x)$ and $v(x)$ are the measurable functions satisfying the conditions

$$0 \leq a(x) \leq \mu_0, \ \textit{for almost all } x \in D, \ \mu_0 = const. > 0 \tag{5}$$

$$v(x) \in L_2(D), \ \left|v(x)\right| \leq b_0 \ \textit{for almost all } x \in D, \ b_0 = const. > 0, \ , \tag{6}$$

respectively.

Definition 1: The function $\psi(x,t) \equiv \psi(x,t;v)$ in the space $B_0 \equiv C^0\left([0,T], \overset{0}{W}{}_2^2(D)\right) \cap C^1\left([0,T], L_2(D)\right)$ is called as a generalized solution of the problem (1)-(3), if it satisfies the equation (1) *for almost all* $(\xi,t) \in S$, and $t \in [0,T]$, the condition (2) *for almost all* $x \in D$ and the condition (3) *for almost all* $(\xi,t) \in S$, where the space B_0 is a Banach space defined as in (Yildirim Aksoy et al., 2016).

RESULTS AND DISCUSSION

In this section, under given conditions, we prove the following theorem which shows the existence and uniqueness of the solutions of the problem (1)-(3).

Theorem 1: Assume that a_1, $a(x)$, $v(x)$ satisfy the conditions (4), (5), (6), respectively and $\varphi \in \overset{0}{W}{}_2^2(\mathrm{D})$, $f \in W_2^{0,1}(\Omega)$ are given functions. Then, the problem (1)-(3) has a unique solution $\psi \in B_0$ satisfying the estimation

$$\left\|\psi(.,t)\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|\frac{\partial \psi(.,t)}{\partial t}\right\|^2_{L_2(\mathrm{D})} \leq c_1\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^6_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^{18}_{\overset{0}{W}{}_2^1(D)}\right) \tag{7}$$

where $c_1 > 0$ is a constant independent from t.

Here, $W_2^1(D)$, $\overset{0}{W}{}_2^1(D)$, $W_2^2(D)$, $\overset{0}{W}{}_2^2(D)$ and $W_2^{0,1}(\Omega)$ are the Sobolev spaces and these spaces are defined in (Ladyzhenskaya, 1985) widely.

Proof: As a fundamental system of functions in the space $\overset{0}{W}{}_2^2(D)$ according to Galerkin's method, the eigenfunctions $u_k = u_k(x)$, $k = 1,2,\ldots$ corresponding to the eigenvalues λ_k of the following spectral problem are taken:

$$Lu_k(x) = -a_0 \frac{d^2u_k(x)}{dx^2} + a(x)u_k(x) = \lambda_k u_k(x), \;\; x \in D$$
$$u_k(x)\big|_\Gamma = 0, \;\; k = 1,2,...$$

As known, the eigenvalues λ_k are real and nonnegative. Also, the eigenfunctions $u_k = u_k(x)$ are real and satisfy the ortogonality condition in the spaces $L_2(D), \;\; \overset{0}{W}{}_2^1(D), \;\; \overset{0}{W}{}_2^2(D)$. Assume that the eigenfunctions $u_k = u_k(x), \;\; k = 1,2,...$ are an-orthonormal basis in the space $L_2(D)$ and satisfy the inequality

$$\|u_k\|_{\overset{0}{W}{}_2^2(D)} \le d_k \,, \;\; k = 1,2,...$$

where $d_k > 0$ for $k = 1,2,...$ are constants.

The approximate solutions of the problem (1)-(3) with Galerkin's method are investigated in the form:

$$\psi^N(x,t) = \sum_{k=1}^{N} C_k^N(t) u_k(x)$$

where the coefficients $C_k^N(t) = \left(\psi^N(.,t), u_k\right)_{L_2(D)} = \left(\psi^N, u_k\right)$ are solutions of Cauchy problem:

$$i\left(\frac{\partial \psi^N(.,t)}{\partial t}, u_k\right)_{L_2(D)} = \left(a_0 \nabla \psi^N(.,t), \nabla u_k\right)_{L_2(D)} + \left(a\psi^N(.,t), u_k\right)_{L_2(D)} +$$
$$+\left(v\psi^N(.,t), u_k\right)_{L_2(D)} - \left(a_1 \left|\psi^N(.,t)\right|^2 \psi^N(.,t), u_k\right)_{L_2(D)} + \left(f, u_k\right)_{L_2(D)}, \;\; k = \overline{1,N} \qquad (8)$$

$$C_k^N(0) = \left(\psi^N(.,0), u_k\right)_{L_2(D)} = \left(\varphi, u_k\right)_{L_2(D)} = \varphi_k, \;\; k = \overline{1,N}. \qquad (9)$$

Since the coefficients of the system of first-order nonlinear ordinary differential equations (8) is continuous, as known from (Pontryagin, 1962), the problem (8)-(9) has locally at least one solution on $[0,T]$. To show the existence of global solution on $[0,T]$ of the problem (8)-(9), we prove the following lemma, which show the uniformly boundedness on $[0,T]$ of all possible solutions of the problem (8)-(9).

