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Archimedean Copula Parameter Estimation with Kendall Distribution Function

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ABSTRACT: In the literature, up to now, it is common that for Gumbel, Clayton and Frank calculated Kendall Distribution function $K(u)$ and to the extent those applications have been made. Kendall distribution functions show stochastic orderings of random vectors. The aim of Kendall distribution function is selected suitable copula function for using data set. For dependence structures of the data set, we calculated Kendall Tau and Spearman Rho values which are nonparametric. Based on this method, parameters of copula are obtained. In this paper, we are made Kendall Distribution function which obtained with the help of generator function of Archimedean copula calculation for Ali Mikhail Haq and Joe and in relation with that simulation study. We used data set which generated dependent generalized pareto distribution (Gp(3,3,3)) for this study. For dependency among these variables, we used Archimedean copula. In connection with this, we define basic properties of copulas and nonparametric methods Kendall Tau, Spearman Rho are given. In this study, to explain the relationship among the variables, five Archimedean copula are selected; Gumbel, Clayton, Frank Joe and Ali Mikhail Haq. Afterwards, we are obtained nonparametric estimation of parameters of these copulas with the help of Kendall Tau. With Kendall distribution function values, we found the suitable Archimedean copula family for this data set.

Key Words: Archimedean copula, copula function, kendall distribution function, kendall tau

Kendall Dağılım Fonksiyonu Yardımıyla Arşimedyan Copula Parametre Tahmini

ÖZET: Literatürde şimdiye kadar Gumbel, Clayton ve Frank için Kendall dağılım fonksiyonu hesaplanmış ve uygulamaları yapılmıştır. Kendall dağılım fonksiyonu tesadüfi vektörlerin stokastik sıralamasını gösterir. Kendall dağılım fonksiyonunun amacı kullanılan veri seti için uygun olan copula fonksiyonunu seçmektir. Veri setinin bağımlılık yapısı için parametrik olmayan Kendall Tau ve Spearman Rho değerlerini hesapladık. Bu yönteme bağlı olarak, copula parametreleri elde edildi. Bu çalışmada Ali Mikhail Haq ve Joe copula için arşimedyan copulanın üreteç fonksiyonu yardımıyla Kendall dağılım fonksiyonunu hesapladık ve bununla ilgili simülasyon çalışması yaptık. Biz bu çalışma için bağımlı genelleştirilmiş pareto (Gp(3,3,3)) dağılımından üretilen veri seti kullandık. Bu değişkenler arasındaki bağımlılık yapısı için Arşimedyan copula kullandık. Bununla bağlantılı olarak, copulanın temel özellikleri tanıtıldı ve nonparametrik Kendall Tau ve Spearman Rho verildi. Bu çalışmada bu değişkenler arasındaki bağımlılık yapısını açıklamak için beş Arşimedyan copula ailesi seçildi; Gumbel, Clayton, Frank Joe ve Ali Mikhail Haq. Devamında Kendall tau yardımıyla bu copulaların parametrelerinin nonparametrik tahmini elde edildi. Kendall dağılım fonksiyonu değerleri ile veri seti için uygun arşimedyan copula bulundu.

Anahtar Kelimeler: Archimedean copula, copula fonsiyonu, kendall dağılım fonksiyonu, kendall tau

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INTRODUCTION

Copulas were first introduced as a part of metric spaces theory. The important features and applications of copulas have been progressing during recent years. (Sklar, 1959) introduced the general notions of a copula (Schweitzer and Wolff, 1981). A copula function links to univariate marginal to their multivariate distribution. Using Copula function, we model connection between random variables. Copula function is analyzing the dependence structure and it is provides degree of dependence structure. Copula is continuous transformation and invariant under increasing. Copulas can use for modeling dependence in several applied fields such as econometric, finance and actuarial studies. Archimedean copula definitions us to minimize the work of multivariate copula to a only univariate function. In this article explores for Gumbel, Clayton and Frank calculated Kendall Distribution function and to the extent that applications have been made. We made $K(u)$ function calculation for Ali Mikhail Haq and Joe and in relation with that simulation study. Throughout the paper we work bivariate Archimedean copulas; Clayton, Gumbel and Frank, Joe and Ali Mikhail Haq.

