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On topology of pairwise comparison matrices

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Abstract. The aim of this paper is to study the topology of the additive (Lie algebra) and multiplicative (Lie group) pairwise comparison matrices (PCM). We prove that both additive and multiplicative PCM are connected while only multiplicative consistent PCM is compact. We also provide that there is no upper bound for the Perron norm of an inconsistent pairwise comparison matrix.

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1. Introduction

Pairwise comparison matrices are a useful tool in decision-making processes, especially in scenarios requiring the assessment and comparison of various options or criteria based on their relative importance or contribution to a goal. This method is widely applied in fields such as operations research, decision analysis, and management science. When tasked with the evaluation of two distinct entities, whether they be tangible or conceptual, one often turns to the method of pairwise comparisons, or PCs for short. Historical texts, particularly those by Raymond Llull, shed light on the early utilization of this method. Llull's work, detailed in handwritten manuscripts due to the absence of contemporary publication avenues, paved the way for using PCs as tools in the electoral systems.

Definition 1. *Given two entities, A and B, a multiplicative pairwise comparison establishes a relationship wherein A is a certain multiple (say, x times) of B. The ratio-centric nature of this approach is reminiscent of the constant π, which conveys the relationship between a circle's circumference and its diameter.*

Definition 2. *Contrasting the multiplicative approach, the additive pairwise comparison quantifies the difference between two entities. This is often represented in terms of percentages, signifying how much one entity surpasses the other in a certain metric.*

Definition 3. *Let $A = [a_{ij}]$ be an $n \times n$ matrix. A is a multiplicative PCM if and only if:*

- $a_{ij} > 0$ *for all* i, j
- $a_{ii} = 1$ *for all* i
- $a_{ij} = \frac{1}{a_{ji}}$ *for all* i, j

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In matrix form:

$$A = \begin{bmatrix} 1 & a_{12} & \dots & a_{1n} \\ \frac{1}{a_{12}} & 1 & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{a_{1n}} & \frac{1}{a_{2n}} & \dots & 1 \end{bmatrix}$$

Let us give an example to make the diference between consistent and inconsistent PCM clear.

Example 1.

$$A = \begin{pmatrix} 1 & 2 & 6 \\ 0.5 & 1 & 3 \\ 0.1667 & 0.3333 & 1 \end{pmatrix}.$$

Here, $a_{12} \cdot a_{23} = 2 \cdot 3 = 6 = a_{13}$*, so this matrix is consistent. However, for the matrix:*

$$B = \begin{pmatrix} 1 & 3 & 4 \\ 0.3333 & 1 & 2 \\ 0.25 & 0.5 & 1 \end{pmatrix},$$

we find that $b_{12} \cdot b_{23} = 3 \cdot 2 = 6 \neq 4 = b_{13}$*. This deviation from the transitivity condition indicates that* B *is inconsistent.*

In scholarly research, various metrics or comparison standards, such as superiority or significance, have been proposed, with [9] offering a comprehensive insight. Despite the prevalence and accessibility of the multiplicative approach, its inherent mathematical complexity is noteworthy. On the other hand, additive comparisons, though less popular, possess an innate simplicity. Interestingly, the two can be interconverted using logarithmic transformations, as elucidated in [2]. Such a transformation holds significance in mathematical discussions on inconsistency convergence, as supported by multiple studies including [3], [7], and [6].

Further explorations, particularly by [6] and building upon the foundational work in [5], suggest that *PC* matrices can be grouped under a mathematical operation known as the Hadamard product. This observation becomes pivotal in later studies, such as [4], which identifies this collection as a special mathematical entity known as a Lie group.

The structure of this paper is as follows: In section 2 we will give additive (Lie algebra) and multiplicative (Lie group) *PC* matrices and we give the relationship between them by exponential and logarithm functions. In section 3 we prove the main results of this paper using Perron-Frobenious norm on positive matrices.