Lemma 1: The solution of Cauchy problem (8)-(9) satisfies the estimation

$$\sum_{k=1}^{N}\left|C_k^N(t)\right|^2+\sum_{k=1}^{N}\left|\frac{dC_k^N(t)}{dt}\right|^2\leq\left\|\psi^N(.,t)\right\|^2_{\overset{0}{W}{}_2^2(D)}+\left\|\frac{\partial\psi^N(.,t)}{\partial t}\right\|^2_{L_2(D)}\leq$$
$$\leq c_2\left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)}+\|f\|^2_{W_2^{0,1}(\Omega)}+\|\varphi\|^6_{\overset{0}{W}{}_2^1(D)}+\|\varphi\|^6_{\overset{0}{W}{}_2^2(D)}+\|f\|^6_{W_2^{0,1}(\Omega)}+\|\varphi\|^{18}_{\overset{0}{W}{}_2^1(D)}\right), \tag{10}$$

for $any\ t\in[0,T]$ and $N=1,2,...$, where $c_2>0$ is a constant independent from N and t.

Proof: If we multiply the k-th equation in system (8) by $\overline{C}_k^N(t)$ and sum the obtained equalities on k from 1 to N, and later integrate over the region Ω_t ., we get

$$\int_{\Omega_t}\left(i\frac{\partial\psi^N}{\partial t}\overline{\psi}^N-a_0\left|\nabla\psi^N\right|^2-a(x)\left|\psi^N\right|^2-v(x)\left|\psi^N\right|^2+a_1\left|\psi^N\right|^4\right)dxd\tau=\int_{\Omega_t}f\overline{\psi}^N dxd\tau. \tag{11}$$

Subtracting the complex conjugate of (11) from itself and using the inequality

$$\left\|\psi^N(.,0)\right\|^2_{L_2(D)}=\sum_{k=1}^{N}\left|C_k^N(0)\right|^2\leq\sum_{k=1}^{\infty}\left|C_k^N(0)\right|^2=\|\varphi\|^2_{L_2(D)} \tag{12}$$

we obtain

$$\left\|\psi^N(.,t)\right\|^2_{L_2(D)}+2\,\mathrm{Im}\,a_1\int_{\Omega_t}\left|\psi^N\right|^4dxd\tau\leq\left\|\psi^N(.,0)\right\|^2_{L_2(D)}+2\int_{\Omega_t}|f|\left|\psi^N\right|dxd\tau$$
$$\leq\|\varphi\|^2_{L_2(D)}+2\int_{\Omega_t}|f|\left|\psi^N\right|dxd\tau$$

and then by Young's inequality it is written that

$$\left\|\psi^N(.,t)\right\|^2_{L_2(D)}+2\,\mathrm{Im}\,a_1\left\|\psi^N\right\|^4_{L_4(\Omega_t)}\leq\|\varphi\|^2_{L_2(D)}+\|f\|^2_{L_2(\Omega)}+\int_0^t\left\|\psi^N(.,\tau)\right\|^2_{L_2(D)}d\tau\ .$$

Thus, in inequality above, using Gronwall's lemma, we get

$$\left\|\psi^N(.,t)\right\|^2_{L_2(D)}+2\,\mathrm{Im}\,a_1\int_0^t\left\|\psi^N(.,\tau)\right\|^4_{L_4(D)}d\tau\leq c_3\left(\|\varphi\|^2_{L_2(D)}+\|f\|^2_{L_2(\Omega)}\right) \tag{13}$$

for any $t\in[0,T]$..

As similar to obtaining the equality (11), if we take the derivative of the system (8) in variable t and multiply the obtained system with $\frac{d\overline{C}_k^N(t)}{dt}$, we get

$$\int_{\Omega_t}\left(i\frac{\partial^2\psi^N}{\partial t^2}\frac{\partial\overline{\psi}^N}{\partial t} - a_0\left|\frac{\partial}{\partial t}\left(\nabla\psi^N\right)\right|^2 - a(x)\left|\frac{\partial\psi^N}{\partial t}\right|^2 - v(x)\left|\frac{\partial\psi^N}{\partial t}\right|^2 \right.$$
$$\left. + a_1\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t}\right)dxd\tau = \int_{\Omega_t}\frac{\partial f}{\partial t}\frac{\partial\overline{\psi}^N}{\partial t}dxd\tau. \tag{14}$$