MATERIAL AND METHOD

2.1 Copula

The copula is defined as a $C:[0,1]^2$ ® $[0,1]$ that ensures the limiting conditions

✓ $C(u,0)=C(0,u)=0$ and $C(u,1)=C(1,u)=u, \forall u \in [0,1]$.

✓ $(u_1,u_2,v_1,v_2) \in [0,1]^4$, such that $u_1 \le u_2, v_1 \le v_2$

$$C(u_2,v_2)-C(u_2,v_1)-C(u_1,v_2)+C(u_1,v_1) ³\geq 0.$$

Ultimately, twice differentiable and 2-increasing property can be replaced by the condition;

$$c(u,v)=\frac{\partial^2 C(u,v)}{\partial u \partial v} ³\geq 0. \tag{1}$$

$c(u,v)$ is density of the copula. In the following, for -uniform random $U_1,U_2,...,U_n$ variables, the joint distribution function C is described;

$$C(u_1,u_2,......,u_n,q)=P(U_1 \le u_1, U_2 \le u_2, \le U_n ....u_n).. \tag{2}$$

Here θ is dependence parameter (Sklar, 1959; Schweitzer and Wolff, 1981;Genest and Mackay, 1986;Frees and Valdes, 1998;Cherubini and Luciano, 2001;Genest and Favre, 2006;Genest et al., 2009).

2.2 Sklar Theorem

Let X and Y be random variables with continuous distribution functions F_X and F_Y, with $F_X(X)$ and $F_Y(Y)$ are uniformly distributed on the space $[0,1]$. At this case, there is a copula so for all $x, y \in R$,

$$F_{XY}(X,Y) = C(F_X(X), F_Y(Y)) \tag{3}$$

The copula C for (X,Y) is the joint distribution function for the pair $F_X(X)$, $F_Y(Y)$ ensured F_X, F_Y continuous (Sklar, 1959;Schweitzer and Wolff, 1981;Genest and Mackay, 1986;Frees and Valdes, 1998;Cherubini and Luciano, 2001;Genest and Favre, 2006;Genest et al., 2009).

2.3 Archimedean Copula

Let φ define a function, $\phi : [0,1] \to [0,\infty]$, that is continuous and this function is supplies:

✓ $\phi(1) = 0, \phi(0) = \infty$.

✓ For all $t \in (0,1)$, $\phi'(t) < 0$, ϕ is decreasing, for all $t \in (0,1) \phi''(t) \geq 0$, ϕ is convex.

ϕ has an inverse $\phi^{-1} : [0,\infty] \to [0,1]$, that is to say this equation has the similar properties out of $\phi^{(-1)}(0) = 1$ and $\phi^{(-1)}(\infty) = 0$. The Archimedean copula is shown by

$$C(u,v) = \phi^{(-1)}[\phi(u) + \phi(v)]. \tag{4}$$

(Schweitzer and Wolff, 1981;Cherubini and Luciano, 2001;Genest and Favre, 2006;Genest et al., 2009)

2.4 Gumbel Copula

The Archimedean copula is defined with the help of generator function $\phi(t) = (-lnt)^{\theta}$, $\theta \geq 1$;

$$C_{\theta}(u,v) = \exp\left(-[(-\ln u)^{\theta} + (-\ln v)^{\theta}]^{1/\theta}\right); 0 \leq u, v \leq 1 \tag{5}$$

θ is the copula parameter restricted to .

2.5 Clayton Copula

The Archimedean copula is defined with the help of generator function $\phi(t) = \dfrac{t^{-\theta} - 1}{\theta}$, $\theta \in [\ 1, \infty) / \{0\}$

$$C_q(u,v) = (u^{-\theta} + v^{-\theta} - 1)^{1/\theta} \tag{6}$$

θ is the copula parameter restricted to $(0, \infty)$.

