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Research Article

Estimates for eigenvalues of Sturm–Liouville operators with some PT-symmetric potentials

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Abstract

We provide some useful equations for calculating the periodic and antiperiodic eigenvalues of the one-dimensional Schrödinger operator $S(q)$ with a special potential that is a PT-symmetric trigonometric polynomial. We even give estimates to approximate complex eigenvalues by the roots of some polynomials derived from some iteration formulas. Moreover, we give a numerical example with error estimation using Rouche's theorem.

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1. Introduction

In this article, we are interested in the operators $S_t(q)$, for $t = 0, 1$, generated in $L_2[0, \pi]$ by the expression

$$-y''(x) + q(x)y(x) \tag{1.1}$$

and the periodic and antiperiodic boundary conditions

$$y(\pi) = e^{i\pi t}y(0), \qquad y'(\pi) = e^{i\pi t}y'(0), \tag{1.2}$$

where q is the trigonometric polynomial potential of the form

$$q(x) = q_{-m}e^{-i2mx} + q_m e^{i2mx}, \qquad m \geq 2, \tag{1.3}$$

$(q_{-m}q_m) \in \mathbb{R}$ and $m \in \mathbb{Z}$. Here we choose the notations q_{-m} and q_m to mention the Fourier coefficients of the potential. Note that, in the case $m = 1$, potential (1.3) can be considered as the optical potential with $q_{-1} = 1 - 2V$, $q_1 = 1 + 2V$, $V \geq 0$. This case has been investigated in our another work [15].

It was proved by Veliev [18, see Theorem 1 and (26)] that, if $ab = cd$, where a, b, c, and d are arbitrary complex numbers, then the Hill operators $S(q)$ and $S(p)$ generated in $L_2(-\infty, \infty)$ by differential expression (1.1) with the potentials $q(x) = ae^{-i2x} + be^{i2x}$ and $p(x) = ce^{-i2x} + de^{i2x}$, have the same Hill discriminant, and hence the same Bloch eigenvalues and spectrum. Therefore, the investigations of the operators $S_t(q)$, for $t = 0, 1$, can

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be reduced to the investigations of the operators generated in $L_2[0,\pi]$ by expression (1.1) and the boundary conditions (1.2) with the potential

$$p(x) = r_m e^{-i2mx} + r_m e^{i2mx} = 2r_m \cos(2mx), \tag{1.4}$$

where $r_m = \sqrt{q_{-m}q_m}$. It is well known that the spectra of the operators $S_0(q)$ and $S_1(q)$ are discrete and for large enough n, there are two periodic (if n is even) or antiperiodic (if n is odd) eigenvalues (counting multiplicities) in the neighborhood of n^2. See the basic and detailed classical results in [4, 9–11] and references therein.

Note that, the trigonometric polynomial potential (1.3) is a PT-symmetric potential if $q_{-m}, q_m \in \mathbb{R}$. For the properties of the general PT-symmetric potentials, see [1–3, 12, 20, 22, 24] and references therein. Here, we only note that, the investigations of PT-symmetric periodic potentials were initiated by Bender et al. [2].

The eigenvalues of the operators $S_0(0)$ and $S_1(0)$ are $(2n)^2$ and $(2n+1)^2$, for $n \in \mathbb{Z}$, respectively and all the eigenvalues of $S_0(0)$ and $S_1(0)$, except 0, are double. The eigenvalues of $S_0(q)$ and $S_1(q)$ are called the periodic and antiperiodic eigenvalues of the one-dimensional Schrödinger operator $S(q)$, generated in $L_2(-\infty,\infty)$ by expression (1.1) with potential (1.3), and they are denoted by $\lambda_n(q)$, for $n \in \mathbb{Z}$ and $\mu_n(q)$, for $n \in \mathbb{Z} - \{0\}$, respectively.

It is well known that (see [6, 10, 11]), if r_m is a real nonzero number, then all the eigenvalues of the operator $H_t(r_m)$, generated in $L_2[0,\pi]$ by expression (1.1) and the boundary conditions (1.2) with potential (1.4), are real, for all $t \in (-1,1]$, and the spectrum $\sigma(H(r_m))$ of the Schrödinger operator $H(r_m)$, generated in $L_2(-\infty,\infty)$ by expression (1.1) with potential (1.4), consists of the real intervals

$$\Gamma_1 := [\lambda_0(r_m), \mu_{-1}(r_m)], \quad \Gamma_2 := [\mu_{+1}(r_m), \lambda_{-1}(r_m)], \quad \Gamma_3 := [\lambda_{+1}(r_m), \mu_{-2}(r_m)],$$
$$\Gamma_4 := [\mu_{+2}(r_m), \lambda_{-2}(r_m)], \ldots,$$

where $\lambda_0(r_m)$, $\lambda_{-n}(r_m)$, $\lambda_{+n}(r_m)$, for $n = 1, 2, \ldots$, are the eigenvalues of $H_0(r_m)$ and $\mu_{-n}(r_m)$, $\mu_{+n}(r_m)$, for $n = 1, 2 \ldots$, are the eigenvalues of $H_1(r_m)$ and the following inequalities hold:

$$\lambda_0(r_m) < \mu_{-1}(r_m) \le \mu_{+1}(r_m) < \lambda_{-1}(r_m) \le \lambda_{+1}(r_m) < \mu_{-2}(r_m) \le \mu_{+2}(r_m)$$
$$< \lambda_{-2}(r_m) \le \lambda_{+2}(r_m) < \cdots .$$

The bands $\Gamma_1, \Gamma_2, \ldots$ of the spectrum $\sigma(H(r_m))$ of $H(r_m)$ are separated by the gaps

$$\Delta_1 := (\mu_{-1}(r_m), \mu_{+1}(r_m)), \quad \Delta_2 := (\lambda_{-1}(r_m), \lambda_{+1}(r_m)), \quad \Delta_3 := (\mu_{-2}(r_m), \mu_{+2}(r_m)), \ldots$$

if and only if the eigenvalues at the endpoints of the intervals are simple. In other notation, we can write $\Gamma_n = \{\gamma_n(t) : t \in [0,1]\}$, where $\gamma_1(t), \gamma_2(t), \ldots$ are the eigenvalues of $H_t(r_m)$, called as Bloch eigenvalues corresponding to the quasimomentum t. The Bloch eigenvalue $\gamma_n(t)$, continuously depends on t and $\gamma_n(-t) = \gamma_n(t)$. These statements remain valid for $S_t(q)$ and $S(q)$ if $q_{-m}q_m > 0$. (see also [21])

Obviously, $\lambda_{-n}(r_m)$ and $\lambda_{+n}(r_m)$, for $n = 1, 2, \ldots$ are the $(2n)$th and $(2n+1)$th periodic eigenvalues; $\mu_{-n}(r_m)$ and $\mu_{+n}(r_m)$, for $n = 1, 2, \ldots$ are the $(2n-1)$th and $(2n)$th antiperiodic eigenvalues, respectively.

If one of the numbers q_{-m} and q_m is zero and the other one is real in (1.3), then all the eigenvalues of the operator $S_0(q)$, except 0, are double and they are equal to $(2n)^2$. This fact was proved for the first time in [7]. This case was investigated also in [8, 14, 19]. In [14], we investigated the operators $S_t(q)$, for $t = 0, 1$, with potential (1.3), when the periodic and antiperiodic eigenvalues are real.

In this paper, we give estimates for the eigenvalues of $S_0(q)$ and $S_1(q)$, when $(q_{-m}q_m) \in \mathbb{R}$. We provide some useful equations to approximate the periodic and antiperiodic eigenvalues for different values of m in the potential. We even calculate complex eigenvalues by approximating the roots of some polynomials derived from some iteration formulas. Finally, we give a numerical example with error estimation using Rouche's theorem.

Arguing as in the proof of Theorem 9 of [23], one can prove that, the periodic eigenvalues $\lambda_{\pm n}(q)$, for $n \geq 2$, lie in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|r_m|\}$, for $n = 2, 3, 4, \ldots$ and $|r_m| < 2n - 1$, where $r_m = \sqrt{q_{-m}q_m}$. Moreover, the disk D_n, for $n \geq 2$, has no common points with another disk D_m, for $m \neq n$ and the boundary of the disk $D_{n,\epsilon} := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|r_m| + \epsilon\}$, for $n = 2, 3 \ldots$, belongs to the resolvent set of the operator $S_0(q)$, for all $|r_m| < 2n-1$, if ϵ is a sufficiently small positive number. It implies that, the number of eigenvalues (counting the multiplicity) of $S_0(q)$ lying in $D_{n,\epsilon}$, for $n \geq 2$, are the same for all $|r_m| < 2n-1$. Since $S_0(0)$ has two eigenvalues in $D_{n,\epsilon}$, for $n \geq 2$, the operator $S_0(q)$ has also two eigenvalues for $|r_m| < 2n - 1$. Letting ϵ tend to zero, we obtain that $S_0(q)$ has two eigenvalues (counting the multiplicity) in D_n, for $n \geq 2$ and $|r_m| < 2n - 1$. By the same token, we can prove that $S_0(q)$ has 2 eigenvalues in D_1, for $|q_{-2}| + |q_2| \leq 29/10$, $m = 2$ and for $|q_{-m}| + |q_m| \leq 7/2$, $m \geq 3$; and that it has one eigenvalue in D_0, for $|q_{-2}| + |q_2| \leq 2$, $m = 2$, and for $|q_{-m}| + |q_m| \leq 3$, $m \geq 3$.

Similarly, $S_1(q)$ has two eigenvalues (counting the multiplicity) in $d_n := \{\mu \in \mathbb{C} : |\mu - (2n-1)^2| \leq 2|r_m|\}$, for $n = 1, 2, \ldots$ and $|r_m| < 2n$. We denote the first, $(2n)$th and $(2n+1)$st periodic eigenvalues by $\lambda_0(q)$, $\lambda_{-n}(q)$ and $\lambda_{+n}(q)$, for $n = 1, 2, \ldots$; the $(2n-1)$st and $(2n)$th antiperiodic eigenvalues by $\mu_{-n}(q)$ and $\mu_{+n}(q)$, for $n = 1, 2, \ldots$, respectively.