Let's subtract the complex conjugate of (14) from itself. Thus, we have

$$\int_{\Omega_t} i\left(\frac{\partial}{\partial t}\left|\frac{\partial\psi^N}{\partial t}\right|^2\right)dxd\tau + \int_{\Omega_t} i\,\mathrm{Im}\,a_1\left[\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t} + \frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\overline{\psi}^N\right)\frac{\partial\psi^N}{\partial t}\right]dxd\tau$$
$$= -\int_{\Omega_t}\mathrm{Re}\,a_1\left[\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t} - \frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\overline{\psi}^N\right)\frac{\partial\psi^N}{\partial t}\right]dxd\tau \tag{15}$$
$$+2i\int_{\Omega_t}\mathrm{Im}\left(\frac{\partial f}{\partial t}\frac{\partial\overline{\psi}^N}{\partial t}\right)dxd\tau.$$

It is clear that

$$\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t} + \frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\overline{\psi}^N\right)\frac{\partial\psi^N}{\partial t} = 2\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 + \left(\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\right)\right)^2$$
$$\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t} - \frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\overline{\psi}^N\right)\frac{\partial\psi^N}{\partial t} = \left(\psi^N\right)^2\left(\frac{\partial\overline{\psi}^N}{\partial t}\right)^2 - \left(\overline{\psi}^N\right)^2\left(\frac{\partial\psi^N}{\partial t}\right)^2,$$

since

$$\frac{\partial}{\partial t}\left(\left|\psi^N\right|^2\psi^N\right)\frac{\partial\overline{\psi}^N}{\partial t} = 2\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 + (\psi^N)^2\left(\frac{\partial\overline{\psi}^N}{\partial t}\right)^2.$$

Using above two equalities in equation (15), we obtain

$$\left\|\frac{\partial\psi^N(.,t)}{\partial t}\right\|^2_{L_2(D)}+2\operatorname{Im}a_1\int\limits_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2dxd\tau+\operatorname{Im}a_1\int\limits_{\Omega_t}\left(\frac{\partial}{\partial t}\left|\psi^N\right|^2\right)^2dxd\tau\leq$$

$$\leq 2\left|\operatorname{Re}a_1\right|\int\limits_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2dxd\tau+2\int\limits_{\Omega_t}\left|\frac{\partial f}{\partial t}\right|\left|\frac{\partial\psi^N}{\partial t}\right|dxd\tau+\left\|\frac{\partial\psi^N(.,0)}{\partial t}\right\|^2_{L_2(D)}.$$

Here, if we take into account the condition (4) and apply Young's inequality, we get

$$\left\|\frac{\partial\psi^N(.,t)}{\partial t}\right\|^2_{L_2(D)}+\tilde{a}\int\limits_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2dxd\tau\leq\left\|\frac{\partial\psi^N(.,0)}{\partial t}\right\|^2_{L_2(D)}+\left\|\frac{\partial f}{\partial t}\right\|^2_{L_2(\Omega)}+\int_0^t\left\|\frac{\partial\psi^N(.,\tau)}{\partial t}\right\|^2_{L_2(D)}d\tau \qquad (16)$$

for any $t\in[0,T]$, where $\tilde{a}=2\left(\operatorname{Im}a_1-\left|\operatorname{Re}a_1\right|\right)\geq 2\left|\operatorname{Re}a_1\right|>0$.

To evaluate the term $\left\|\frac{\partial\psi^N(.,0)}{\partial t}\right\|^2_{L_2(D)}$ in (16), let's write the system (8) in the form:

$$i\left(\frac{\partial\psi^N(.,t)}{\partial t},u_k\right)_{L_2(D)}=-\left(a_0\Delta\psi^N,u_k\right)_{L_2(D)}+\left(a\psi^N(.,t),u_k\right)_{L_2(D)}$$
$$+\left(v\psi^N(.,t),u_k\right)_{L_2(D)}-\left(a_1\left|\psi^N(.,t)\right|^2\psi^N(.,t),u_k\right)_{L_2(D)}+\left(f,u_k\right)_{L_2(D)},\ k=\overline{1,N}. \qquad (17)$$

Let's multiply the k-th equation by $\frac{d\overline{C}_k^N(0)}{dt}$ taking $t=0$ in the system (17). As similar to obtaining the equality (11) we obtain

$$\int\limits_D i\left|\frac{\partial\psi^N(x,0)}{\partial t}\right|^2dx=\int\limits_D\Big[-a_0\Delta\psi^N(x,0)+a(x)\psi^N(x,0)+v(x)\psi^N(x,0)$$
$$+a_1\left|\psi^N(x,0)\right|^2\psi^N(x,0)+f(x,0)\Big]\frac{\partial\overline{\psi}^N(x,0)}{\partial t}dx$$

and by Young's inequality and the condition (6)

$$\left\|\frac{\partial\psi^N(.,0)}{\partial t}\right\|^2_{L_2(D)} \le 4\left\|L\psi^N(.,0)\right\|^2_{L_2(D)} + 4\int_D\left(|v(x)|\left|\psi^N(x,0)\right|\right)^2 dx$$
$$+4|a_1|^2\int_D\left|\psi^N(.,0)\right|^6 dx + 4\int_D|f(x,0)|^2 dx$$
$$\le 4\left\|L\psi^N(.,0)\right\|^2_{L_2(D)} + 4b_0^2\left\|\psi^N(.,0)\right\|^2_{L_2(D)}$$
$$+4|a_1|^2\left\|\psi^N(.,0)\right\|^6_{L_6(D)} + 4\|f(.,0)\|^2_{L_2(D)}. \qquad (18)$$