2.6 Frank Copula

The Archimedean copula is defined with the help of generator function; $\phi(t) = -\ln \frac{-e^{-\theta\tau} - 1}{e^{-\theta} - 1}$, $\theta \in R \,/\, \{0\}$;

$$C_\theta(u,v) = -\frac{1}{\theta} \ln \left(1 + \frac{\left(e^{-\theta u} - 1\right)\left(e^{-\theta v} - 1\right)}{\left(e^{-\theta} - 1\right)}\right) \tag{7}$$

θ is the copula parameter restricted to $(0, \infty)$.

2.7 Ali Mikhail Haq Copula

The Archimedean copula is defined with the help of generator function; $\varphi(t) = \ln \; 1 - [\theta(1-t)] \,/ t$

$$C_\theta(u,v) = \frac{uv}{1 - \theta\,(1-u)(1-v)} \tag{8}$$

θ is the copula parameter restricted to $[-1, \infty]$

2.8 Joe Copula

The Archimedean copula is defined with the help of generator function; $\varphi(t) = -\ln\left[1 - (1-t)^\theta\right]$

$$C_\theta(u,v) = 1 - \left[(1-u)^\theta + (1-v)^\theta - ((1-u)^\theta (1-v)^\theta\right]^{1/\theta} \tag{9}$$

θ is the copula parameter restricted to $[1, \infty]$.

2.9 The Nonparametric Estimation

The Archimedean Copula submits each copula has statement that connects its parameters to associated Kendal Tau and Spearman Rho. In this study only the relationships contain Kendal Tau that is given in table.

Table 1.The link between Archimedean copulas and Kendall Tau

Family	Range of θ	t
Gumbel	$\theta \in [1, \infty)$	$\frac{\theta - 1}{\theta}$
Clayton	$\theta \in [0, \infty)$	$\frac{\theta}{\theta + 2}$
Frank	$\theta \in (-\infty, \infty)$	$1 - \frac{4}{\theta}\left[1 - D_1(\theta)\right]$
Ali Mikhail Haq	$\theta \in [-1, 1]$	$\frac{3\theta - 2}{3\theta} - \frac{2(1-\theta)^2 \ln(1-\theta)}{3\theta^2}$
Joe	$\theta \in [1, \infty)$	$1 + \frac{4}{\theta} D_J(\theta)$

Here D is debye functions. $D_J(\theta) = \int_{t=0}^{1} \frac{\left[\ln\left(1 - t^{\theta}\right)\right]\left(1 - t^{\theta}\right)}{t^{\theta - 1}} dt$.

2.10 Kendall Distribution Function and Properties

In the past, it proposed a nonparametric method for forecasting the dependence function of a double of random variables for Archimedean copula (Genest and Rivest, 1993). The state of emphasizing a probability model for independent observations $(x_1, y_1), ¼ , (x_n, y_n)$ from a bivariate non Gaussian distribution function *H(X,Y)* might be reduced by denoted H and its marginal of F_X and F_Y, its related dependence function C. C is the association copula with generator φ and Kendall Distribution function the function given by

$$K(u) = \Pr\{C(U_1, ..., U_n \leq u\} \quad (10)$$

(Genest and Rivest, 1993) gives that if *C* is Archimedean copula, forecast of Archimedean copula is singly defined by function on the space (0,1);

$$K(u) = u - \frac{\phi(u)}{\phi'(u)} \tag{11}$$

a nonparametric estimation of K is shown by

$$K_n(u) = \sum_{j=1}^{n} I\{U_j \leq u\} / n+1 \tag{12}$$

To define the generator function ϕ, we show the paces; to forecast Kendall Tau value utilizing the non-parametric estimation and nonparametric forecast of K. For $K_n(u)$ nonparametric estimation of $K(u)$

i) The nonparametric forecast of Archimedean copula Kendall Tau correlation coefficient using

ii) Define the pseudo-observations $U_i = F_n(X_i, Y_i) = \sum_{j=1}^{n} I\left[\{X_j \leq X_i, Y_j \leq Y_i\}\right] / n+1, \quad i=1,2,\ldots,n$

$$K_n(u) = \frac{(U_i \leq u)}{n+1} = \frac{\text{number of } U_i \leq u}{n+1} \tag{13}$$

iii) Form a parametric estimation of K

$$K(u) = u - \frac{\phi(u)}{\phi'(u)} \tag{14}$$

iv) The election of Archimedean copula that is suitable for the data may be done by minimum a range

$$\int \left[K_{\phi_n}(u) - K_n(u)\right]^2 dK_n(u) \tag{15}$$

(Schweitzer and Wolff, 1981;Frees and Valdes, 1998;Cherubini and Luciano, 2001;Genest et al., 2009).

