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Some Properties of the Generalization of the p-Adic Factorial and the p-Adic Gamma Function

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Abstract

In this work, we investigate several properties of the generalized p-adic gamma function Γ_q. We demonstrate remarkable identities and special values of Γ_q. We also derive a novel representation of Γ_q via its Mahler expansion and establish relationships among the coefficients within this expansion. To simplify this study, we introduce the definition of the q-adic factorial and establish its properties. In addition, some congruences are derived for this new concept.

Keywords: Mahler expansions; p-adic gamma function; p-adic factorial; p-adic number.

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p-Adik Faktöriyelin ve p-Adik Gama Fonksiyonunun Genellemesinin Bazı Özellikleri

Öz

Bu çalışmada, genelleştirilmiş p-adik gama fonksiyonu Γ_q'nun çeşitli özelliklerini araştırıyoruz. Γ_q'nun dikkate değer özdeşliklerini ve özel değerlerini gösteriyoruz. Ayrıca Γ_q'nun Mahler açılımı aracılığıyla yeni bir gösterimini türetiyor ve bu açılımdaki katsayılar arasında ilişkiler kuruyoruz. Bu çalışmayı basitleştirmek için q-adik faktöriyel tanımını tanıtıyor ve özelliklerini belirliyoruz. Ek olarak, bu yeni kavram için bazı kongrüanslar türetiyoruz.

Anahtar Kelimeler: Mahler açılımı; p-adik gama fonksiyonu; p-adik faktöriyel; p-adik sayı.

1. Introduction

In 1897, Kurt Hensel introduced the p-adic numbers, which are an extension of the rational numbers for a given prime number p. These numbers have become essential in modern mathematics, due to their wide applications in number theory, analysis, mathematical physics, and cryptography. The unique representation of the p-adic numbers is given by the infinite series:

$$\mathrm{x} = \sum_{\mathrm{j}=\mathrm{m}}^{+\infty} \mathrm{x_j p^j} \quad , \qquad \text{where } \mathrm{x_j} \in \{0,1,\dots,\mathrm{p}-1\} \text{ and } \mathrm{m} \in \mathbb{Z}\,.$$

The p-adic gamma function Γ_p is the p-adic analog of the classical gamma function. In 1975, Morita [1] defined Γ_p as a unique continuous extension of the following sequence:

$$\Gamma_p(n+1) = (-1)^{n+1} \prod_{\substack{k=1 \\ (p,k)=1}}^{n} k \quad , \qquad \text{where } n \in \mathbb{N}.$$

Several generalizations of the p-adic gamma function have been proposed (see [2-5]), one of which was presented by Kaori Ota in 1994 [4]. To facilitate the study of the generalized p-adic hypergeometric function, Ota defined the generalized p-adic gamma function Γ_q, where $q = p^t$ and t is a positive integer, by the formula:

$$\Gamma_q(\alpha+1) = \Gamma_p(h_\ell(\alpha)+1).$$

such that $\alpha = \sum_{j=0}^{+\infty} \alpha_j p^j$ and $h_\ell(\alpha) = \sum_{j=\ell}^{+\infty} \alpha_j p^{j-\ell}$ for $\ell \in \mathbb{N}$.

The representation of generalized hypergeometric functions in terms of generalized gamma functions is particularly important. Such representations consistently yield results of notable mathematical significance, especially in applied contexts. The profound utility of these relationships manifests across multiple disciplines, including quantum field theory, analytic number theory, and cryptography.

N. Koblitz [3] proposed another extension of the p-adic gamma function, using the same notation Γ_q for his modified function. However, this shared notation does not lead to confusion. Both Koblitz's and Ota's generalized p-adic gamma functions were presented without complete characterizations of their properties.

In this paper, we introduce the concept of a q-adic factorial and establish several congruences and inequalities related to this new construct (see Definition 3.1, Lemma 3.3, Proposition 3.6, Proposition 3.7, Corollary 3.8, and Corollary 3.9). Furthermore, we present a definition of the generalized p-adic gamma function Γ_q and we demonstrate some of its combinatorial properties, similar to those of the p-adic gamma function (see Proposition 3.15, Proposition 3.16, and Corollary 3.17). Additionally, we propose a Mahler expansion for Γ_q and prove the relationships between its coefficients (see Proposition 3.18). Finally, we provide several numerical examples to illustrate our results (see Examples 3.4 and 3.13). These results offer new insights into the combinatorial and analytic structure of p-adic special functions and paving the way for further developments.

2. Preliminaries

2.1. Notations

In this work, we use the following concepts: $p \in \{2,3,5,7,11,13,17,...\}$ is a prime number, $\mathbb{Z}$ denotes the set of integers, $\mathbb{Z}_-$ (resp. $\mathbb{Z}_+$) represents the set of negative integers (resp. the positive integers), $\mathbb{Z}^*$ is the set of non-zero integers. The set of non-negative integers is denoted by $\mathbb{N}$, the field of rational numbers is $\mathbb{Q}$, and $\mathbb{R}$ is the field of real numbers. The absolute value in $\mathbb{R}$ is denoted by $|.|$, and the real integer part is denoted by $[.]$.