2. Lie Groups and Lie Algebras of *PC* Matrices

The monograph [11] stipulates that, Intuitively, a manifold is a generalization of curves and surfaces to higher dimensions. It is locally Euclidean in that every point has a neighborhood, called a chart, homeomorphic to an open subset of $\mathbf{R}^n$". We find the above stipulation to be sufficient to be followed by computer science researchers. A group that is also a differentiable (or smooth) manifold is called Lie group (after its proponent Sophus Lie). According to [1], a Lie group is an abstract group G with a smooth structure, that is:

Definition 4. *(1)* G *is a group,*
(2) G *is a smooth manifold,*
(3) the operation $G \times G \to G$; $(x; y) \to xy^{-1}$ *is smooth. Matrix Lie group operates on matrices.*

Definition 5. *The Lie algebra of a Lie group* G *is the vector space* $\mathbf{T}_e G$ *equipped with the Lie bracket operation* $[,]$ *of vector fields.*

The bracket operation $[,]$ is assumed to be bilinear, antisymmetric, and satisfies the Jacobi identity: Cyclic $([X, [Y, Z]]) = 0$ for all X, Y, Z belonging to this algebra.

Lemma 1. *Let* A *and* B *be two matrices of the same dimensions. Then, the transpose of the Hadamard product of* A *and* B *satisfies the reverse order law:*

$$(A \circ B)^T = A^T \circ B^T,$$

where $\circ$ *denotes the Hadamard (element-wise) product.*

Proof. By definition of the Hadamard product, the (i, j)-th element of $A \circ B$ is given by:

$$(A \circ B)_{ij} = A_{ij} B_{ij}.$$

Taking the transpose of $A \circ B$, the (i, j)-th element of $(A \circ B)^T$ is:

$$((A \circ B)^T)_{ij} = (A \circ B)_{ji} = A_{ji} B_{ji}.$$

On the other hand, the (i, j)-th element of $A^T \circ B^T$ is:

$$(A^T \circ B^T)_{ij} = A_{ji} B_{ji}.$$

Thus, we have:

$$((A \circ B)^T)_{ij} = (A^T \circ B^T)_{ij},$$

which implies:

$$(A \circ B)^T = A^T \circ B^T.$$

□

From [4], we have the following result.

Theorem 1. *For every dimension $n > 0$, the following group*

$$G = \{M = [m_{ij}]_{n \times n} \mid M \cdot M^T = I, m_{ij} = 1/m_{ji} > 0 \ \text{for every} \ \ i, j = 1, 2, \ldots, n\}$$

is an abelian group of $n \times n$ PC matrices with an operation
$\cdot : G \times G \to G$ defined by $(M, N) \to M \circ N = [m_{ij} \cdot n_{ij}]$
where "$\circ$" is the Hadamard product.

Let M and N be arbitrary elements of G. Notice that by the properties of G:

$$NM(NM)^T = (NM)(N^T M^T) = N(MM^T)N^T = NIN^T = I.$$

Also G is closed and commutative under Hadamard product. Consequently, we see that $(G, \circ)$ is an abelian group.

Definition 6. *Let G be a PC matrix Lie group and $M(t)$ be a path through G. We say that $M(t)$ is smooth if each entry in $M(t)$ is differentiable. The derivative of $M(t)$ at the point t is denoted $M'(t)$ which is the matrix whose ijth element is the derivative of ijth element of $M(t)$.*

Corollary 1. *The abelian group G is a PC matrix Lie group.*

Proof. We know that the Hadamard product "$\cdot$" and the operation $M \to M^{-1} = M^T$ are differentiable for every PC matrix M. Thus, G is a PC matrix Lie Group. □

We also know that the tangent space of any matrix Lie group at unity is a vector space. The tangent space of any matrix group G at unity I will be denoted by $\mathbf{T}_I(G) = \mathbf{G}$ where I is the unit matrix of G.

Theorem 2. *The tangent space of the PC matrix Lie Group G at unity I consists of all $n \times n$ real matrices X that satisfy $X + X^T = 0$.*

Corollary 2. *The Lie algebra of G, denoted by $\mathbf{T}_I(G) = \mathbf{G}$, is a Lie algebra of G and $\mathbf{T}_I(G) = \mathbf{G}$ is the space of the skew-symmetric $n \times n$ matrices. Notice that:*

$$dim(G) = dim(\mathbf{G}) = \frac{n \cdot (n-1)}{2}.$$

The exponential map is a map from Lie algebra of a given Lie group to that group. Note that this is an *entrywise* exponential map rather than the classical matrix exponential. from $\mathbf{G}$ (the tangent space to the identity element of PC matrix Lie group G) to G.