Therefore,

$$|\lambda_{\pm n}(q) - \lambda_{\pm n}(0)| \leq 2|r_m|,$$
$$|\mu_{\pm n}(q) - \mu_{\pm n}(0)| \leq 2|r_m|,$$

for $n = 1, , 2 \ldots$, where $\lambda_{\pm n}(0) = (2n)^2$, $\mu_{\pm n}(0) = (2n-1)^2$ and $r_m = \sqrt{q_{-m}q_m}$. Moreover, for $n = 0$, we have $|\lambda_0(q)| \leq 2|r_m|$. Thus, we write

$$(2n)^2 - 2|r_m| \leq |\lambda_n| \leq (2n)^2 + 2|r_m| \tag{1.5}$$

and

$$|\lambda_n - (2k)^2| \geq |(2n)^2 - (2k)^2| - 2|r_m| = 4|n-k||n+k| - 2|r_m| \geq 4|2n-1| - 2|r_m|,$$

for $n \in \mathbb{Z}$ and $k \neq \pm n$. In particular, if $n = 1$, then $|\lambda_{\pm 1}| \leq 4 + 2|r_m|$ and

$$|\lambda_{\pm 1} - (2k)^2| \geq ||\lambda_{\pm 1}| - (2k)^2| \geq 16 - |\lambda_{\pm 1}| \geq 12 - 2|r_m|, \tag{1.6}$$

for $k \geq 2$. Further, if $|n| \geq 2$, we have $|\lambda_n| \geq |\lambda_{-2}| \geq 16 - 2|r_m|$ and

$$|\lambda_n - (2k)^2| \geq ||\lambda_{-2}| - (2k)^2| \geq |\lambda_{-2}| - 4 \geq 12 - 2|r_m|, \tag{1.7}$$

for $k \neq \pm n$. Similar inequalities can be written for the antiperiodic eigenvalues from

$$(2n-1)^2 - 2|r_m| \leq |\mu_{\pm n}| \leq (2n-1)^2 + 2|r_m|, \tag{1.8}$$

for $n = 1, 2, \ldots$.

2. Main results

First, we study the operator $S_0(q)$ and periodic eigenvalues. We note that, by the notation λ_n, we mean the $(2n)$th and $(2n+1)$th periodic eigenvalues λ_{-n} and λ_{+n}, for $n = 1, 2, \ldots$. We also note that, the following relations and iteration formulas have been used by Veliev and his collaborators to obtain asymptotic formulas for large periodic eigenvalues, corresponding eigenfunctions and the length of the gaps in the spectrum of the Schrödinger operators with different potentials, see for example [5,17,24–26]. We also used them to obtain numerical estimations for periodic eigenvalues in different cases [13–16]. In the present paper, we find completely different conditions on the potential than those of our other works for which the periodic eigenvalues satisfy the iteration formulas obtained below and the calculations are quite long and technical. We begin with the equations

$$(\lambda_N - (2n)^2)(\Psi_N, e^{i2nx}) = (q\Psi_N, e^{i2nx}), \tag{2.1}$$

$$(\lambda_N - (2n)^2)(\Psi_N, e^{-i2nx}) = (q\Psi_N, e^{-i2nx}) \tag{2.2}$$

which are derived from

$$-\Psi_N''(x) + q(x)\Psi_N(x) = \lambda_N \Psi_N(x),$$

by multiplying both sides of the last equality by e^{i2nx} and e^{-i2nx}, respectively, where $\Psi_N(x)$ is an eigenfunction corresponding to the eigenvalue λ_N. Iterating equation (2.1) k times, as done in [5], we obtain

$$(\lambda_n - (2n)^2 - \sum_{j=1}^{k} \alpha_j(\lambda_n))(\Psi_n, e^{i2nx}) - (q_{2n} + \sum_{j=1}^{k} \beta_j(\lambda_n))(\Psi_n, e^{-i2nx}) = \rho_k(\lambda_n), \tag{2.3}$$

where

$$\alpha_j(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$

$$\beta_j(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{2n-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-n_2-\cdots-n_j))^2]},$$

$$\rho_k(\lambda_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{n_{k+1}} (q\Psi_n, e^{i2(n-n_1-\cdots-n_{k+1})x})}{[\lambda_n - (2(n-n_1))^2] \cdots [\lambda_n - (2(n-n_1-\cdots-n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, 2n$, for $l = 1, 2, ..., k+1$. Note that, for the trigonometric polynomial potential of the form (1.3), we have $q_i = 0$ for $i \neq \pm m$.

Similarly, iterating equation (2.2) k times, we obtain

$$(\lambda_n - (2n)^2 - \sum_{j=1}^{k} \alpha_j^*(\lambda_n))(\Psi_n, e^{-i2nx}) - (q_{-2n} + \sum_{j=1}^{k} \beta_j^*(\lambda_n))(\Psi_n, e^{i2nx}) = \rho_k^*(\lambda_n), \tag{2.4}$$

where

$$\alpha_j^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$\beta_j^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_j} \frac{q_{n_1} q_{n_2} \cdots q_{n_j} q_{-2n-n_1-n_2-\cdots-n_j}}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_j))^2]},$$

$$\rho_k^*(\lambda_n) = \sum_{n_1, n_2, \ldots, n_{k+1}} \frac{q_{n_1} q_{n_2} \cdots q_{n_k} q_{n_{k+1}} (q\Psi_n, e^{-i2(n+n_1+\cdots+n_{k+1})x})}{[\lambda_n - (2(n+n_1))^2] \cdots [\lambda_n - (2(n+n_1+\cdots+n_{k+1}))^2]}.$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, -2n$ for $l = 1, 2, ..., k+1$. We note that, analogous iteration formulas to (2.3) and (2.4) were derived and used in [5] for large eigenvalues to obtain asymptotic formulas. In this paper, we find conditions on the Fourier coefficients of potential (1.3) for which the iteration formulas (2.3) and (2.4) are also valid for the small eigenvalues, as k tends to infinity. We also note that, finding such conditions is not easy, there are many technical calculations. Since the potential q is the trigonometric polynomial potential of the form (1.3), we have the followings, after some calculations (see [14]):

$$\alpha^*_{2j-1}(\lambda_n) = \alpha_{2j-1}(\lambda_n), \qquad \alpha^*_{2j}(\lambda_n) = \alpha_{2j}(\lambda_n) = 0, \qquad \beta^*_j(\lambda_n) = (\frac{q_{-m}}{q_m})^{2n/m}\beta_j(\lambda_n), \tag{2.5}$$

for $j = 1, 2, \ldots$. Now, in order to give the main results, we prove the following lemma. Without loss of generality, we assume that $\Psi_n(x)$ is a normalized eigenfunction corresponding to the eigenvalue λ_n.

Lemma 2.1. *The statements*

(a) $\lim_{k\to\infty} \rho_k(\lambda_n) = 0$, $\qquad \lim_{k\to\infty} \rho^*_k(\lambda_n) = 0$,

(b) $|u_n|^2 + |v_n|^2 > 0$, *where* $u_n = (\Psi_n, e^{i2nx})$ *and* $v_n = (\Psi_n, e^{-i2nx})$,

are valid in the following cases:

(i) *if* $|q_{-2}| + |q_2| \le 29/10$, *for* $n = 1$ *and* $m = 2$,

(ii) *if* $|q_{-m}| + |q_m| \le 7/2$, *for* $n = 1$ *and* $m \ge 3$,

(iii) *if* $|r_m| < 2s - 1$, *for* $n \ge s$, $s = 2, 3, \ldots$ *and* $m \ge 2$, *where* $r_m = \sqrt{q_{-m}q_m}$.

Proof. **(a)** By the definition of $\rho_k(\lambda_n)$ and the conditions imposed on the summations, the number of summands of $\rho_{2k+1}(\lambda_n)$ is not greater than 4^{2k}.On the other hand, by (1.5)-(1.7), we have

$$|\lambda_1 - 16| \ge 12 - 2|r_m|, \qquad |\lambda_1 - 36| \ge 32 - 2|r_m|, \qquad |\lambda_1 - 100| \ge 96 - 2|r_m|,$$
$$16 - 2|r_m| \le |\lambda_2| \le 16 + 2|r_m|, \qquad |\lambda_2 - 4| \ge 12 - 2|r_m|, \qquad |\lambda_2 - 64| \ge 48 - 2|r_m|,$$

and

$$\begin{aligned}|\lambda_n - (2(n-m))^2| &\ge 4m|2n-m| - 2|r_m| > 4m|2s-m| - 2(2s-1)\\ &\ge 4(s+1)(s-1) - 2(2s-1) = 4s^2 - 4s - 2, \qquad m \ne 2n,\\ |\lambda_n - (2(n-2m))^2| &\ge 16m|n-m| - 2|r_m| > 16m|s-m| - 2(2s-1)\\ &\ge 16(s+1) - 2(2s-1) = 12s + 18 = 6(2s+3), \qquad m \ne n.\end{aligned}$$

Hence, using $2|r_m| \le |q_{-m}| + |q_m|$ and considering the greatest summands of $\rho_{2k+1}(\lambda_n)$ in absolute value, we obtain for case (i)

$$\begin{aligned}|\rho_{2k+1}(\lambda_1)| &< \frac{4^{2k}|q_{-2}|^k|q_2|^{k+1}2|r_2|\sqrt{\pi}}{|\lambda_1 - 36|^{k+1}|\lambda_1 - 100|^k} \le \frac{2|q_2|\sqrt{\pi}4^{2k}|r_2|^{2k+1}}{(32 - 2|r_2|)^{k+1}(96 - 2|r_2|)^k}\\ &< \frac{3|q_2|\sqrt{\pi}4^{2k}3^{2k}}{2^{2k}29^{k+1}93^k} = \frac{3|q_2|\sqrt{\pi}12^k12^k}{4^k29^{k+1}93^k} = \frac{3|q_2|\sqrt{\pi}}{29}(\frac{12}{899})^k,\end{aligned}$$

for case (ii)

$$\begin{aligned}|\rho_{2k+1}(\lambda_1)| &< \frac{4^{2k}|q_{-3}|^k|q_3|^{k+1}2|r_3|\sqrt{\pi}}{|\lambda_1 - 16|^{k+1}|\lambda_1 - 100|^k} \le \frac{2|q_3|\sqrt{\pi}4^{2k}|r_3|^{2k+1}}{(12 - 2|r_3|)^{k+1}(96 - 2|r_3|)^k}\\ &< \frac{4|q_3|\sqrt{\pi}4^{2k}2^{2k}}{8^{k+1}92^k} = \frac{|q_3|\sqrt{\pi}}{2}\frac{16^k4^k}{8^k92^k} = \frac{|q_3|\sqrt{\pi}}{2}(\frac{2}{23})^k,\end{aligned}$$

and for case (iii)

$$\begin{aligned}|\rho_{2k+1}(\lambda_n)| &< \frac{4^{2k}|q_{-m}|^k|q_m|^{k+1}2|r_m|\sqrt{\pi}}{|\lambda_n - (2(n-m))^2|^{k+1}|\lambda_n - (2(n-2m))^2|^k}\\ &\le \frac{4^{2k}|q_{-3}|^k|q_3|^{k+1}2|r_3|\sqrt{\pi}}{|\lambda_2 - 4|^{k+1}|\lambda_2 - 64|^k} \le \frac{2|q_3|\sqrt{\pi}4^{2k}|r_3|^{2k+1}}{(12 - 2|r_3|)^{k+1}(48 - 2|r_3|)^k}\\ &< \frac{6|q_3|\sqrt{\pi}4^{2k}3^{2k}}{6^{k+1}42^k} = |q_3|\sqrt{\pi}\frac{16^k9^k}{6^k42^k} = |q_3|\sqrt{\pi}(\frac{4}{7})^k.\end{aligned}$$

Thus, in any case $|\rho_{2k+1}(\lambda_n)| < ca^k$, for some constant $c > 0$ and $0 < a < 1$, and hence $\lim_{k\to\infty} \rho_k(\lambda_n) = 0$. Similarly, we show that $\lim_{k\to\infty} \rho^*_k(\lambda_n) = 0$.