$$f(x;k,\mu,\sigma) = \frac{1}{\sigma}\left(1 + k\frac{(x-m)}{\sigma}\right)^{-1-1/k} \tag{16}$$

where, $k \neq 0$ is shape parameter, $\mu \in (-\infty, \infty)$ is state parameter, $\sigma \in (0,\infty)$ is scale parameter. The cumulative distribution functions of the generalized Pareto distribution;

Table 2. Kendall Distribution functions of Archimedean copulas

Family	Generator $\phi(u)$	Generator first derivative $\phi'(u)$	The distribution function $K(u) = u - \frac{\phi(u)}{\phi'(u)}$
Gumbel	$(-\ln(u))^{\theta}$	$-\theta(\ln u)^{\theta-1}\frac{1}{u}$	$u - \frac{(u \ln u)}{\theta}$
Clayton	$u^{-\theta} - 1$	$-\theta u u^{-\theta-1}$	$u - \frac{(u^{\theta+1} - u)}{\theta}$
Frank	$-\ln\left(\frac{e^{-\theta u} - 1}{e^{-\theta} - 1}\right)$	$\frac{\theta}{1 - e^{\theta u}}$	$u - \frac{\ln \frac{e^{-\theta u} - 1}{e^{-\theta} - 1}}{\theta}(e^{-\theta u} - 1)$
Ali Mikhail Haq	$\ln[1 - \theta(1-u)]/u$	$\frac{\theta u - \ln[1-\theta(1-u)][1-\theta(1-u)]}{u^2[1-\theta(1-u)]}$	$u - \frac{\ln[1-\theta(1-u)]u[1-\theta(1-u)]}{\theta u - \ln[1-\theta(1-u)][1-\theta(1-u)]}$
Joe	$-\ln[1-(1-t)^{\theta}]$	$\left[\frac{\theta(1-t)^{\theta-1}}{[1+(1-t)^{\theta}}\right]$	$t - \frac{\ln[1-(1-t)^{\theta}][1-(1-t)^{\theta}]}{\theta(1-t)^{\theta-1}}$

2.11 Generalized Pareto Distribution

The probability density function of the generalized Pareto distribution is shown;

$$F_{(k,m,s)}(x) = \begin{cases} 1 - \left(1 + \frac{k(x-m)}{s}\right)^{-1-1/k} & for\ k \neq 0 \\ 1 - \exp\left(-\frac{x-m}{s}\right) & for\ k = 0 \end{cases}. \tag{17}$$

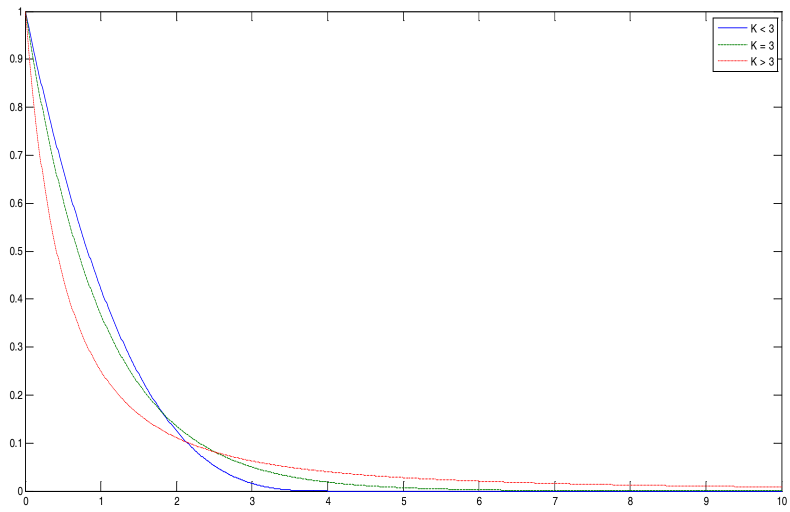


Figure 1. Change chart for k (k=3,k<3 and k>3) shape parameter of the generalized Pareto distribution