Since

$$\left\|\psi^N(.,t)\right\|^6_{L_6(D)} \le \beta^6\left(\left\|\nabla\psi^N(.,t)\right\|^4_{L_2(D)}\left\|\psi^N(.,t)\right\|^2_{L_2(D)}\right) \text{ for any } t\in[0,T] \qquad (19)$$

according to the known inequality in (Ladyzhenskaya, 1985 p.34), where $\beta = 3^{2/3} > 0$ is a constant, we write the inequality

$$\left\|\psi^N(.,0)\right\|^6_{L_6(D)} \le \beta^6\left(\left\|\nabla\psi^N(.,0)\right\|^4_{L_2(D)}\left\|\psi^N(.,0)\right\|^2_{L_2(D)}\right) \le c_4\left\|\psi^N(.,0)\right\|^6_{\overset{0}{W}{}_2^1(D)} \le c_5\|\varphi\|^6_{\overset{0}{W}{}_2^1(D)} \qquad (20)$$

In inequality (18), if we use the estimation (13), the inequalities (20),

$$\left\|L\psi^N(.,0)\right\|^2_{L_2(D)} \le c_6\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)},$$

$$\|f(.,t)\|^2_{L_2(D)} \le c_7\|f\|^2_{W_2^{0,1}(\Omega)} \text{ for any } t\in[0,T], \qquad (21)$$

we obtain

$$\left\|\frac{\partial\psi^N(.,0)}{\partial t}\right\|^2_{L_2(D)} \le c_8\left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^1(D)}\right), \qquad (22)$$

where $c_j > 0,\ j=\overline{3,8}$ are constants. Thus, if we use the inequality (22) in (16)

$$\left\|\frac{\partial\psi^N(.,t)}{\partial t}\right\|^2_{L_2(D)} + a\int_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 dx d\tau$$
$$\le c_9\left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^1(D)}\right) + \int_0^t\left\|\frac{\partial\psi^N(.,\tau)}{\partial t}\right\|^2_{L_2(D)} d\tau \qquad (23)$$

and then apply Gronwall's lemma, we get

$$\left\| \frac{\partial \psi^N(.,t)}{\partial t} \right\|^2_{L_2(D)} \le c_{10} \left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^1(D)} \right) \tag{24}$$

for any $t \in [0,T]$ .. If we apply the estimation (24) to (23), we get

$$\left\| \frac{\partial \psi^N(.,t)}{\partial t} \right\|^2_{L_2(D)} + a \int_{\Omega_t} \left|\psi^N\right|^2 \left| \frac{\partial \psi^N}{\partial t} \right|^2 dx d\tau \le c_{11} \left(\|\varphi\|^2_{\overset{0}{W}{}_2^2(D)} + \|f\|^2_{W_2^{0,1}(\Omega)} + \|\varphi\|^6_{\overset{0}{W}{}_2^1(D)} \right) \tag{25}$$

for any t □ $[0,T]$, where $c_9, c_{10}, c_{11} > 0$ are the constants.

To evaluate the term $\nabla \psi^N$, let's multiply the k-th equation in the system (8) by $\frac{d\overline{C}_k^N(t)}{dt}$, sum the obtained equalities on k from 1 to N and integrate over the region Ω_t. Thus, summing the complex conjugate of obtained equality with itself, we get

$$a_0 \int_{\Omega_t} \frac{\partial}{\partial t} \left|\nabla \psi^N\right|^2 dx d\tau + \int_{\Omega_t} \frac{\partial}{\partial t} \left(a(x) \left|\psi^N\right|^2 \right) dx d\tau = - \int_{\Omega_t} \frac{\partial}{\partial t} \left(v(x) \left|\psi^N\right|^2 \right) dx d\tau$$
$$+ \int_{\Omega_t} \left(a_1 \left|\psi^N\right|^2 \psi^N \frac{\partial \overline{\psi}^N}{\partial t} + \overline{a}_1 \left|\psi^N\right|^2 \overline{\psi}^N \frac{\partial \psi^N}{\partial t} \right) dx d\tau - 2 \int_{\Omega_t} \mathrm{Re} \left(f \frac{\partial \overline{\psi}^N}{\partial t} \right) dx d\tau.$$