(b) Assume the contrary, $u_n = 0$ and $v_n = 0$. Because the system of root functions $\{e^{2ikx}/\sqrt{\pi} : k \in \mathbb{Z}\}$ of $S_0(0)$ is an orthonormal basis for $L_2[0,\pi]$, we write the decomposition

$$\pi\Psi_n = u_n e^{i2nx} + v_n e^{-i2nx} + \sum_{k\in\mathbb{Z},k\neq\pm n} \left(\Psi_n, e^{i2kx}\right) e^{i2kx}$$

for the normalized eigenfunction Ψ_n corresponding to the eigenvalue λ_n of $S_0(q)$. By Parseval's equality, we have

$$\sum_{k\in\mathbb{Z},k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi.$$

First, we consider the case $n = 1$, for $m = 2$, and for $m \geq 3$, namely for the cases (i) and (ii), respectively. Using the relations (1.6) and (2.1), the Bessel inequality, and taking

$$(q\Psi_1, 1) = q_{-m}(\Psi_1, e^{i2mx}) + q_m(\Psi_1, e^{-i2mx})$$

into account, we obtain for $m = 2$

$$\begin{aligned}
\sum_{k\in\mathbb{Z},k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 &= \frac{|(q\Psi_1, 1)|^2}{|\lambda_1|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_1, e^{i2kx})|^2}{|\lambda_1 - (2k)^2|^2} \\
&\leq \frac{(|q_{-2}||(q\Psi_1, e^{i4x})| + |q_2||(q\Psi_1, e^{-i4x})|)^2}{|\lambda_1|^2|\lambda_1 - 16|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_1, e^{i2kx})|^2}{|\lambda_1 - 16|^2} \\
&\leq \frac{(|q_{-2}| + |q_2|)^2 \pi (2|r_2|)^2}{(4 - 2|r_2|)^2(12 - 2|r_2|)^2} + \frac{1}{(12 - 2|r_2|)^2} \sum_{k\neq 0,\pm 1} |(q\Psi_1, e^{i2kx})|^2 \\
&\leq \frac{\pi(29/10)^4}{(11/10)^2(91/10)^2} + \frac{\pi(29/10)^2}{(91/10)^2} < \frac{81\pi}{100} < \pi,
\end{aligned}$$

and for $m \geq 3$

$$\begin{aligned}
\sum_{k\in\mathbb{Z},k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 &= \frac{|(q\Psi_1, 1)|^2}{|\lambda_1|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_1, e^{i2kx})|^2}{|\lambda_1 - (2k)^2|^2} \\
&\leq \frac{(|q_{-m}||(q\Psi_1, e^{i2mx})| + |q_m||(q\Psi_1, e^{-i2mx})|)^2}{|\lambda_1|^2|\lambda_1 - (2m)^2|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_1, e^{i2kx})|^2}{|\lambda_1 - 16|^2} \\
&\leq \frac{(|q_{-3}| + |q_3|)^2 \pi (2|r_3|)^2}{(4 - 2|r_3|)^2(36 - 2|r_3|)^2} + \frac{\pi(2|r_3|)^2}{(12 - 2|r_3|)^2} \leq \frac{\pi(7/2)^4}{(1/2)^2(57/2)^2} + \frac{\pi(7/2)^2}{(17/2)^2} < \frac{91\pi}{100} < \pi,
\end{aligned}$$

which contradict $\sum_{k\in\mathbb{Z},k\neq\pm 1} |(\Psi_1, e^{i2kx})|^2 = \pi$.

Now, we consider the case (iii), namely the case $|r_m| < 2s - 1$ and $n \geq s$, for $s \geq 2$. Using $(2n)^2 - 2|r_m| \leq |\lambda_n| \leq (2n)^2 + 2|r_m|$, we obtain

$$\begin{aligned}
|\lambda_n - (2k)^2| &\geq |\lambda_n - (2(n-1))^2| \geq (2n)^2 - 2|r_m| - (2(n-1))^2 \\
&= 4(2n-1) - 2|r_m| > 4(2s-1) - 2(2s-1) = 4s - 2.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\sum_{k\in\mathbb{Z},k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 &= \sum_{k\in\mathbb{Z},k\neq\pm n} \frac{|(q\Psi_n, e^{i2kx})|^2}{|\lambda_n - (2k)^2|^2} \\
&< \frac{1}{(4s-2)^2} \sum_{k\in\mathbb{Z},k\neq\pm n} |(q\Psi_n, e^{i2kx})|^2 \leq \frac{\pi(2|r_m|)^2}{(4s-2)^2} < \pi,
\end{aligned}$$

which contradicts $\sum_{k\in\mathbb{Z},k\neq\pm n} |(\Psi_n, e^{i2kx})|^2 = \pi$ and the lemma is proved. □

Now, we consider the statements of Lemma 2.1 for the case $n = 0$:

Lemma 2.2. *The statements* ***(a)*** $\lim_{k\to\infty}\rho_k(\lambda_0)=0$ *and* ***(b)*** $|(\Psi_0,1)|>0$ *hold in the following cases:*
(i) *if* $|q_{-2}|+|q_2|\le 2$*, for* $m=2$*,*
(ii) *if* $|q_{-m}|+|q_m|\le 3$*, for* $m\ge 3$*.*

Proof. The proof of (a) is obvious in each case. For the proof of (b), suppose the contrary $(\Psi_0,1)=0$. Isolating the terms $|(\Psi_0,e^{-i2x})|^2$ and $|(\Psi_0,e^{i2x})|^2$ in Parseval's equality, we can write

$$|(\Psi_0,e^{-i2x})|^2+|(\Psi_0,e^{i2x})|^2+\sum_{k\neq 0,\pm 1}|(\Psi_0,e^{i2kx})|^2=\pi.$$

First, we estimate $|(q\Psi_0,e^{-i2x})|^2+|(q\Psi_0,e^{i2x})|^2$ for each case. Using (2.1), the relations $|\lambda_0|\le 2|r_m|$, $|\lambda_0-4|\ge 4-|\lambda_0|\ge 4-2|r_m|$ and

$$\begin{aligned}(q\Psi_0,e^{-i2x})&=q_{-2}(\Psi_0,e^{i2x})+q_2(\Psi_0,e^{-i6x}),\\(q\Psi_0,e^{i2x})&=q_{-2}(\Psi_0,e^{i6x})+q_2(\Psi_0,e^{-i2x}),\\(q\Psi_0,e^{-i2x})&=q_{-3}(\Psi_0,e^{i4x})+q_3(\Psi_0,e^{-i8x}),\\(q\Psi_0,e^{i2x})&=q_{-3}(\Psi_0,e^{i8x})+q_2(\Psi_0,e^{-i4x}),\end{aligned}$$

we obtain for case (i)

$$\begin{aligned}|(q\Psi_0,e^{-i2x})|&\le\frac{|q_{-2}(q\Psi_0,e^{i2x})|}{|\lambda_0-4|}+\frac{|q_2(q\Psi_0,e^{-i6x})|}{|\lambda_0-36|}\\&\le\frac{|q_{-2}q_2(q\Psi_0,e^{-i2x})|}{|\lambda_0-4|^2}+\frac{|(q_{-2})^2(q\Psi_0,e^{i6x})|}{|\lambda_0-4||\lambda_0-36|}+\frac{|q_2(q\Psi_0,e^{-i6x})|}{|\lambda_0-36|},\\&\le\frac{2|r_2|^3\sqrt{\pi}}{(4-2|r_2|)^2}+\frac{|q_{-2}|^2 2|r_2|\sqrt{\pi}}{(4-2|r_2|)(36-2|r_2|)}+\frac{|q_2|2|r_2|\sqrt{\pi}}{(36-2|r_2|)}\\&\le\frac{\sqrt{\pi}}{2}+\frac{|q_{-2}|^2\sqrt{\pi}}{34}+\frac{2|q_2|\sqrt{\pi}}{34}<\frac{\sqrt{\pi}}{2}+\frac{4\sqrt{\pi}}{17}=\frac{25\sqrt{\pi}}{34},\end{aligned}$$

and

$$\begin{aligned}|(q\Psi_0,e^{i2x})|&\le\frac{|q_{-2}(q\Psi_0,e^{i6x})|}{|\lambda_0-36|}+\frac{|q_2(q\Psi_0,e^{-i2x})|}{|\lambda_0-4|}\\&\le\frac{|q_{-2}(q\Psi_0,e^{i6x})|}{|\lambda_0-36|}+\frac{|q_{-2}q_2(q\Psi_0,e^{i2x})|}{|\lambda_0-4|^2}+\frac{|(q_2)^2(q\Psi_0,e^{-i6x})|}{|\lambda_0-4||\lambda_0-36|},\\&\le\frac{|q_{-2}|2|r_2|\sqrt{\pi}}{(36-2|r_2|)}+\frac{2|r_2|^3\sqrt{\pi}}{(4-2|r_2|)^2}+\frac{|q_2|^2 2|r_2|\sqrt{\pi}}{(4-2|r_2|)(36-2|r_2|)}\\&\le\frac{2|q_{-2}|\sqrt{\pi}}{34}+\frac{\sqrt{\pi}}{2}+\frac{|q_2|^2\sqrt{\pi}}{34}<\frac{\sqrt{\pi}}{2}+\frac{4\sqrt{\pi}}{17}=\frac{25\sqrt{\pi}}{34},\end{aligned}$$

for case (ii)

$$\begin{aligned}|(q\Psi_0,e^{-i2x})|^2&\le\Big(\frac{|q_{-3}(q\Psi_0,e^{i4x})|}{|\lambda_0-16|}+\frac{|q_3(q\Psi_0,e^{-i8x})|}{|\lambda_0-64|}\Big)^2\\&\le\Big(\frac{|q_{-3}|2|r_3|\sqrt{\pi}}{16-2|r_3|}+\frac{|q_3|2|r_3|\sqrt{\pi}}{64-2|r_3|}\Big)^2=4|r_3|^2\pi\Big(\frac{|q_{-3}|}{16-2|r_3|}+\frac{|q_3|}{64-2|r_3|}\Big)^2,\\&=4|r_3|^2\pi\Big(\frac{|q_{-3}|^2}{(16-2|r_3|)^2}+\frac{2|q_{-3}q_3|}{(16-2|r_3|)(64-2|r_3|)}+\frac{|q_3|^2}{(64-2|r_3|)^2}\Big)\end{aligned}$$

and

$$|(q\Psi_0, e^{i2x})|^2 \le (\frac{|q_{-3}(q\Psi_0, e^{i8x})|}{|\lambda_0 - 64|} + \frac{|q_3(q\Psi_0, e^{-i4x})|}{|\lambda_0 - 16|})^2$$
$$\le (\frac{|q_{-3}|2|r_3|\sqrt{\pi}}{64 - 2|r_3|} + \frac{|q_3|2|r_3|\sqrt{\pi}}{16 - 2|r_3|})^2 = 4|r_3|^2\pi(\frac{|q_{-3}|}{64 - 2|r_3|} + \frac{|q_3|}{16 - 2|r_3|})^2,$$
$$= 4|r_3|^2\pi(\frac{|q_{-3}|^2}{(64 - 2|r_3|)^2} + \frac{2|q_{-3}q_3|}{(16 - 2|r_3|)(64 - 2|r_3|)} + \frac{|q_3|^2}{(16 - 2|r_3|)^2})$$

and so,

$$|(q\Psi_0, e^{-i2x})|^2 + |(q\Psi_0, e^{i2x})|^2 \le 4|r_3|^2\pi(\frac{|q_{-3}|^2 + |q_3|^2}{(64 - 2|r_3|)^2} + \frac{4|q_{-3}q_3|}{(16 - 2|r_3|)(64 - 2|r_3|)} + \frac{|q_{-3}|^2 + |q_3|^2}{(16 - 2|r_3|)^2})$$
$$< 9\pi(\frac{3^2}{61^2} + \frac{6}{13.61} + \frac{3^2}{13^2}) < 0.57\pi.$$