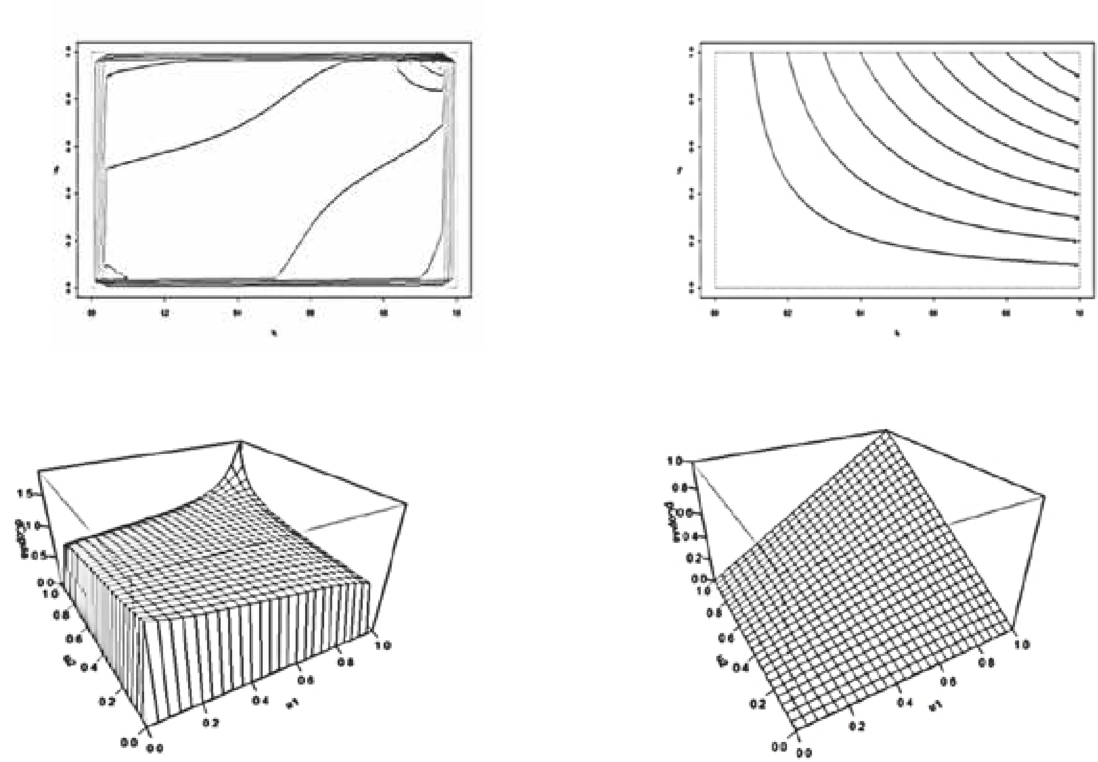
RESULTS AND DISCUSSION

The first section is the method that has been proposed by given (Genest and Rivest, 1993). In this section, out of Pearson correlation coefficient, another one measures of dependence denominated correlation. it is based on Kendall Tau that has been nonparametric measures of dependence. We have seen that the doubled correlations are all positive. Namely, Kendall Tau value is positive. This study consists of estimation of Archimedean copula. (Genest and Mackay, 1986) simplified method and leads to guess the parameters of Archimedean copula that focuses on Kendall Tau value. This study, up to now, in previous studies, in the literature it is common that for Gumbel, Clayton and Frank calculated Kendall Distribution function and to the extent that applications have been made. We made Kendall Distribution function calculation for Ali Mikhail Haq and Joe and in relation with that simulation study. In this simulation study, we generated dependent gamma distribution $X \sim Gp(3,3,3)$ and $Y \sim Gp(3,3,3)$. Here *n=400* data were used. We calculated Kendall Tau value in such that 0.074 for (X,Y) .Using this, that is shown the parameters of copulas obtained and consequences are given table 3.