Since $\mathrm{Re}\, a_1 < 0$ and

$$a_1 \left|\psi^N\right|^2 \psi^N \frac{\partial \overline{\psi}^N}{\partial t} + \overline{a}_1 \left|\psi^N\right|^2 \overline{\psi}^N \frac{\partial \psi^N}{\partial t} = \frac{\mathrm{Re}\, a_1}{2} \frac{\partial}{\partial t} \left(\left|\psi^N\right|^4 \right) - 2\, \mathrm{Im}\, a_1 \, \mathrm{Im} \left(\left|\psi^N\right|^2 \psi^N \frac{\partial \overline{\psi}^N}{\partial t} \right),$$

we obtain the inequality

$$
\begin{aligned}
&a_0\left\|\nabla\psi^N(.,t)\right\|^2_{L_2(D)}+\frac{|\operatorname{Re}a_1|}{2}\left\|\psi^N(.,t)\right\|^4_{L_4(D)}\le a_0\left\|\nabla\psi^N(.,0)\right\|^2_{L_2(D)}+\mu_0\left\|\psi^N(.,0)\right\|^2_{L_2(D)}\\
&+\frac{|\operatorname{Re}a_1|}{2}\left\|\psi^N(.,0)\right\|^4_{L_4(D)}+\int_D|v(x)|\left|\psi^N(x,t)\right|^2dx+\int_D|v(x)|\left|\psi^N(x,0)\right|^2dx+\int_D|f(x,0)|^2dx\\
&+\int_D|f(x,t)|^2dx+\int_D\left|\psi^N(x,0)\right|^2dx+\int_D\left|\psi^N(x,t)\right|^2dx+\left\|\psi^N\right\|^2_{L_2(\Omega)}+\left\|\frac{\partial f}{\partial t}\right\|^2_{L_2(\Omega)}\\
&+2\operatorname{Im}a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\psi^N\frac{\partial\bar\psi^N}{\partial t}\right|dxd\tau+\mu_0\left\|\psi^N(.,t)\right\|^2_{L_2(D)}.
\end{aligned}
$$

from the last equality. Here, using the Cauchy-Schwarz inequality, the inequalities (6), (21),

$$\left\|\nabla\psi^N(.,0)\right\|^2_{L_2(D)}\le c_{12}\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^1(D)}$$

and the estimation (13), we obtain

$$
\begin{aligned}
&a_0\left\|\nabla\psi^N(.,t)\right\|^2_{L_2(D)}+\frac{|\operatorname{Re}a_1|}{2}\left\|\psi^N(.,t)\right\|^4_{L_4(D)}\le\frac{|\operatorname{Re}a_1|}{2}\left\|\psi^N(.,0)\right\|^4_{L_4(D)}\\
&+c_{13}\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^1(D)}+\left\|f\right\|^2_{W_2^{0,1}(\Omega)}\right)+2\operatorname{Im}a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\psi^N\frac{\partial\bar\psi^N}{\partial t}\right|dxd\tau.
\end{aligned}
\tag{26}
$$

In (26), if we use the inequality

$$\left\|\psi^N(.,0)\right\|^4_{L_4(D)}\le c_{14}\left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)}+c_{15}\left\|\varphi\right\|^2_{L_2(D)}$$

from the known inequality in (Ladyzhenskaya, 1985 p.34), we get

$$
\begin{aligned}
&a_0\left\|\nabla\psi^N(.,t)\right\|^2_{L_2(D)}+\frac{|\operatorname{Re}a_1|}{2}\left\|\psi^N(.,t)\right\|^4_{L_4(D)}\le 2\operatorname{Im}a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\psi^N\frac{\partial\bar\psi^N}{\partial t}\right|dxd\tau\\
&\qquad+c_{16}\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^1(D)}+\left\|f\right\|^2_{W_2^{0,1}(\Omega)}+\left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)}\right),
\end{aligned}
$$

where the constant $c_j>0,\ j=\overline{12,16}$ are indepentent from N and t. Here, since

$$2\,\mathrm{Im}\,a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\psi^N\frac{\partial\bar{\psi}^N}{\partial t}\right|dxd\tau\le \mathrm{Im}\,a_1\int_0^t\left\|\psi^N(.,\tau)\right\|_{L_4(D)}^4 d\tau+\mathrm{Im}\,a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 dxd\tau$$

$$\le c_{17}\left(\left\|\varphi\right\|_{L_2(D)}^2+\left\|f\right\|_{L_2(\Omega)}^2\right)+\mathrm{Im}\,a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 dxd\tau$$

from the estimation (13), we have from the above inequality

$$a_0\left\|\nabla\psi^N(.,t)\right\|_{L_2(D)}^2+\frac{\left|\mathrm{Re}\,a_1\right|}{2}\left\|\psi^N(.,t)\right\|_{L_4(D)}^4\le \mathrm{Im}\,a_1\int_{\Omega_t}\left|\psi^N\right|^2\left|\frac{\partial\psi^N}{\partial t}\right|^2 dxd\tau$$
$$+c_{18}\left(\left\|\varphi\right\|_{\overset{0}{W}{}_2^1(D)}^2+\left\|f\right\|_{W_2^{0,1}(\Omega)}^2+\left\|\varphi\right\|_{\overset{0}{W}{}_2^1(D)}^6\right). \tag{27}$$