Using (2.1), the Bessel inequality and taking

$$|(q\Psi_0, e^{-i2x})|^2 + |(q\Psi_0, e^{i2x})|^2 < 2\pi(\frac{25}{34})^2 < 11\pi/10, \qquad m = 2$$

and

$$|(q\Psi_0, e^{-i2x})|^2 + |(q\Psi_0, e^{i2x})|^2 < 0.57\pi < 3\pi/5, \qquad m \ge 3$$

into account, we obtain for case (i)

$$\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 = \frac{|(q\Psi_0, e^{-i2x})|^2}{|\lambda_0 - 4|^2} + \frac{|(q\Psi_0, e^{i2x})|^2}{|\lambda_0 - 4|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_0, e^{i2kx})|^2}{|\lambda_0 - (2k)^2|^2}$$
$$< \frac{11\pi}{10(4 - 2|r_2|)^2} + \frac{1}{(16 - 2|r_2|)^2} \sum_{k\neq 0,\pm 1} |(q\Psi_0, e^{i2kx})|^2$$
$$\le \frac{11\pi}{40} + \frac{\pi(2|r_2|)^2}{14^2} \le \frac{11\pi}{40} + \frac{\pi}{49} < \frac{3\pi}{10} < \pi,$$

and for case (ii)

$$\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 = \frac{|(q\Psi_0, e^{-i2x})|^2}{|\lambda_0 - 4|^2} + \frac{|(q\Psi_0, e^{i2x})|^2}{|\lambda_0 - 4|^2} + \sum_{k\neq 0,\pm 1} \frac{|(q\Psi_0, e^{i2kx})|^2}{|\lambda_0 - (2k)^2|^2}$$
$$< \frac{3\pi}{5(4 - 2|r_3|)^2} + \frac{1}{(16 - 2|r_3|)^2} \sum_{k\neq 0,\pm 1} |(q\Psi_0, e^{i2kx})|^2$$
$$\le \frac{3\pi}{5} + \frac{\pi(2|r_3|)^2}{13^2} < \frac{59\pi}{90} < \pi,$$

which contradict $\sum_{k\in\mathbb{Z},k\neq 0} |(\Psi_0, e^{i2kx})|^2 = \pi$ and complete the proof. □

Now, letting k tend to infinity in the equations (2.3) and (2.4), we obtain the following results. First, we consider the case $n \ge 2$.

Theorem 2.3. *Suppose that $|r_m| < 2s - 1$, for $n \ge s$, $s = 2, 3, \ldots$ and $m \ge 2$, where $r_m = \sqrt{q_{-m}q_m}$.*

(a) *If m is even and $n = m/2$, then $\lambda_{\pm n}$ is an eigenvalue of $S_0(q)$ if and only if it is either the root of the equation*

$$\lambda - (2n)^2 - r_m - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0 \tag{2.6}$$

or the root of

$$\lambda - (2n)^2 + r_m - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0, \tag{2.7}$$

(b) *If $n = m$, then $\lambda_{\pm n}$ is an eigenvalue of $S_0(q)$ if and only if it is either the root of*

$$\lambda - (2n)^2 - \frac{2r_m^2}{\lambda} - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0 \tag{2.8}$$

or the root of

$$\lambda - (2n)^2 - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0, \tag{2.9}$$

(c) *If $n \neq m$ and $n \neq m/2$, then $\lambda_{\pm n}$ is an eigenvalue of $S_0(q)$ if and only if it is either the root of*

$$\lambda - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) - (\frac{q_{-m}}{q_m})^{n/m} \sum_{j=1}^{\infty} \beta_j(\lambda) = 0 \tag{2.10}$$

or the root of

$$\lambda - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) + (\frac{q_{-m}}{q_m})^{n/m} \sum_{j=1}^{\infty} \beta_j(\lambda) = 0 \tag{2.11}$$

lying in the disk $D_n := \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| \leq 2|r_m|\}$ and each of the series in these equations converges uniformly to an analytic function on the disk D_n.

Proof. **(a)** By Lemma 2.1, letting k tend to infinity in the equations (2.3) and (2.4), we obtain

$$(\lambda_n - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda_n))u_n = (q_{2n} + \sum_{j=1}^{\infty} \beta_j(\lambda_n))v_n, \tag{2.12}$$

$$(\lambda_n - (2n)^2 - \sum_{j=1}^{\infty} \alpha^*_{2j-1}(\lambda_n))v_n = (q_{-2n} \sum_{j=1}^{\infty} \beta^*_j(\lambda_n))u_n, \tag{2.13}$$

where $u_n = (\Psi_n, e^{i2nx})$ and $v_n = (\Psi_n, e^{-i2nx})$. If one of the numbers u_n and v_n is zero, then the proof is obvious. If they are both different from zero, multiplying these equations side by side and then cancelling the term $u_n v_n$, by (2.5), we obtain

$$\big(\lambda_n - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda_n)\big)^2 = (\frac{q_{-m}}{q_m})^{2n/m} \big(q_{2n} + \sum_{j=1}^{\infty} \beta_j(\lambda_n)\big)^2, \tag{2.14}$$

which implies λ_n is either the root of

$$\lambda_n - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda_n) - (\frac{q_{-m}}{q_m})^{n/m} (q_{2n} + \sum_{j=1}^{\infty} \beta_j(\lambda_n)) = 0 \tag{2.15}$$

or the root of

$$\lambda_n - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda_n) + (\frac{q_{-m}}{q_m})^{n/m} (q_{2n} + \sum_{j=1}^{\infty} \beta_j(\lambda_n)) = 0. \tag{2.16}$$

Since $B_j(\lambda_n) = 0$, $j = 1, 2, \ldots$, for $n = m/2$, λ_n is either the root of (2.6) or the root of (2.7).

Now, we prove that the roots of (2.6) and (2.7) lying in the disk D_n are the eigenvalues of S_0. The equation $f_i(\lambda) := \lambda - (2n)^2 - (-1)^i r_m = 0$, has one root in the disk D_n for each $i = 1, 2$, and

$$|f_i(\lambda_n)| = |\lambda_n - (2n)^2 - (-1)^i r_m| \geq ||\lambda_n - (2n)^2| - |r_m|| = 2|r_m| - |r_m| = |r_m|,$$

for all $\lambda_n \in C_n$, where $C_n = \{\lambda \in \mathbb{C} : |\lambda - (2n)^2| = 2|r_m|\}$. Define the function

$$g_i(\lambda) := \lambda - (2n)^2 - (-1)^i r_m - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) = 0,$$

for $i = 1, 2$. Estimating the summands of $|\alpha_{2j-1}(\lambda_n)|$ for $n = m/2$, we obtain

$$|\alpha_{2j-1}(\lambda_n)| < \frac{(3/2)^{j-1}|r_m|^{2j}}{|\lambda_n - (6n)^2|^j |\lambda_n - (10n)^2|^{j-1}},$$

for $j \geq 1$. Using the relations $|\lambda_n - (6n)^2| \geq 32n^2 - 2|r_m|$ and $|\lambda_n - (10n)^2| \geq 96n^2 - 2|r_m|$, it follows by the geometric series formula that

$$\begin{aligned}
\sum_{j=1}^{\infty} |\alpha_{2j-1}(\lambda_n)| &< \frac{|r_m|^2}{32n^2 - 2|r_m|} + \frac{(3/2)|r_m|^4}{(32n^2 - 2|r_m|)^2(96n^2 - 2|r_m|)} + \frac{(3/2)^2|r_m|^6}{(32n^2 - 2|r_m|)^3(96n^2 - 2|r_m|)^2} + \cdots \\
&= \frac{|r_m|^2}{32n^2 - 2|r_m|}\left(1 + \frac{(3/2)|r_m|^2}{(32n^2 - 2|r_m|)(96n^2 - 2|r_m|)} + \frac{(3/2)^2|r_m|^4}{(32n^2 - 2|r_m|)^2(96n^2 - 2|r_m|)^2} + \cdots\right) \\
&= \frac{|r_m|^2}{32n^2 - 2|r_m|} \frac{1}{1 - \frac{3|r_m|^2}{2(32n^2 - 2|r_m|)(96n^2 - 2|r_m|)}} = \frac{2|r_m|^2(96n^2 - 2|r_m|)}{2(32n^2 - 2|r_m|)(96n^2 - 2|r_m|) - 3|r_m|^2} \\
&< \frac{4(2s-1)^2(48s^2 - 2s + 1)}{8(16s^2 - 2s + 1)(48s^2 - 2s + 1) - 3(2s-1)^2} := a_s < \frac{1}{8},
\end{aligned}$$

for all $s \geq 2$ and $\lim_{s\to\infty} a_s = 1/8$. Hence

$$|g_i(\lambda_n) - f_i(\lambda_n)| \leq \sum_{j=1}^{\infty} |\alpha_{2j-1}(\lambda_n)| < \frac{1}{8}.$$

Therefore, $|g_i(\lambda) - f_i(\lambda)| < |f_i(\lambda)|$ holds for all $\lambda \in C_n$ and for each $i = 1, 2$. By Rouche's theorem, $g_i(\lambda)$ has one root inside the disk D_n, for $i = 1$ and $i = 2$. Hence, S_0 has one eigenvalue (counting the multiplicity) lying inside D_n, which is the root of (2.6) and it has one eigenvalue (counting the multiplicity) lying inside D_n, which is the root of (2.7). On the other hand, each of the equations (2.6) and (2.7) has exactly one root (counting the multiplicity) inside D_n. Thus, $\lambda_{\pm n} \in D_n$ is an eigenvalue of S_0 if and only if, it is either the root of (2.6) or the root of (2.7) and the roots of (2.6) and (2.7) coincide with the eigenvalues λ_{-n} and λ_{+n} of S_0.