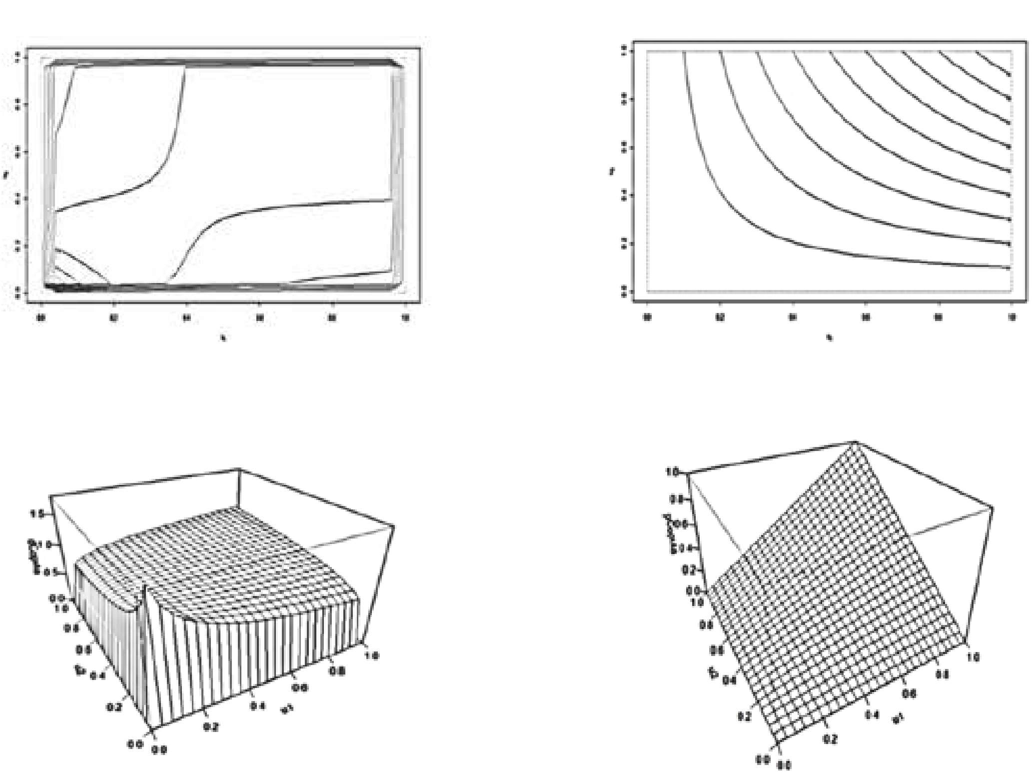
Table 3. Nonparametric estimation of Archimedean copula

Dependency parameter	Gumbel	Clayton	Frank	Ali Mikhail Haq	Joe
$\hat{\theta}$	1.079914	0.1598272	0.6689712	0.3061617	2.139303