for any $t\in[0,T]$, where $c_{18}>0$ is independent from N and t. Since $\tilde{a}>0$ and $\mathrm{Im}\,a_1>0$, using (25) in (27), we get

$$a_0\left\|\nabla\psi^N(.,t)\right\|_{L_2(D)}^2+\frac{\left|\mathrm{Re}\,a_1\right|}{2}\left\|\psi^N(.,t)\right\|_{L_4(D)}^4\le c_{19}\left(\left\|\varphi\right\|_{\overset{0}{W}{}_2^2(D)}^2+\left\|f\right\|_{W_2^{0,1}(\Omega)}^2+\left\|\varphi\right\|_{\overset{0}{W}{}_2^1(D)}^6\right) \tag{28}$$

for any $t\in[0,T]$, , where $c_{19}>0$ is a constant.

Finally, to evaluate the derivation $\Delta\psi^N$, , multiplying the k-th equation in the system (17) by $\lambda_k\bar{C}_k^N$, , we have

$$\int_D\left[i\frac{\partial\psi^N}{\partial t}L\bar{\psi}^N-\left|L\psi^N\right|^2-v(x)\psi^N L\bar{\psi}^N+a_1\left|\psi^N\right|^2\psi^N L\bar{\psi}^N-fL\bar{\psi}^N\right]dx=0$$

and thus it is written that

$$\left\|L\psi^N\right\|^2_{L_2(D)} \le 4\left\|\frac{\partial\psi^N(.,t)}{\partial t}\right\|^2_{L_2(D)} + 4b_0^2\left\|\psi^N(.,t)\right\|^2_{L_2(D)} + 4\left|a_1\right|^2\left\|\psi^N(.,t)\right\|^6_{L_6(D)} + 4\left\|f\right\|^2_{L_2(D)}. \tag{29}$$

If we use the inequality

$\left\|L\psi^N(.,t)\right\|^2_{L_2(D)} \ge \frac{a_0^2}{2}\left\|\Delta\psi^N(.,t)\right\|^2_{L_2(D)} - \left\|a(x)\psi^N(.,t)\right\|^2_{L_2(D)}$ and the condition (5), the estimations (13), (24) in the (27), we get

$$a_0^2\left\|\Delta\psi^N(.,t)\right\|^2_{L_2(D)} \le c_{20}\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(0,l)} + \left\|f\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(0,l)}\right) + 4\left|a_1\right|^2\left\|\psi^N(.,t)\right\|^6_{L_6(D)}. \tag{30}$$

for any $t\in[0,T]$. For the term $\left\|\psi^N(.,t)\right\|^6_{L_6(D)}$ in the (30), from (19), it is written that

$\left\|\psi^N(.,t)\right\|^6_{L_6(D)} \le \beta^6\left(\left\|\psi^N(.,t)\right\|^6_{\overset{0}{W}{}_2^1(D)}\right)$ for any $t\in\left[0,T\right]$

which implies that

$$\left\|\psi^N(.,t)\right\|^6_{L_6(D)} \le c_{21}\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^6_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^{18}_{\overset{0}{W}{}_2^1(D)}\right) \tag{31}$$

from the estimations (13) and (28). If we apply the inequality (31) to (30), we obtain

$$\left\|\Delta\psi^N(.,t)\right\|^2_{L_2(D)} \le c_{22}\left(\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^6_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^{18}_{\overset{0}{W}{}_2^1(D)}\right). \tag{32}$$

Thus, combining the estimations (13), (24), (28) and (32), we can easily written that

$$\begin{aligned}\left\|\psi(.,t)\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|\frac{\partial\psi(.,t)}{\partial t}\right\|^2_{L_2(\mathrm{D})} \le c_{23}\Bigg(&\left\|\varphi\right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^2_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^1(D)} \\ &+ \left\|\varphi\right\|^6_{\overset{0}{W}{}_2^2(D)} + \left\|f\right\|^6_{W_2^{0,1}(\Omega)} + \left\|\varphi\right\|^{18}_{\overset{0}{W}{}_2^1(D)}\Bigg)\end{aligned} \tag{33}$$

for any $t \in [0,T]$, where $c_j > 0\ j = \overline{20,23}$ are constants. Since

$$\sum_{k=1}^{N} \left| C_k^N(t) \right|^2 + \sum_{k=1}^{N} \left| \frac{dC_k^N(t)}{dt} \right|^2 \le \left\| \psi^N(.,t) \right\|^2_{\overset{0}{W}{}_2^2(D)} + \left\| \frac{\partial \psi^N(.,t)}{\partial t} \right\|^2_{L_2(D)}$$

if we take $c_2 = c_{23}$ in (33), it seem that the functions $\psi^N(x,t)$ satisfy the estimation (10), that is, Cauchy problem (8)-(9) has one global solution on [0,T].