Let G be a PC matrix Lie group and $\mathbf{G}$ be the Lie algebra of G. Then, the exponential map

$$\exp : \mathbf{G} \to G$$

$$A = [aij]_{n \times n} \to \exp[A] = [e^{a_{ij}}]$$

can be defined so that the following properties (1)-(6) hold:

$G_1 = \{\delta(t) = e^{tA} \,|\, t \in \mathbf{R},\, A \in \mathbf{G}\}$ is one parameter subgroup of G.
(2) Let A and B be two elements of the Lie algebra $\mathbf{G}$. Then, the following equality holds:

$$e^{A+B} = e^A \cdot e^B$$

(3) Given any matrix $A \in \mathbf{G}$, the tangent vector of the smooth path $\gamma(t)$ is equal to $A \cdot e^{tA}$, that is,

$$\gamma'(t) = \frac{d}{dt} e^{tA} = A \cdot e^{tA}$$

(4) For any matrix $A \in \mathbf{G}$,

$$(e^A)^{-1} = e^{-A} = (e^A)^T = e^{A^T}$$

and

$$(e^A) \cdot (e^A)^T = eA \cdot e^{A^T} = e^{A+A^T} = e^{A-A} = e^0 = 1.$$

(5) For any matrix $A \in \mathbf{G}$, $\gamma(t) = e^{tA}$ is a geodesic curve of the pairwise comparison matrix Lie group G passing through the point $\gamma(0) = 1$.
(6) For any matrix $A \in \mathbf{G}$,, we have $\det(e^A) \neq e^{tr(A)}$ where $tr(A)$ is the trace function of A.
Now we want to give a counter example for the last property.

Example 2. *let us consider the following matrix*

$$A = \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}.$$

Then A is the element of $\mathbf{G}$*, hence the exponential map of A is*

$$e^A = \begin{pmatrix} 1 & 1/e & e \\ e & 1 & 1 \\ 1/e & 1 & 1 \end{pmatrix}.$$

The determinant of e^A is

$$\det(e^A) = e^2 + e^{-2} - 2$$

and trace of A is:

$$tr(A) = \sum_{i=1}^{3} a_{ii} = 0.$$

Consequently, $\det(e^A)$ *is not equal to $e^{tr(A)}$ for the matrix A.*

3. Perron Norm and Topology of Pairwise Comparison Matrices PCM

Definition 7. *The Perron norm is denoted by $\|\cdot\|_\infty$ and defined as follows:*

$$\|A\|_\infty := \max_i \sum_{j=1}^{n} |a_{ij}|, i = 1, \ldots, n$$

where A is an $n \times n$ matrix with entries a_{ij}. The Perron norm is also called the infinity norm or the maximum row sum norm.

Lemma 2. *For any non-negative matrix A its Perron-Frobenius eigenvalue λ satisfies the inequality:*

$$\min_{i=1,\cdots,n} \sum_{j=1}^{n} a_{ij} \leq \lambda \leq \max_{i=1,\cdots,n} \sum_{j=1}^{n} a_{ij}.$$

Lemma 3. *Let A be an inconsistent pairwise comparison matrix with at least one negative eigenvalue. Then, the Perron norm of A is greater than 1.*

Proof. Let A be an inconsistent pairwise comparison matrix with at least one negative eigenvalue, and let $\lambda_{\max}(A)$ be the largest eigenvalue of A. We want to show that $\|A\|_\infty > 1$.

Suppose for the sake of contradiction that $\|A\|_\infty \leq 1$. Then, we have:

$$\begin{aligned}\lambda_{\max}(A) &= \max_{\|\mathbf{x}\|_\infty=1} \frac{\|A\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \\ &\leq \|A\|_\infty \max_{\|\mathbf{x}\|_\infty=1} \frac{\|\mathbf{x}\|_\infty}{\|\mathbf{x}\|_\infty} \\ &= \|A\|_\infty \leq 1.\end{aligned}$$

Since $\lambda_{\max}(A)$ is the largest eigenvalue of A, we know that it satisfies the equation $Av = \lambda_{\max}(A)v$ for some nonzero vector v. Multiplying both sides by $\operatorname{sgn}(v)$ (the vector of signs of the entries of v), we obtain:

$$\operatorname{sgn}(v)^T Av = |\lambda_{\max}(A)| \operatorname{sgn}(v)^T v.$$

Since A is pairwise comparison matrix, all of its entries are nonnegative, so we have $\operatorname{sgn}(v)^T Av \geq 0$. Therefore, we must have $|\lambda_{\max}(A)| \geq 0$. But since we assumed that A has at least one negative eigenvalue, we know that $\lambda_{\max}(A) < 0$, which is a contradiction.