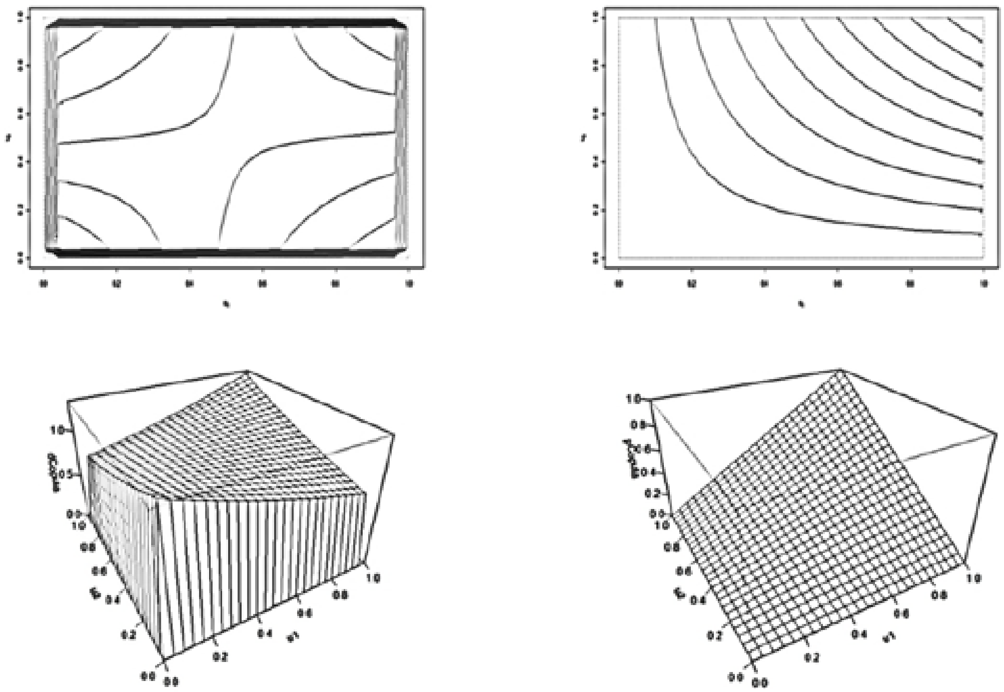
The following figures are obtained by using the values in table 3.



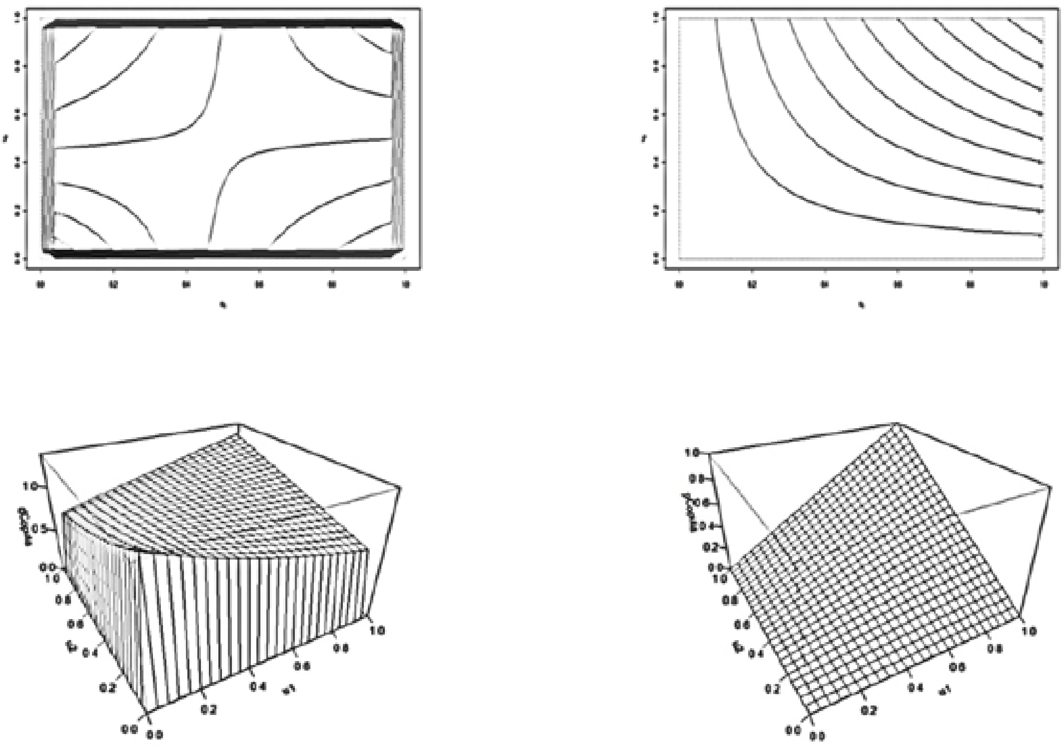
Figures 2. For Gumbel Copula $\theta = 1.079914$, respectively two and three dimensional density and pobability function.