Let's define a family of functions $\ell_{N,k}(t) = \left(\psi^N(.,t), u_k \right)_{L_2(D)}$, $k, N = 1,2,...$ by means of the functions $\psi^N(x,t)$. . From (10), it follows that $\ell_{N,k}(t)$ and $\frac{d\ell_{N,k}(t)}{dt}$ are uniformly bounded on $[0,T]$. Also, as similar to the paper (Yildirim Aksoy et al., 2016), the equicontinuity of the functions $\ell_{N,k}(t)$ and $\frac{d\ell_{N,k}(t)}{dt}$ for fixed k and $N \ge k$, , $k, N = 1,2,...$ is shown on $[0,T]$.

In this way, from Ascoli-Arzela's theorem (Hsieh and Sibuya, 1999), we can extract the subsequences

$\left\{ \ell_{N_m,k}(t) \right\}$, $\left\{ \frac{d\ell_{N_m,k}(t)}{dt} \right\}$, $m = 1,2,...$ from sequences $\left\{ \ell_{N,k}(t) \right\}$, $\left\{ \frac{d\ell_{N,k}(t)}{dt} \right\}$, , respectively, such that

$$\ell_{N_m,k}(t) \xrightarrow{uniformly} \ell_k(t), \quad \frac{d\ell_{N_m,k}(t)}{dt} \xrightarrow{uniformly} \frac{d\ell_k(t)}{dt} \text{ on } [0,T].$$

Now, let's define the function $\psi(x,t) = \sum_{k=1}^{\infty} \ell_k(t) u_k(x)$ using the function $\ell_k(t)$ which implies that $\frac{\partial \psi(x,t)}{\partial t} = \sum_{k=1}^{\infty} \frac{d\ell_k(t)}{dt} u_k(x)$. . It is easily shown that the subsequences $\left\{ \psi^{N_m}(x,t) \right\}$ and $\left\{ \frac{\partial \psi^{N_m}(x,t)}{\partial t} \right\}$ are uniformly with respect to t weakly converge to the functions $\psi(x,t)$ and $\frac{\partial \psi(x,t)}{\partial t}$ in $\overset{0}{W}{}_2^2(D)$ and $L_2(D)$, respectively, i.e., limit relations

$$\left\{ \psi^{N_m} \right\} \xrightarrow{weakly} \psi(x,t) \text{ in } \overset{0}{W}{}_2^2(D), \quad \left\{ \frac{\partial \psi^{N_m}}{\partial t} \right\} \xrightarrow{weakly} \frac{\partial \psi(x,t)}{\partial t} \text{ in } L_2(D) \tag{34}$$

is written. Thus, since $\left\{ \psi^{N_m} \right\} \in B_0$, $\psi \in C^0\left([0,T], W_2^2(D) \right) \cap C^1\left([0,T], L_2(D) \right)$. .

As similar to the paper (Yildirim Aksoy et al., 2016), it is shown that $\psi(x,t)$ satisfies the equation (1) *for almost all* $x \in D$ and *any* $t \in [0,T]$, , the condition (2) *for almost all* $x \in D$ and the condition (3) *for almost all* $(\xi,t) \in S$ .Thus, $\psi(x,t) \in B_0$.

If we take the lower limit of inequality (10) for $N = N_m$ and $m \to \infty$ by by using the relations (34), we get that function $\psi(x,t)$ satisfies the estimation (7).

Finally, let's prove the uniqueness of the solution of the problem (1)-(3) in B_0 . For purpose, let's take two different solutions of the problem (1)-(3) such as $\psi(x,t)$ and $\phi(x,t)$ in B_0 . Then, the function $\omega(x,t) = \psi(x,t) - \phi(x,t)$ holds the following boundary value problem:

$$i\frac{\partial \omega}{\partial t} + a_0 \Delta \omega - a(x)\omega - v(x)\omega + a_1\left(|\psi|^2 + |\phi|^2\right)\omega + a_1 \psi\phi\bar{\omega} = 0, \quad (x,t) \in \Omega \tag{35}$$

$$\omega(x,0) = 0, \quad x \in D, \quad \omega(\xi,t)\big|_S = 0, \quad (\xi,t) \in S. \tag{36}$$

To show the uniqueness of the solution, let's multiply the equation (35) with $\bar{\text{w}}(x,t)$ and integrate over $\square_t$. Thus, we obtain

$$\int_{\Omega_t}\left[i\frac{\partial \omega}{\partial t}\bar{\omega} - a_0|\nabla\omega|^2 - a(x)|\omega|^2 - v(x)|\omega|^2 + a_1\left(|\psi|^2 + |\phi|^2\right)|\omega|^2 + a_1\psi\phi(\bar{\omega})^2\right]dxd\tau = 0. \tag{37}$$