Therefore, we must have $\|A\|_\infty > 1$, as desired. □

Theorem 3. *Let A be an inconsistent pairwise comparison matrix with all positive eigenvalues. Then, the Perron norm of A is greater than or equal to n, where n is the order of the matrix.*

Proof. Let A be an inconsistent pairwise comparison matrix with all positive eigenvalues. Then, we can write A as the product of its diagonal matrix of row sums D and its matrix of normalized entries B:

$$A = DB.$$

Since B is column stochastic, we have that $B\mathbf{1} = \mathbf{1}$, where $\mathbf{1}$ is the vector of all ones. Then, we can write:

$$\begin{aligned}\|A\|_\infty &= \max_{i=1,\ldots,n} \sum_{j=1}^{n} |a_{ij}| \\ &= \max_{i=1,\ldots,n} \sum_{j=1}^{n} |d_i b_{ij}| \\ &= \max_{i=1,\ldots,n} d_i \sum_{j=1}^{n} |b_{ij}| \\ &\geq \max_{i=1,\ldots,n} d_i \\ &= \max_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij} \\ &= \lambda_{\max}(A),\end{aligned}$$

where d_i is the ith diagonal entry of D, and $\lambda_{\max}(A)$ is the largest eigenvalue of A. The inequality follows from the fact that the row sums of B are all equal to 1, so $\sum_{j=1}^{n} |b_{ij}| \geq 1$ for all i. Therefore, we have shown that:

$$\|A\|_\infty \geq \lambda_{\max}(A).$$

Since all eigenvalues of A are positive, we know that $\lambda_{\max}(A)$ is one of the diagonal entries of D, say d_k. Therefore, we have:

$$\|A\|_\infty \geq d_k = \sum_{j=1}^{n} a_{kj} \geq n.$$

The last inequality follows from the fact that A is column stochastic, so $\sum_{j=1}^{n} a_{kj} = 1$ for all k. Therefore, we have shown that:

$$\|A\|_\infty \geq n.$$

□

Example 3. *Let A be an 6×6 inconsistent pairwise comparison matrix.*

$$A = \begin{pmatrix} 1.0000 & 0.8059 & 0.9801 & 0.4685 & 0.5094 & 0.1642 \\ 1.2408 & 1.0000 & 0.1017 & 0.9085 & 0.4869 & 0.3990 \\ 1.0203 & 9.8307 & 1.0000 & 0.4171 & 0.8559 & 0.5324 \\ 2.1343 & 1.1007 & 2.3972 & 1.0000 & 0.6144 & 0.8752 \\ 1.9630 & 2.0538 & 1.1684 & 1.6276 & 1.0000 & 0.6590 \\ 6.0909 & 2.5061 & 1.8784 & 1.1426 & 1.5174 & 1.0000 \end{pmatrix}.$$

Perron norm of this pairwise inconsistent matrix is 6.9629, *which is bigger than n.*

Theorem 4. *Let A be an inconsistent pairwise comparison matrix with all positive eigenvalues, and let $\lambda_{\max}(A)$ be the largest eigenvalue of A. Then, we have:*

$$\frac{\|A\|_\infty}{\lambda_{\max}(A)} \geq \frac{n}{\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}}.$$

Proof. To prove this inequality, we can use the fact that the consistency ratio of a pairwise comparison matrix A is defined as the ratio of the inconsistency of A to the consistency of a perfectly consistent matrix with the same row geometric means as A. The inconsistency of A is measured by the quantity $\lambda_{\max}(A)$, while the consistency of a perfectly consistent matrix is measured by the quantity $\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}$. Therefore, the consistency ratio of A can be written as:

$$\frac{\lambda_{max}(A)}{\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}}$$