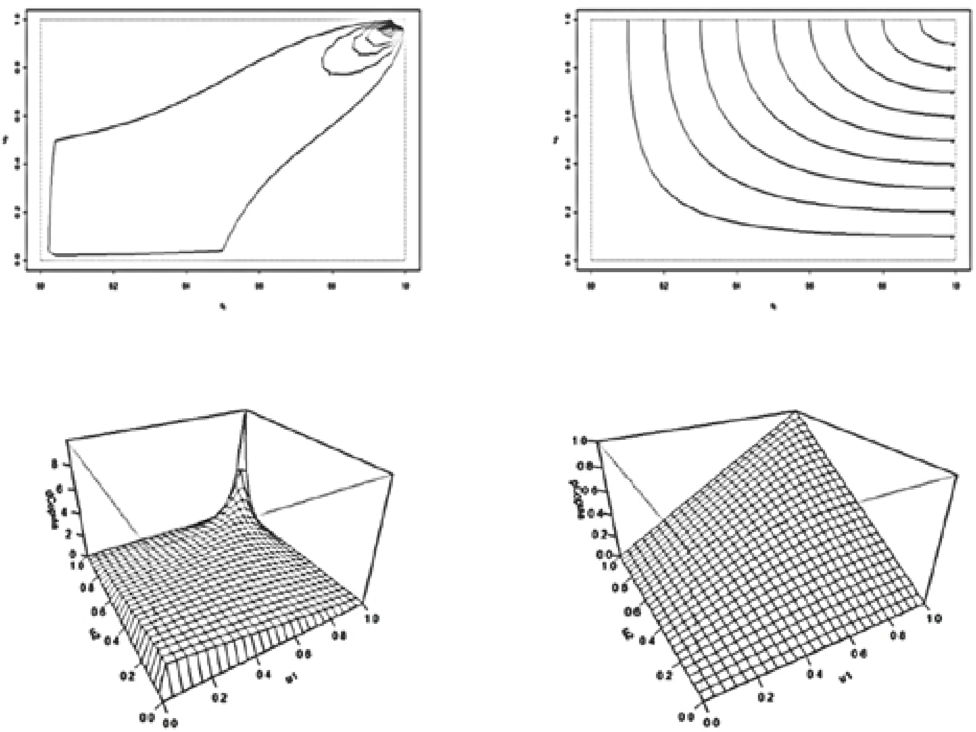
Figures 3. For Clayton Copula $\theta = 0.1598272$, respectively two and three dimensional density and probability function



Figures 4. For Frank Copula $\theta = 0,6689712$, respectively two and three dimensional density and probablity function



Figures 5. For Ali Mikhail Haq Copula $\theta = 0.3061617$, respectively two and three dimensional density and probability function



Figures 6. For Joe Copula $\theta = 2.139303$, respectively two and three dimensional density and probablity function.

Finally, this study consists of forming a fit copula to the data. The consequences of estimations are given table 4.

Table 4. Fitting a suitable copula the data

pairs	Gumbel	Clayton	Frank	Ali Mikhail Haq	Joe
(X,Y)	0.00077860	0.0000010	0.00343404	0.00069857	0.00082700

CONCLUSION

In our paper, we modeled the dependence structure between $X \sim Gp(3,3,3)$ an $Y \sim Gp(3,3,3)$. utilizing Archimedean copula. According to table 4, $K_n(u)$ the nonparametric estimation of is obtained by utilizing pseudo-observations, and utilizing table 2, for Gumbel, Clayton, Frank, Ali Mikhail Haq and Joe respectively $K_G(u)$ $K_C(u)$,$K_F(u)$,$K_{AMH}(u)$and $K_J(u)$ values calculated. In table 4 $K_n(u)$ value compared Consequently, using the square distance measure, with 0.0000010 value, Clayton copula shows better suitability than Gumbel, Frank, Ali Mikhail Haq and Joe.

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