Subtracting the complex conjugate of (37) from itself and using the condition (36), we obtain

$$\|\omega(.,t)\|^2_{L_2(D)} + 2\,\mathrm{Im}\, a_1 \int_{\Omega_t}\left(|\psi|^2 + |\phi|^2\right)|\omega|^2\, dxd\tau = -2\int_{\Omega_t} \mathrm{Im}\left(a_1\psi\phi(\bar{\omega})^2\right)dxd\tau$$

which is equivalent to

$$\|\omega(.,t)\|^2_{L_2(D)} + 2\,\mathrm{Im}\, a_1 \int_{\Omega_t}\left(|\psi|^2 + |\phi|^2\right)|\omega|^2\, dxd\tau \le 2|a_1| \int_{\Omega_t} |\psi||\phi||\omega|^2 dxd\tau$$

$$\le |a_1| \int_{\Omega_t}\left(|\psi|^2 + |\phi|^2\right)|\omega|^2 dxd\tau.$$

Since $|a_1| \le |\mathrm{Re}\,a_1| + |\mathrm{Im}\,a_1| = |\mathrm{Re}\,a_1| + \mathrm{Im}\,a_1$ and $\mathrm{Im}\,a_1 \ge 2|\mathrm{Re}\,a_1|$ it is written that $|a_1| \le \frac{3}{2}\mathrm{Im}\,a_1$.. Thus,

$$\|\omega(.,t)\|^2_{L_2(D)} + \frac{1}{2}\mathrm{Im}\,a_1 \int_{\Omega_t} \left(|\psi|^2 + |\phi|^2\right)|\omega|^2 \, dxd\tau \le 0$$

for any $t \in [0,T]$., which implies that $\|\omega(.,t)\|^2_{L_2(D)} = 0$ for any $t \in [0,T]$.. Hence,

$$\psi(x,t) = \phi(x,t) \text{ for any } t \in [0,T], \text{ almost all } x \in D$$

i.e., the problem (1)-(3) has a unique solution in B_0.

CONCLUSION

As a result, we obtained that the solution of the considered initial boundary value problem exists and it has a unique solution. Also, we shown that the solution of the partial differential equation satisfies a priori estimation. Different from the previous studies we studied a two-dimensional nonlinear Schrödinger equation. So, this study is more comprehensive than previous.

REFERENCES

Bu C, 1994. An initial-buondary value problem of the nonlinear Schrödinger equation, Appl. Anal. 53: 241-254.

Bu C, Tsuyata K, Zhang C, 2005. Nonlinear Schrödinger equation with inhomogeneous Dirichlet-Boundary data. J. Math. Phys., 46: 083504.

Hashimoto H, Ono H, 1972. H. Nonlinear modulation of Gravity Waves. J. Phys. Soc. Jpn., 33: 805-811.

Holmer J, 2005. The initial-boundary value problem for the 1-d nonlinear Schrödinger equation on the half-line. Diff. Integ. Equation, 18: 647-668.

Hsieh P F, Sibuya Y, 1999. Basic Theory of Ordinary Differential Equations, Springer Verlag, New York. 468p.

Iskenderov A D, Yagubov G Y, 2007. Optimal control Problem with unbounded potential for multidimensional, nonlinear and nonstationary Schrödinger equation. Proceedings of the Lankaran State University, Natural Sciences series. 3-56.

Kaikina E I, 2013. Inhomogeneous Neumann initial-boundary value problem for the nonlinear Schrödinger equation. Journal of Differential Equation, 255: 3338-3356.

Kelley P L,1965. Self-focusing of optical beams. Pyhsical Review Letters, 15: 1005-1008.

Ladyzhenskaya O A, 1985. The Boundary Value Problems of Mathematical Physics, Springer Verlag. 322p.

Mahmudov N M, 2007. Solvability of Boundary Value Problems for a Schrödinger Equation with Pure Imaginary Coefficient in the Nonlinear Part of This Equation. Proceedings of IMM of NAS of Azerbaijan, Vol.27, issue 35: 25-36.

Pontryagin L S, 1962. Ordinary Differential Equtions. Addison-Wesley Publishing Comp., (translated from the Russian).

Schimizu K, Ichikawa Y H, 1972. Automodulation of Ion Oscillation Modes in Plasma. J. Phys. Soc. Jpn., 33: 789-792.

Stewartson K, Stuart J T, 1971. A Nonlinear Instability Theory for a Wave System in Plane Poiseuille Flow. J. of Fluid Mechanic, 48(3): 529-545.

Strauss W, Bu C, 2001. An Inhomogeneous Boundary Value Problem for Nonlinear Schrödinger Equations. Journal of Differential Equations, 173: 79-91.

Talanov V I, 1965. Self-focusing of wave beams in nonlinear media. Soviet Physics-JETP Letters, 2: 138-141.

Tsutsumi M, 1991. On Global Solutions to the Initial Boundary Value Problem for Nonlinear Schrödinger Equation in Exterior Domain. Comm. Partial Diffential Equations, 6: 885-907.

Yildirim Aksoy N, Kocak Y, Ozeroglu Y, 2016. On the solvability of initial boundary value problems for nonlinear time-dependent Schrödinger equations. Quaestiones Mathematicae, 39(6): 751-771.