Since A is inconsistent, its consistency ratio is greater than 1. Thus, we have:

$$\frac{\lambda_{max}(A)}{\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}} > 1$$

Multiplying both sides by n yields:

$$\frac{n\lambda_{max}(A)}{\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}} > n$$

Using the fact that the infinity norm of A is greater than or equal to its row sums, we have:

$$\frac{\|A\|_\infty}{\lambda_{max}(A)} \geq 1$$

Multiplying both sides by n yields:

$$\frac{n\|A\|_\infty}{\lambda_{max}(A)} \geq n$$

Combining the above inequalities, we get:

$$\frac{\|A\|_\infty}{\lambda_{\max}(A)} \geq \frac{n}{\min_{i=1,\ldots,n} \sum_{j=1}^{n} a_{ij}}.$$

□

Example 4. *Let A be an 4×4 inconsistent pairwise comparison matrix.*

$$A = \begin{pmatrix} 1.0000 & 0.1540 & 0.0627 & 0.1305 \\ 6.4931 & 1.0000 & 0.3371 & 0.9015 \\ 15.9500 & 2.9663 & 1.0000 & 0.2203 \\ 7.6609 & 1.1092 & 4.5385 & 1.0000 \end{pmatrix}.$$

In this example, $\lambda_{max}(A) = 4.7790$ and $\|A\|_\infty = 20.1367$. Then, $\frac{\|A\|_\infty}{\lambda_{\max}(A)} = 4.2136$. On the other hand, $n = 4$ and $\min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij} = 1.3472$. Thus,

$$\frac{\|A\|_\infty}{\lambda_{\max}(A)} = 4.2136 > \frac{n}{\min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}} = 2.9690.$$

Theorem 5. *Let A be an inconsistent pairwise comparison matrix with all positive eigenvalues, and let $\lambda_{\max}(A)$ be the largest eigenvalue of A. Then, we have:*

$$\frac{\lambda_{\max}}{2} \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij} < \|A\|_\infty$$

Proof. Suppose that A is an inconsistent pairwise comparison matrix with all positive eigenvalues, and let $\lambda_{\max}(A)$ be its largest eigenvalue. We can use the fact that A is inconsistent to obtain a lower bound for its Perron norm.

Recall that the consistency ratio of A is defined as $\frac{\lambda_{\max}(A)}{\min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}}$. Since A is inconsistent, its consistency ratio is greater than 1. Thus, we have:

$$\frac{\lambda_{max}(A)}{\min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}} > 1$$

Multiplying both sides by $\min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}$ yields:

$$\lambda_{max}(A) > \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}$$

Using the triangle inequality, we have:

$$\|A\|_\infty \geq \max_{i=1,\ldots,n} \sum_{j=1}^n a_{ij} - \max_{i=1,\ldots,n} \sum_{j=1}^n (-a_{ij}) \geq 2 \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}$$

Therefore, we have:

$$\frac{\|A\|_\infty}{\lambda_{\max}(A)} \geq \frac{2 \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}}{\lambda_{\max}} > 1$$

This inequality can be rearranged to obtain a lower bound for the Perron norm of A:

$$\|A\|_\infty > \frac{\lambda_{\max}(A)}{2} \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}$$

Therefore, we have shown that the Perron norm of an inconsistent pairwise comparison matrix with all positive eigenvalues is bounded below by

$$\frac{\lambda_{\max}(A)}{2} \min_{i=1,\ldots,n} \sum_{j=1}^n a_{ij}.$$

□

Theorem 6. *There is no upper bound for the Perron norm of an inconsistent pairwise comparison matrix.*

Proof. First, let us define what we mean by an inconsistent pairwise comparison matrix. A pairwise comparison matrix is said to be inconsistent if there exist at least one pair of elements for which the ratio of their weights is not consistent with the ratio of their weights as compared to a third element. In other words, if $A_{i,j}$ denotes the weight given by element i compared to element j, and $A_{j,k}$ denotes the weight given by element j compared to element k, then an inconsistent pairwise comparison matrix is one for which there exist i, j, and k such that $A_{i,j} \cdot A_{j,k} \neq A_{i,k}$.

Now, let A be an inconsistent pairwise comparison matrix of size $n \times n$. Let λ be the largest eigenvalue of A, and let v be a corresponding eigenvector with non-negative entries (i.e., a positive eigenvector). We can assume that the entries of v are normalized so that their sum is 1.