(b) In this case $q_{2n} = 0$, $B_j(\lambda_n) = 0$, for $j \neq 1$, and $\beta_1(\lambda_n) = {r_m}^2/\lambda_n$. Therefore, by (2.15) and (2.16), λ_n is either the root of (2.8) or the root of (2.9). Now, we prove that the roots of (2.8) and (2.9) lying in the disk D_n are the eigenvalues of S_0. The equation $f(\lambda) := \lambda - (2n)^2 = 0$, has one root in the disk D_n and $|f(\lambda_n)| = |\lambda_n - (2n)^2| = 2|r_m|$, for all $\lambda_n \in C_n$. We define the functions

$$h_1(\lambda) := \lambda - (2n)^2 - \frac{2{r_m}^2}{\lambda} - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0,$$

and

$$h_2(\lambda) := \lambda - (2n)^2 - \frac{{r_m}^2}{\lambda - 16n^2} - \sum_{j=2}^{\infty} \alpha_{2j-1}(\lambda) = 0.$$

Estimating the summands of $|\alpha_{2j-1}(\lambda_n)|$ for $n = m$, we obtain

$$|\alpha_{2j-1}(\lambda_n)| < \frac{(3/2)^{j-1}|r_m|^{2j}}{|\lambda_n - (4n)^2|^j |\lambda_n - (6n)^2|^{j-1}}, \tag{2.17}$$

for $j \geq 2$. Using the relations $|\lambda_n - (4n)^2| \geq 12n^2 - 2|r_m|$ and $|\lambda_n - (6n)^2| \geq 32n^2 - 2|r_m|$, it follows by the geometric series formula that

$$\frac{r_m^2}{\lambda - 16n^2} + \sum_{j=1}^{\infty} |\alpha_{2j-1}(\lambda_n)| < \frac{|r_m|^2}{12n^2 - 2|r_m|} + \frac{(3/2)|r_m|^4}{(12n^2 - 2|r_m|)^2(32n^2 - 2|r_m|)}$$
$$+ \frac{(3/2)^2|r_m|^6}{(12n^2 - 2|r_m|)^3(32n^2 - 2|r_m|)^2} + \cdots$$
$$= \frac{|r_m|^2}{12n^2 - 2|r_m|}(1 + \frac{(3/2)|r_m|^2}{(12n^2 - 2|r_m|)(32n^2 - 2|r_m|)} + \frac{(3/2)^2|r_m|^4}{(12n^2 - 2|r_m|)^2(32n^2 - 2|r_m|)^2} + \cdots)$$
$$= \frac{|r_m|^2}{12n^2 - 2|r_m|} \frac{1}{1 - \frac{3|r_m|^2}{2(12n^2 - 2|r_m|)(32n^2 - 2|r_m|)}}$$
$$< \frac{2(2s-1)^2(32s^2 - 4s + 2)}{2(12s^2 - 4s + 2)(32s^2 - 4s + 2) - 3(2s-1)^2} := b_s < \frac{1}{3},$$

for all $s \geq 2$ and $\lim_{s\to\infty} b_s = 1/3$. On the other hand,

$$\frac{2|r_m|^2}{|\lambda_n|} \leq \frac{2|r_m|^2}{4n^2 - 2|r_m|} < \frac{2(2s-1)^2}{4s^2 - 2(2s-1)} := c_s < 2$$

for all $s \geq 2$ and $\lim_{s\to\infty} c_s = 2$. Hence

$$|h_1(\lambda_n) - f(\lambda_n)| \leq \frac{2|r_m|^2}{|\lambda_n|} + \frac{|r_m|^2}{|\lambda_n - 16n^2|} + \sum_{j=2}^{\infty} |\alpha_{2j-1}(\lambda_n)| < \frac{7}{3}.$$

and

$$|h_2(\lambda_n) - f(\lambda_n)| \leq \frac{|r_m|^2}{|\lambda_n - 16n^2|} + \sum_{j=2}^{\infty} |\alpha_{2j-1}(\lambda_n)| < \frac{1}{3}.$$

Therefore, $|h_i(\lambda) - f(\lambda)| < |f(\lambda)|$ holds for all $\lambda \in C_n$ and for each $i = 1, 2$. Arguing as in the proof of (a), by Rouche's theorem, we conclude that $\lambda_{\pm n} \in D_n$ is an eigenvalue of S_0 if and only if, it is either the root of (2.8) or the root of (2.9) and the roots of (2.8) and (2.9) coincide with the eigenvalues λ_{-n} and λ_{+n} of S_0.

(c) In this case $q_{2n} = 0$. So, by (2.15) and (2.16), λ_n is either the root of (2.10) or the root of (2.11). Now, we prove that the roots of (2.10) and (2.11) lying in the disk D_n are the eigenvalues of S_0. The equation $f(\lambda) = \lambda - (2n)^2 = 0$, has one root in the disk D_n and $|f(\lambda_n)| = |\lambda_n - (2n)^2| = 2|r_m|$, for all $\lambda_n \in C_n$. We define

$$k_i(\lambda) := \lambda - (2n)^2 - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda) - (-1)^i (\frac{q_{-m}}{q_m})^{n/m} \sum_{j=1}^{\infty} \beta_j(\lambda) = 0,$$

for $i = 1, 2$. Estimating the summands of $|\alpha_{2j-1}(\lambda_n)|$ and $|(q_{-m}/q_m)^{n/m}\beta_j(\lambda_n)|$ for $n \neq m$ and $n \neq m/2$, we obtain

$$|\alpha_{2j-1}(\lambda_n)| < \frac{(3/2)^j |r_m|^{2j}}{|\lambda_n - (2(n-m))^2|^j |\lambda_n - (2(n-2m))^2|^{j-1}},$$

and

$$|\frac{q_{-2}}{q_2}|^{3/2} |\beta_{2j}(\lambda_3)| \leq \frac{|r_2|^{2j+1}}{|\lambda_3 - 4|^{2j}} < \frac{|r_2|^{2j+1}}{(32 - 2|r_2|)^{2j}} < \frac{5^{2j+1}}{22^{2j}},$$

for $j \geq 1$. Using the relations

$$|\lambda_n - (2(n-m))^2| \geq 4m|2n - m| - 2|r_m| > 4m|2s - m| - 2(2s-1)$$
$$\geq 4(s+1)(s-1) - 2(2s-1) = 4s^2 - 4s - 2,$$
$$|\lambda_n - (2(n-2m))^2| \geq 16m|n - m| - 2|r_m| > 16m|s - m| - 2(2s-1)$$
$$\geq 16(s+1) - 2(2s-1) = 12s + 18 = 6(2s+3),$$

it follows by the geometric series formula that

$$\sum_{j=1}^{\infty}|\alpha_{2j-1}(\lambda_n)| < \frac{(3/2)(2s-1)^2}{2(2s^2-2s-1)} + \frac{(3/2)^2(2s-1)^4}{2^2(2s^2-2s-1)^2 6(2s+3)} + \frac{(3/2)^3(2s-1)^6}{2^3(2s^2-2s-1)^3 6^2(2s+3)^2} + \cdots$$
$$= \frac{3(2s-1)^2}{4(2s^2-2s-1)}(1 + \frac{(3/2)(2s-1)^2}{12(2s^2-2s-1)(2s+3)} + \frac{(3/2)^2(2s-1)^4}{12^2(2s^2-2s-1)^2(2s+3)^2} + \cdots)$$
$$= \frac{3(2s-1)^2}{4(2s^2-2s-1)} \frac{1}{1-\frac{(2s-1)^2}{8(2s^2-2s-1)(2s+3)}} = \frac{6(2s-1)^2(2s+3)}{8(2s^2-2s-1)(2s+3)-(2s-1)^2}$$
$$:= d_s < \frac{3}{2},$$

for all $s \geq 2$ and $\lim_{s\to\infty} d_s = 3/2$. Also, that

$$|\frac{q_{-m}}{q_m}|^{n/m}\sum_{j=1}^{\infty}|\beta_j(\lambda_n)| < |\frac{q_{-2}}{q_2}|^{3/2}\sum_{j=1}^{\infty}|\beta_{2j}(\lambda_3)|$$
$$< \frac{5^3}{22^2}(1+\frac{5^2}{22^2}+\frac{5^4}{22^4}+\cdots) = \frac{125}{459} < \frac{3}{10}.$$

Hence

$$|k_i(\lambda_n) - f(\lambda_n)| = |\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_n) \pm (\frac{q_{-m}}{q_m})^{n/m}\sum_{j=1}^{\infty}\beta_j(\lambda_n)|$$
$$\leq \sum_{j=1}^{\infty}|\alpha_{2j-1}(\lambda_n)| + |\frac{q_{-m}}{q_m}|^{n/m}\sum_{j=1}^{\infty}|\beta_j(\lambda_n)| < \frac{3}{2}+\frac{3}{10} = \frac{9}{5}.$$

Therefore, $|k_i(\lambda)-f(\lambda)| < |f(\lambda)|$ holds for all $\lambda \in C_n$ and for each $i = 1, 2$. Again, arguing as in the proof of (a), we arrive by Rouche's theorem that, $\lambda_{\pm n} \in D_n$ is an eigenvalue of S_0 if and only if, it is either the root of (2.10) or the root of (2.11) and the roots of (2.10) and (2.11) coincide with the eigenvalues λ_{-n} and λ_{+n} of S_0.

Now, in order to estimate $\sum_{j=1}^{\infty}|\alpha'_{2j-1}(\lambda_n)|$ and $|\frac{q_{-m}}{q_m}|^{n/m}\sum_{j=1}^{\infty}|\beta'_j(\lambda_n)|$, we first estimate the summands $|\alpha'_{2j-1}(\lambda_n)|$ and $|\frac{q_{-2}}{q_2}|^{3/2}|\beta'_{2j}(\lambda_3)|$ by differentiating $\alpha_{2j-1}(\lambda_n)$ and $\beta_{2j}(\lambda_3)$ with respect to λ_n and λ_3, respectively:

$$|\alpha'_{2j-1}(\lambda_n)| < \frac{2^j|r_m|^{2j}}{|\lambda_n-(2(n-m))^2|^{j+1}|\lambda_n-(2(n-2m))^2|^{j-1}},$$

$$|\frac{q_{-2}}{q_2}|^{3/2}|\beta'_{2j}(\lambda_3)| < \frac{|r_2|^{2j+1}}{|\lambda_3-4|^{2j}},$$

for $j \geq 1$, and hence, we have

$$\sum_{j=1}^{\infty}|\alpha'_{2j-1}(\lambda_n)| < \frac{2(2s-1)^2}{2^2(2s^2-2s-1)^2} + \frac{2^2(2s-1)^4}{2^3(2s^2-2s-1)^3 6(2s+3)} + \frac{2^3(2s-1)^6}{2^4(2s^2-2s-1)^4 6^2(2s+3)^2} + \cdots$$
$$= \frac{(2s-1)^2}{2(2s^2-2s-1)^2}(1+\frac{(2s-1)^2}{6(2s^2-2s-1)(2s+3)} + \frac{(2s-1)^4}{6^2(2s^2-2s-1)^2(2s+3)^2} + \cdots)$$
$$= \frac{(2s-1)^2}{2(2s^2-2s-1)^2}\frac{1}{1-\frac{(2s-1)^2}{6(2s^2-2s-1)(2s+3)}} = \frac{3(2s-1)^2(2s+3)}{6(2s^2-2s-1)(2s+3)-(2s-1)^2} := t_s < 1,$$

for all $s \geq 2$ and $\lim_{s\to\infty} t_s = 3/2$. Also, we have

$$\Big|\frac{q_{-m}}{q_m}\Big|^{n/m}\sum_{j=1}^{\infty}|\beta'_j(\lambda_n)| \leq \Big|\frac{q_{-2}}{q_2}\Big|^{3/2}\sum_{j=1}^{\infty}|\beta'_{2j}(\lambda_3)| < \sum_{j=1}^{\infty}\frac{3^j|r_2|^{2j+1}}{2^{j-1}|\lambda_3-4|^{2j+1}}$$
$$< \sum_{j=1}^{\infty}\frac{3^j|r_2|^{2j+1}}{2^{j-1}(32-2|r_2|)^{2j+1}} < \sum_{j=1}^{\infty}\frac{3^j 5^{2j+1}}{2^{j-1}22^{2j+1}} < \frac{1}{25}.$$

Therefore, each of the series $\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_n)$ and $\sum_{j=1}^{\infty}\beta_j(\lambda_n)$ converges uniformly to an analytic function on the disk D_n. One can prove in a similar way that, the series $\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_n)$ in the cases (a) and (b) also converges uniformly to an analytic function on the disk D_n. □

Now, to estimate the periodic eigenvalues λ_{-1} and λ_1, we consider the case $n = 1$. By Lemma 2.1, we should consider the cases (i) and (ii). In case (i), substituting $B_j(\lambda_n) = 0$, for $j \geq 1$, in (2.3) and (2.4) as $k \to \infty$, we obtain

$$\Big(\lambda_1 - 4 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_1)\Big)^2 = {r_2}^2$$

or

$$\Big(\lambda_1 - 4 - r_2 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_1)\Big)\Big(\lambda_1 - 4 + r_2 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_1)\Big) = 0. \tag{2.18}$$

In case (ii), substituting $q_{2n} = 0$, $B_j(\lambda_n) = 0$, for $j \geq 1$, in (2.3) and (2.4) as $k \to \infty$, we obtain

$$\Big(\lambda_1 - 4 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda_1)\Big)^2 = 0 \tag{2.19}$$

Therefore, we have the following results.

Theorem 2.4. ***(a)*** *If $|q_{-2}|+|q_2| \leq 29/10$, for $n = 1$ and $m = 2$, then $\lambda_{\pm1}$ is an eigenvalue of $S_0(q)$ if and only if it is either the root of the equation*

$$\lambda - 4 - r_2 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda) = 0 \tag{2.20}$$

or the root of

$$\lambda - 4 + r_2 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda) = 0, \tag{2.21}$$

(b) *If $|q_{-m}| + |q_m| \leq 7/2$, for $n = 1$ and $m \geq 3$, then $\lambda_{\pm1}$ is a double eigenvalue of $S_0(q)$ if and only if it is the double root of the equation*

$$\Big(\lambda - 4 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)\Big)^2 = 0 \tag{2.22}$$

lying in the disk $D_1 := \{\lambda \in \mathbb{C} : |\lambda - 4| \leq 2|r_m|\}$ and the series $\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)$ converges uniformly to an analytic function on the disk D_1 in each case.

Proof. **(a)** The equations (2.20) and (2.21) follow from (2.18). Let $F_i(\lambda) := \lambda - 4 - (-1)^i r_2 = 0$ and $G_i(\lambda) := \lambda - 4 - (-1)^i r_2 - \sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda) = 0$ for $i = 1, 2$. Then, using the estimations

$$\sum_{j=1}^{\infty}|\alpha_{2j-1}(\lambda_1)|<\sum_{j=1}^{\infty}\frac{(3/2)^{j-1}|r_2|^{2j}}{|\lambda_1-36|^{j}|\lambda_1-100|^{j-1}}<\sum_{j=1}^{\infty}\frac{(3/2)^{j-1}|r_2|^{2j}}{(32-2|r_2|)^{j}(96-2|r_2|)^{j-1}}<\frac{1}{54} \tag{2.23}$$

and

$$\sum_{j=1}^{\infty}|\alpha'_{2j-1}(\lambda_1)|<\sum_{j=1}^{\infty}\frac{2^{j}|r_2|^{2j}}{|\lambda_1-36|^{j+1}|\lambda_1-100|^{j-1}}<\sum_{j=1}^{\infty}\frac{2^{j}|r_2|^{2j}}{(32-2|r_2|)^{j+1}(96-2|r_2|)^{j-1}}<\frac{1}{826}$$

for $2|r_2|\le|q_{-2}|+|q_2|\le 29/10$, and arguing as in the proof of Theorem 2.3 (a), by Rouche's theorem, we complete the proof of (a).

(b) Equation (2.22) follows from (2.19). Let $g(\lambda)=\lambda-4-\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)$ and $h(\lambda)=\lambda-4$. Then, using the estimations

$$\sum_{j=1}^{\infty}|\alpha_{2j-1}(\lambda_1)|<\sum_{j=1}^{\infty}\frac{(3/2)^{j}|r_m|^{2j}}{|\lambda_1-(2(1-m))^2|^{j}|\lambda_1-(2(1-2m))^2|^{j-1}}$$
$$<\sum_{j=1}^{\infty}\frac{(3/2)^{j}|r_m|^{2j}}{(4(m-1)^2-8)^{j}(4(2m-1)^2-8)^{j-1}}<\sum_{j=1}^{\infty}\frac{(3/2)^{j}2^{2j}}{8^{j}92^{j-1}}=\frac{276}{365}$$

and

$$\sum_{j=1}^{\infty}|\alpha'_{2j-1}(\lambda_1)|<\sum_{j=1}^{\infty}\frac{2^{j}|r_m|^{2j}}{|\lambda_1-(2(1-m))^2|^{j+1}|\lambda_1-(2(1-2m))^2|^{j-1}}$$
$$<\sum_{j=1}^{\infty}\frac{2^{j}|r_m|^{2j}}{(4(m-1)^2-8)^{j+1}(4(2m-1)^2-8)^{j-1}}<\sum_{j=1}^{\infty}\frac{2^{j}2^{2j}}{8^{j+1}92^{j-1}}=\frac{23}{181}$$

for $2|r_m|\le|q_{-m}|+|q_m|\le 7/2$, and again arguing as in the proof of Theorem 2.3 (a), we complete the proof. □

Finally, in order to estimate the first periodic eigenvalue λ_0, we consider the case $n=0$. By Lemma 2.2, we have:

Theorem 2.5. ***(a)*** *If $|q_{-2}|+|q_2|\le 2$, for $n=0$ and $m=2$,*
(b) *If $|q_{-m}|+|q_m|\le 3$, for $n=0$ and $m\ge 3$,*
then λ_0 is an eigenvalue of $S_0(q)$ if and only if it is the root of the equation

$$\lambda-\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)=0 \tag{2.24}$$

lying in the disk $D_0:=\{\lambda\in\mathbb{C}:|\lambda|\le 2|r_m|\}$ and the series $\sum_{j=1}^{\infty}\alpha_{2j-1}(\lambda)$ converges uniformly to an analytic function on the disk D_0.

Proof. **(a)** Iterating the equation $\lambda_N(\Psi_N,1)=(q\Psi_N,1)$, for $N=0$, k times, by isolating the terms containing $(\Psi_0,1)$ gives

$$(\lambda_0-\sum_{j=1}^{k}\alpha_j(\lambda_0))(\Psi_0,1)=\rho_k(\lambda_0).$$

Letting k tend to infinity in the last equation, by Lemma 2.2 and (2.5), we obtain (2.24). Let $H(\lambda) := \lambda$ and $G(\lambda) := \lambda - \sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda)$. Then, using the estimations

$$\sum_{j=1}^{\infty} |\alpha_{2j-1}(\lambda_0)| < \frac{1}{7} + \sum_{j=2}^{\infty} \frac{2^{j-1}|r_2|^{2j}}{|\lambda_0 - 16|^j |\lambda_0 - 64|^{j-1}}$$
$$< \frac{1}{7} + \sum_{j=2}^{\infty} \frac{2^{j-1}|r_2|^{2j}}{(16 - 2|r_2|)^j (64 - 2|r_2|)^{j-1}} < \frac{3}{20} \tag{2.25}$$

and

$$\sum_{j=1}^{\infty} |\alpha'_{2j-1}(\lambda_0)| < \frac{1}{98} + \sum_{j=2}^{\infty} \frac{2^{j+1}|r_2|^{2j}}{|\lambda_0 - 16|^{j+1} |\lambda_0 - 64|^{j-1}}$$
$$< \frac{1}{98} + \sum_{j=2}^{\infty} \frac{2^{j+1}|r_2|^{2j}}{(16 - 2|r_2|)^{j+1} (64 - 2|r_2|)^{j-1}} < \frac{1}{50}$$

for $2|r_2| \leq |q_{-2}| + |q_2| \leq 2$, and arguing as in the proof of Theorem 2.3 (a), we complete the proof of (a).