Consider the matrix $B = A + \epsilon I$, where ϵ is a small positive constant and I is the identity matrix of size $n \times n$. We can think of B as a perturbation of A by adding a small positive constant to each of its diagonal entries. It is easy to see that B is a consistent pairwise comparison matrix, since the diagonal entries are now all 1 and the pairwise comparisons remain unchanged.

The largest eigenvalue of B is given by $\lambda + \epsilon$, and a corresponding eigenvector is given by v. Since v has non-negative entries, the corresponding eigenvector for $\lambda + \epsilon$ also has non-negative entries. Moreover, for ϵ small enough, $\lambda + \epsilon$ is still strictly greater than any other eigenvalue of B.

Therefore, the Perron norm of B, which is equal to $\lambda + \epsilon$, can be made arbitrarily large by choosing ϵ sufficiently small. Since B is a consistent pairwise comparison matrix, this shows that there is no upper bound for the Perron norm of an inconsistent pairwise comparison matrix A.

Therefore, we have proved that there is no upper bound for the Perron norm of an inconsistent pairwise comparison matrix. □

Proposition 1. *A subset S of normed linear space X is said to be bounded if it is bounded with respect to the metric induced by the norm.*

Proposition 2. *Show that a subset S in a normed space X is bounded iff there exists $c > 0$ such that $\|x\| \leq c$ for every $x \in S$.*

Corollary 3. *The Lie group of $n \times n$ multiplicative pairwise matrices PCM is not bounded by the Perron norm. However, multiplicative consistent pairwise matrices PCM is bounded.*

Proof. Let A be a consistent $n \times n\, PC$ matrix. The the norm of A is n. So it is bounded. If A is an inconsistent $n \times n\, PC$ matrix. Then, by Theorem 6 and Proposition 2, the $n \times n$ multiplicative inconsistent pairwise matrices PCM is not bounded by the Perron norm. □

Theorem 7. *The Lie group (Lie algebra) of $n \times n$ multiplicative (additive respectively) pairwise matrices PCM is (topologically) closed.*

Proof. Let $\mathbf{G}$ be the Lie algebra of additive pairwise comparison matrices PCM. If $H = [h_{ij}] \in \mathbf{G}$ then $h_{ij} = -h_{ji}$. It implies that $h_{ii} = 0$. So the trace of H is $tr(H) = 0$ for every additive PCM. So the trace function on additive pairwise comparison matrices is constant 0, then it is continuous. Since $\{0\}$ is a closed set then inverse image of $\{0\}$ under trace function is closed too. Hence the Lie algebra of additive pairwise comparison matrices $\mathbf{G}$ is closed. On the other hand, we know that the map $\exp : \mathbf{G} \to G$ is a closed map and onto, therefore the Lie group of $n \times n$ multiplicative pairwise matrices G is (topologically) closed. □

Since a closed and bounded set is compact, then we have the following result.

Corollary 4. *The subset $n \times n$ multiplicative consistent pairwise matrices G is compact. However, the Lie group (Lie algebra) of $n \times n$ multiplicative (additive respectively) pairwise matrices PCM is not compact.*

Theorem 8. *The Lie group (Lie algebra) of $n \times n$ multiplicative (additive respectively) pairwise matrices PCM is connected.*

Proof. Since the singleton set $\{0\}$ is a connected then inverse image of $\{0\}$ under trace function is connected too. Hence, the Lie algebra of additive pairwise comparison matrices $\mathbf{G}$ is connected. On the other hand, we know that the map $\exp : \mathbf{G} \to G$ is a continuous map and onto, therefore the Lie group of $n \times n$ multiplicative pairwise matrices G is connected too because continuous image of a connected set is connected. □

4. Conclusion

Our paper demonstrates that the multiplicative PC matrices (not the elements of a PC matrix) generate a Lie group for the Hadamard product. Lie algebras of these Lie groups are identified here. It has been shown that Lie algebras form spaces of skew-symmetric matrices. Then, we prove that both additive and multiplicative PCM are connected while only consistent multiplicative PCM is compact. Also, we find that there is no upper bound for the Perron norm of an inconsistent pairwise comparison matrix.

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