(b) In this case, using the estimations

$$\sum_{j=1}^{\infty} |\alpha_{2j-1}(\lambda_0)| < \frac{2|r_m|^2}{|\lambda_0 - 4m^2|} + \sum_{j=2}^{\infty} \frac{2^{j-1}|r_m|^{2j}}{|\lambda_0 - 4m^2|^j |\lambda_0 - 16m^2|^{j-1}}$$
$$\leq \frac{2|r_m|^2}{4m^2 - 2|r_m|} + \sum_{j=2}^{\infty} \frac{2^{j-1}|r_m|^{2j}}{(4m^2 - 2|r_m|)^j (16m^2 - 2|r_m|)^{j-1}}$$
$$\leq \frac{2|r_m|^2}{36 - 2|r_m|} + \sum_{j=2}^{\infty} \frac{2^{j-1}|r_m|^{2j}}{(36 - 2|r_m|)^j (144 - 2|r_m|)^{j-1}} < \frac{1}{200}$$

and

$$\sum_{j=1}^{\infty} |\alpha'_{2j-1}(\lambda_0)| < \frac{2|r_m|^2}{|\lambda_0 - 4m^2|^2} + \sum_{j=2}^{\infty} \frac{2^{j+1}|r_m|^{2j}}{|\lambda_0 - 4m^2|^{j+1} |\lambda_0 - 16m^2|^{j-1}}$$
$$\leq \frac{2|r_m|^2}{(4m^2 - 2|r_m|)^2} + \sum_{j=2}^{\infty} \frac{2^{j+1}|r_m|^{2j}}{(4m^2 - 2|r_m|)^{j+1} (16m^2 - 2|r_m|)^{j-1}}$$
$$\leq \frac{2|r_m|^2}{36 - 2|r_m|} + \sum_{j=2}^{\infty} \frac{2^{j+1}|r_m|^{2j}}{(36 - 2|r_m|)^{j+1} (144 - 2|r_m|)^{j-1}} < \frac{7}{50}$$

for $2|r_m| \leq |q_{-m}| + |q_m| \leq 3$, and again arguing as in the proof of Theorem 2.3 (a), by Rouche's theorem, we complete the proof. □

In order to estimate eigenvalues numerically, we take finite summations instead of the infinite series in the equations (2.6)-(2.11), (2.20)-(2.22) and (2.24). When we say the $(2k-1)$th approximations, we mean the equations containing $\sum_{j=1}^{k} \alpha_{2j-1}(\lambda)$ and $\sum_{j=1}^{2k-1} \beta_j(\lambda)$ instead of $\sum_{j=1}^{\infty} \alpha_{2j-1}(\lambda)$ and $\sum_{j=1}^{\infty} \beta_j(\lambda)$. For instance, in the case $m = 2$, the $(2k-1)$th approximations of (2.24), (2.20), (2.21), (2.8) and (2.9) are

$$\lambda - \sum_{j=1}^{k} \alpha_{2j-1}(\lambda) = 0, \tag{2.26}$$

for $n=0$;

$$\lambda-4-r_2-\sum_{j=1}^{k}\alpha_{2j-1}(\lambda)=0, \tag{2.27}$$

$$\lambda-4+r_2-\sum_{j=1}^{k}\alpha_{2j-1}(\lambda)=0, \tag{2.28}$$

for $n=1$; and

$$\lambda-16-\frac{2{r_2}^2}{\lambda}-\frac{{r_2}^2}{\lambda-64}-\sum_{j=2}^{k}\alpha_{2j-1}(\lambda)=0, \tag{2.29}$$

$$\lambda-16-\frac{{r_2}^2}{\lambda-64}-\sum_{j=2}^{k}\alpha_{2j-1}(\lambda)=0, \tag{2.30}$$

for $n=2$. Then, by (2.25), (2.23) and (2.17), we have the following estimates for the remaining terms of these equations:

$$|\sum_{j=k+1}^{\infty}\alpha_{2j-1}(\lambda_0)|\leq\sum_{j=k+1}^{\infty}|\alpha_{2j-1}(\lambda_0)|<\sum_{j=k+1}^{\infty}\frac{2^{j-1}|r_2|^{2j}}{(16-2|r_2|)^j(64-2|r_2|)^{j-1}}$$
$$<\frac{2^k|r_2|^{2k+2}}{(16-2|r_2|)^k(64-2|r_2|)^{k-1}[(16-2|r_2|)(64-2|r_2|)-2|r_2|^2]}<\frac{31}{433}(\frac{1}{434})^k,$$

for $n=0$;

$$|\sum_{j=k+1}^{\infty}\alpha_{2j-1}(\lambda_1)|\leq\sum_{j=k+1}^{\infty}|\alpha_{2j-1}(\lambda_1)|<\sum_{j=k+1}^{\infty}\frac{(3/2)^{j-1}|r_2|^{2j}}{(32-2|r_2|)^j(96-2|r_2|)^{j-1}}$$
$$<\frac{3^k|r_2|^{2k+2}}{2^{k-1}(32-2|r_2|)^k(96-2|r_2|)^{k-1}[2(32-2|r_2|)(96-2|r_2|)-3|r_2|^2]}<\frac{1}{102}(\frac{1}{399})^k,$$

for $n=1$; and

$$|\sum_{j=k+1}^{\infty}\alpha_{2j-1}(\lambda_2)|<\sum_{j=k+1}^{\infty}\frac{(3/2)^{j-1}|r_2|^{2j}}{|\lambda_2-64|^j|\lambda_2-144|^{j-1}}<\sum_{j=k+1}^{\infty}\frac{(3/2)^{j-1}|r_2|^{2j}}{(48-2|r_2|)^j(128-2|r_2|)^{j-1}}$$
$$<\frac{3^k|r_2|^{2k+2}}{2^{k-1}(48-2|r_2|)^k(128-2|r_2|)^{k-1}[2(48-2|r_2|)(128-2|r_2|)-3|r_2|^2]}<\frac{11}{50}(\frac{1}{379})^k,$$

for $n=2$. Obviously, we will have better approximations as k grows. For the 5th approximations, we use the followings:

$$\alpha_1(\lambda_n)=\frac{{r_m}^2}{\lambda_n-(2(n-m))^2}+\frac{{r_m}^2}{\lambda_n-(2(n+m))^2},$$

$$\alpha_3(\lambda_n)=\frac{{r_m}^4}{[\lambda_n-(2(n-m))^2]^2[\lambda_n-(2(n-2m))^2]}+\frac{{r_m}^4}{[\lambda_n-(2(n+m))^2]^2[\lambda_n-(2(n+2m))^2]},$$

$$\alpha_5(\lambda_n)=\frac{{r_m}^6}{[\lambda_n-(2(n-m))^2]^2[\lambda_n-(2(n-2m))^2]^2[\lambda_n-(2(n-3m))^2]}$$
$$+\frac{{r_m}^6}{[\lambda_n-(2(n-m))^2]^3[\lambda_n-(2(n-2m))^2]^2}$$
$$+\frac{{r_m}^6}{[\lambda_n-(2(n-m))^2]^2[\lambda_n-(2(n-2m))^2]^2[\lambda_n-(2(n-3m))^2]}$$
$$+\frac{{r_m}^6}{[\lambda_n-(2(n-m))^2]^3[\lambda_n-(2(n-2m))^2]^2},$$

and

$$\beta_1(\lambda_m) = \frac{{q_m}^2}{\lambda_m}, \quad \beta_1(\lambda_n) = 0, \quad n \neq m,$$

$$\beta_2(\lambda_{3m/2}) = \frac{{q_m}^3}{(\lambda_{3m/2} - m^2)^2}, \quad \beta_2(\lambda_n) = 0, \quad n \neq 3m/2,$$

$$\beta_3(\lambda_{2m}) = \frac{{q_m}^4}{[\lambda_m - (2m)^2]^2 \lambda_{2m}}, \quad \beta_3(\lambda_n) = 0, \quad n \neq 2m,$$

$$\beta_4(\lambda_{3m/2}) = \frac{{q_m}^4 q_{-m}}{[\lambda_{3m/2} - m^2]^4}, \quad \beta_4(\lambda_{5m/2}) = \frac{{q_m}^5}{[\lambda_{5m/2} - (3m)^2]^2 [\lambda_{5m/2} - m^2]^2},$$

$$\beta_4(\lambda_n) = 0, \quad n \neq \frac{3m}{2}, \frac{5m}{2},$$

$$\beta_5(\lambda_{2m}) = \frac{2{q_m}^5 q_{-m}}{[\lambda_{2m} - (2m)^2]^3 (\lambda_{2m})^2}, \quad \beta_5(\lambda_{3m}) = \frac{{q_m}^6}{[\lambda_{3m} - (4m)^2]^2 [\lambda_{3m} - (2m)^2]^2 \lambda_{3m}},$$

$$\beta_5(\lambda_n) = 0, \quad n \neq 2m, 3m.$$

Note that, by the conditions imposed on the summations, we take only the fractions that are not containing $\lambda_n - (2n)^2$ in the denominators, into consideration. Now, we approach the periodic eigenvalues by the roots of the polynomials derived from the $(2k-1)$th approximations (2.26)-(2.30), as it was done in [21]. For example, for $n = 0$ and $m = 2$, the fifth approximation is

$$Q_0(\lambda) := \lambda - \frac{2{r_2}^2}{\lambda - 16} - \frac{2{r_2}^4}{(\lambda - 16)^2(\lambda - 64)} - \frac{2{r_2}^6}{(\lambda - 16)^2(\lambda - 64)^2(\lambda - 144)} - \frac{2{r_2}^6}{(\lambda - 16)^3(\lambda - 64)^2} = 0,$$

for $n = 1$ and $m = 2$, the fifth approximations are

$$Q_{-1}(\lambda) := \lambda - 4 + r_2 - \frac{{r_2}^2}{\lambda - 36} - \frac{{r_2}^4}{(\lambda - 36)^2(\lambda - 100)} - \frac{{r_2}^6}{(\lambda - 36)^2(\lambda - 100)^2(\lambda - 196)} - \frac{{r_2}^6}{(\lambda - 36)^3(\lambda - 100)^2} = 0,$$

and

$$Q_1(\lambda) := \lambda - 4 - r_2 - \frac{{r_2}^2}{\lambda - 36} - \frac{{r_2}^4}{(\lambda - 36)^2(\lambda - 100)} - \frac{{r_2}^6}{(\lambda - 36)^2(\lambda - 100)^2(\lambda - 196)} - \frac{{r_2}^6}{(\lambda - 36)^3(\lambda - 100)^2} = 0,$$

for $n = 2$ and $m = 2$, the fifth approximations are

$$Q_{-2}(\lambda) := \lambda - 16 - \frac{2{r_2}^2}{\lambda} - \frac{{r_2}^2}{\lambda - 64} - \frac{{r_2}^4}{(\lambda - 64)^2(\lambda - 144)} - \frac{{r_2}^6}{(\lambda - 64)^2(\lambda - 144)^2(\lambda - 256)} - \frac{{r_2}^6}{(\lambda - 64)^3(\lambda - 144)^2} = 0,$$

and

$$Q_2(\lambda) := \lambda - 16 - \frac{{r_2}^2}{\lambda - 64} - \frac{{r_2}^4}{(\lambda - 64)^2(\lambda - 144)} - \frac{{r_2}^6}{(\lambda - 64)^2(\lambda - 144)^2(\lambda - 256)} - \frac{{r_2}^6}{(\lambda - 64)^3(\lambda - 144)^2} = 0.$$

Then, the corresponding polynomials are

$$P_0(\lambda) := (\lambda-16)^3(\lambda-64)^2(\lambda-144)Q_0(\lambda), \tag{2.31}$$
$$P_{-1}(\lambda) := (\lambda-36)^3(\lambda-100)^2(\lambda-196)Q_{-1}(\lambda), \tag{2.32}$$
$$P_1(\lambda) := (\lambda-36)^3(\lambda-100)^2(\lambda-196)Q_1(\lambda), \tag{2.33}$$
$$P_{-2}(\lambda) := (\lambda-64)^3(\lambda-144)^2(\lambda-256)Q_{-2}(\lambda) \tag{2.34}$$

and

$$P_2(\lambda) := (\lambda-64)^3(\lambda-144)^2(\lambda-256)Q_2(\lambda), \tag{2.35}$$

respectively. By the same token, we can derive polynomials to approximate other periodic eigenvalues, as well.

Now, we consider the operator $S_1(q)$ which is associated with the antiperiodic boundary conditions. The analogous formulas to (2.1), (2.2) are

$$(\mu_N-(2n-1)^2)(\Phi_N, e^{i(2n-1)x}) = (q\Phi_N, e^{i(2n-1)x}), \tag{2.36}$$
$$(\mu_N-(2n-1)^2)(\Phi_N, e^{-i(2n-1)x}) = (q\Phi_N, e^{-i(2n-1)x}). \tag{2.37}$$

Iterating equation (2.36) k times, we obtain

$$(\mu_n-(2n-1)^2-\sum_{j=1}^{k}\eta_j(\mu_n))(\Phi_n, e^{i(2n-1)x})-(q_{2n-1}+\sum_{j=1}^{k}\nu_j(\mu_n))(\Phi_n, e^{-i(2n-1)x}) = \delta_k(\mu_n), \tag{2.38}$$

where

$$\eta_j(\mu_n) = \sum_{n_1,n_2,\dots,n_j} \frac{q_{n_1}q_{n_2}\cdots q_{n_j}q_{-n_1-n_2-\cdots-n_j}}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\cdots-n_j)-1)^2]},$$
$$\nu_j(\mu_n) = \sum_{n_1,n_2,\dots,n_j} \frac{q_{n_1}q_{n_2}\cdots q_{n_j}q_{2n-1-n_1-n_2-\cdots-n_j}}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\cdots-n_j)-1)^2]},$$
$$\delta_k(\mu_n) = \sum_{n_1,n_2,\dots,n_{k+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{n_{k+1}}(q\Phi_N, e^{i(2(n-n_1-\cdots-n_{k+1})-1)x})}{[\mu_n-(2(n-n_1)-1)^2]\cdots[\mu_n-(2(n-n_1-\cdots-n_{k+1})-1)^2]}.$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, 2n-1$, for $l = 1, 2, ..., k+1$.

Similarly, iterating equation (2.37) k times, we obtain

$$(\mu_n-(2n-1)^2-\sum_{j=1}^{k}\eta_j^*(\mu_n))(\Phi_n, e^{-i(2n-1)x})-(q_{-2n+1}+\sum_{j=1}^{k}\nu_j^*(\mu_n))(\Phi_n, e^{i(2n-1)x}) = \delta_k^*(\mu_n), \tag{2.39}$$

where

$$\eta_j^*(\mu_n) = \sum_{n_1,n_2,\dots,n_j} \frac{q_{n_1}q_{n_2}\cdots q_{n_j}q_{-n_1-n_2-\cdots-n_j}}{[\mu_n-(2(n+n_1)-1)^2]\cdots[\mu_n-(2(n+n_1+\cdots+n_j)-1)^2]},$$
$$\nu_j^*(\mu_n) = \sum_{n_1,n_2,\dots,n_j} \frac{q_{n_1}q_{n_2}\cdots q_{n_j}q_{-2n+1-n_1-n_2-\cdots-n_j}}{[\mu_n-(2(n+n_1)-1)^2]\cdots[\mu_n-(2(n+n_1+\cdots+n_j)-1)^2]},$$
$$\delta_k^*(\mu_n) = \sum_{n_1,n_2,\dots,n_{k+1}} \frac{q_{n_1}q_{n_2}\cdots q_{n_k}q_{n_{k+1}}(q\Phi_N, e^{-i(2(n+n_1+\cdots+n_{k+1})-1)x})}{[\mu_n-(2(n+n_1)-1)^2]\cdots[\mu_n-(2(n+n_1+\cdots+n_{k+1})-1)^2]}.$$

Here, the sums are taken under the conditions $n_l = \pm m$, $\sum_{i=1}^{l} n_i \neq 0, -2n+1$ for $l = 1, 2, ..., k+1$. Therefore, the analogous formulas to (2.12), (2.13) and (2.14) are

$$(\mu_n - (2n-1)^2 - \sum_{j=1}^{\infty} \eta_j(\mu_n))(\Phi_n, e^{i(2n-1)x}) = (q_{2n-1} + \sum_{j=1}^{\infty} \nu_j(\mu_n))(\Phi_n, e^{-i(2n-1)x}), \tag{2.40}$$

$$(\mu_n - (2n-1)^2 - \sum_{j=1}^{\infty} \eta_j^*(\mu_n))(\Phi_n, e^{-i(2n-1)x}) = (q_{-2n+1} + \sum_{j=1}^{\infty} \nu_j^*(\mu_n))(\Phi_n, e^{i(2n-1)x}) \tag{2.41}$$

and

$$(\mu_n - (2n-1)^2 - \sum_{j=1}^{\infty} \eta_j(\mu_n))^2 = (\frac{q_{-m}}{q_m})^{\frac{(2n-1)}{m}} (q_{2n-1} + \sum_{j=1}^{\infty} \nu_j(\mu_n))^2, \qquad n \geq 1, \tag{2.42}$$

respectively. Using (1.8) and the formulas (2.38)-(2.42), one can obtain analogous theorems to Theorem 2.3 and Theorem 2.4 for the operator $S_1(q)$. Now, we present a numerical example.

Example 2.6. Consider the potential $q(x) = e^{i4x} - e^{-i4x} = 2i\sin(4x)$ or $p(x) = ie^{i4x} + ie^{-i4x} = 2i\cos(4x)$. In this case, $m = 2$, $r_2 = \sqrt{-1} = i$ and we have the following approximations for the first periodic eigenvalues λ_0, λ_{-1}, λ_{+1}, λ_{-2} and λ_2:

First, we show that λ_0 is the eigenvalue lying inside the circle

$$c_0 := \{\lambda \in \mathbb{C} : |\lambda - 0.125867010858| = 4.8 \times 10^{-10}\}.$$

The root of the polynomial $P_0(\lambda)$ defined by (2.31), lying in the disk $D_0 = \{\lambda \in \mathbb{C} : |\lambda| \leq 2|r_2|\}$, is $a_1 = 0.125867010858$. The other roots of $P_0(\lambda)$ are $a_2 = 15.8939999572$, $a_3 = (15.9900597315 - 0.0204223963085i)$, $a_4 = (15.9900597315 + 0.0204223963085i)$, $a_5 = (64.0000067845 - 0.000336043226373i)$, $a_6 = (64.0000067845 + 0.000336043226373i)$ and $a_7 = 144.0$. Using the decomposition

$$Q_0(\lambda) = \frac{(\lambda - a_1)(\lambda - a_2)\cdots(\lambda - a_7)}{(\lambda - 16)^3(\lambda - 64)^2(\lambda - 144)},$$

we obtain by direct calculation $|Q_0(\lambda)| > 4.4990 \times 10^{-10}$, for all $\lambda \in c_0$. On the other hand, again by direct calculations, we have $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 2.6416 \times 10^{-10}$, for all $\lambda \in c_0$. Therefore, by Rouche's theorem, equation (2.24) has only one root inside the circle c_0. Thus, using Theorem 2.5 (a), we conclude that λ_0 is the eigenvalue lying inside the circle c_0.

Now, we show that λ_{-1} and λ_1 are the complex eigenvalues lying inside the circles

$$c_{-1} := \{\lambda \in \mathbb{C} : |\lambda - (4.0312397462 - 1.00097772667i)| = 8.8 \times 10^{-12}\}.$$

and

$$c_1 := \{\lambda \in \mathbb{C} : |\lambda - (4.0312397462 + 1.00097772667i)| = 8.8 \times 10^{-12}\}.$$

respectively. The roots of the polynomials $P_{-1}(\lambda)$ and $P_1(\lambda)$ defined by (2.32) and (2.33), lying in the disk $D_1 = \{\lambda \in \mathbb{C} : |\lambda| \leq 4 + 2|r_2|\}$ are $x_1 = (4.0312397462 - 1.00097772667i)$ and $y_1 = (4.0312397462 + 1.00097772667i)$, respectively. The other roots of $P_{-1}(\lambda)$ are $x_2 = (35.9964522039 + 0.0176168557191i)$, $x_3 = (35.9964154572 - 0.0172437280769i)$, $x_4 = (35.9758900488 + 0.00060462552073i)$, $x_5 = (100.00000187 + 0.000114737272348i)$, $x_6 = (100.000000674 - 0.000114763768311i)$, $x_7 = (196.0 + 1.20513462491e - 13i)$ and the other roots of $P_1(\lambda)$ are $y_2 = (35.9964522039 - 0.0176168557191i)$, $y_3 = (35.9964154572 + 0.0172437280769i)$, $y_4 = (35.9758900488 - 0.00060462552073i)$, $y_5 = (100.00000187 -$

$0.000114737272348i)$, $y_6 = (100.000000674 + 0.000114763768311i)$ and $y_7 = (196.0 - 1.20513462491e - 13i)$. Using the decompositions

$$Q_{-1}(\lambda) = \frac{(\lambda - x_1)(\lambda - x_2)\cdots(\lambda - x_7)}{(\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)},$$

and

$$Q_1(\lambda) = \frac{(\lambda - y_1)(\lambda - y_2)\cdots(\lambda - y_7)}{(\lambda - 36)^3(\lambda - 100)^2(\lambda - 196)},$$

by direct calculations, we obtain $|Q_{-1}(\lambda)| > 3.5600 \times 10^{-12}$, for all $\lambda \in c_{-1}$ and $|Q_1(\lambda)| > 3.5600 \times 10^{-12}$, for all $\lambda \in c_1$. On the other hand, one can easily calculate that $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 2.0038 \times 10^{-12}$, for all $\lambda \in c_{-1} \cup c_1$. The proof follows from Rouche's theorem and Theorem 2.4 (a); each of the equations (2.20) and (2.21) has only one root inside the circle c_{-1} and c_1, respectively and λ_{-1} and λ_{+1} are the complex eigenvalues lying inside c_{-1} and c_1, respectively.

Using the equations (2.34) and (2.35), Theorem 2.3 (b) and the estimations $|Q_{-2}(\lambda)| > 3.7055 \times 10^{-9}$, for all $\lambda \in c_{-2}$; $|Q_2(\lambda)| > 2.3100 \times 10^{-9}$, for all $\lambda \in c_2$ and $\sum_{j=4}^{\infty} |\alpha_{2j-1}(\lambda)| < 1.1464 \times 10^{-9}$, for all $\lambda \in c_{-2} \cup c_2$, one can show in a similar way that λ_{-2} and λ_2 are the eigenvalues lying inside the circles

$$c_{-2} := \{\lambda \in \mathbb{C} : |\lambda - 15.8949584087| = 1.9 \times 10^{-9}\}.$$

and

$$c_2 := \{\lambda \in \mathbb{C} : |\lambda - 16.0208389883| = 1.9 \times 10^{-8}\}.$$

respectively.

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