The p-adic valuation is defined as $v_p(0) = +\infty$ and for $n \in \mathbb{Z}^*$ by $v_p(n) = \max\{r \in \mathbb{Z} \,/ p^r \text{ divides } n\}$. For $\frac{n}{m} \in \mathbb{Q}$, the p-adic valuation is given by $v_p\left(\frac{n}{m}\right) = v_p(n) - v_p(m)$.

The p-adic absolute value of an element in $\mathbb{Q}$ is given as

$$|a|_p = \begin{cases} p^{-\nu_p(a)} & \text{if } a \neq 0 \\ 0 & \text{if } a = 0 \end{cases}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic absolute value $|.|_p$ yields a field, known as the field of p-adic numbers, denoted by $\mathbb{Q}_p$. The p-adic absolute value on $\mathbb{Q}_p$ is the extension of that on $\mathbb{Q}$. The set of p-adic integers, denoted $\mathbb{Z}_p$, contains the p-adic numbers that satisfy $|a|_p \leq 1$. For more details about p-adic numbers, we refer to the classical book of Gouvêa [6].

2.2. The p-adic Factorial and the p-adic Gamma Function

In this part, we present the concept of the p-adic factorial and the p-adic gamma function, along with several fundamental properties.

Definition 2.1. [7] We define the p-adic factorial by $0!_p = 1$, and for $n \in \mathbb{N}$ is given by

$$n!_p = \prod_{\substack{m=1 \\ p \nmid m}}^{n} m. \tag{1}$$

For more details and properties of this concept see [7].

The p-adic gamma function has found significant applications in dynamic systems and string theory. Various mathematicians [8-10] have utilized this function to explore properties of polynomials.

Definition 2.2. [1] The p-adic gamma function is a function $\Gamma_p : \mathbb{Z}_p \to \mathbb{Z}_p$ that extends the following sequence: $\Gamma_p(n+1) = (-1)^{n+1} n!_p$, for $n \in \mathbb{Z}_+$. Which we have, for $x \in \mathbb{Z}_p$:

$$\Gamma_p(x) = \lim_{n \to x} \Gamma_p(n).$$

Here, we cite the properties of Γ_p that we will give their analogue for the generalization of Γ_p in the next section.

Proposition 2.3. [11] The function Γ_p holds the following properties

1. $\Gamma_p(0) = 1$, $\Gamma_p(1) = -1$, $\Gamma_p(2) = 1$.
2. $\Gamma_p(n+1) = (-1)^{n+1} n!_p$, $\forall n \in \mathbb{N}$.

Proposition 2.4. [11] Let $n \geq 1$ be a positive integer with its p-adic expansion given by $\sum_{j=0}^{\ell} n_j p^j$ and we suppose the sum of digits is $S_n = \sum_{j=0}^{\ell} n_j$. Then, For $\mu = \left[\frac{n}{p}\right]$, we have $\Gamma_p(n+1) = \frac{(-1)^{n+1}\, n!}{\mu!\, p^{\mu}}$. In particular $\Gamma_p(p^k) = \frac{(-1)^p\, p^k!}{p^{k-1}!\, p^{p^{k-1}}}$,

1. For $m \in \mathbb{N}$ such that $0 \le \lambda < p$, we have

$$\Gamma_p(np+\lambda+1) = \frac{(-1)^{np+\lambda+1}\,(np+\lambda)!}{n!\; p^n}.$$

2. We have the relation between $n!$ and Γ_p

$$n! = (-1)^{n+1-\ell}(-p)^{\frac{n-S_n}{p-1}} \prod_{i=0}^{\ell} \Gamma_p\left(\left[\frac{n}{p^i}\right]+1\right).$$

Theorem 2.5. [11] (Mahler Expansion of Γ_p)

For $x \in \mathbb{Z}_p$, let the Mahler expansion of Γ_p be given by the series $\Gamma_p(x+1) = \sum_{k=0}^{+\infty} \alpha_k \binom{x}{k}$, where the symbol $\binom{x}{\eta}$ is defined by $\binom{x}{0} = 1$ and $\binom{x}{\eta} = \frac{x(x-1)\ldots(x-\eta+1)}{\eta!}$, $\eta = 1,2,\ldots$

Then, the coefficients α_k satisfy the following relationship

$$\exp\left(x + \frac{x^p}{p}\right)\frac{1-x^p}{1-x} = \sum_{k=0}^{+\infty}(-1)^{k+1}\alpha_k \frac{x^k}{k!}. \tag{2}$$

3. Main Results and Proofs

Drawing on the work presented in [7], we will introduce a q-adic factorial and establish a few congruences and inequalities related to this new concept. We also provide a definition for the generalization of the p-adic gamma function and demonstrate some of its properties, including the Mahler expansion. Throughout this section, we consider $q = p^t$, where t is a positive integer.

3.1. The q-adic Factorial

Definition 3.1. The q-adic factorial is defined by $0!_q = 1$ and for $n > 1$ is equal to

$$n!_q = \prod_{\substack{k=1 \\ q \nmid k}}^{n} k. \tag{3}$$

Remark 3.2. If $1 \le n \le q-1$, then for all $1 \le k \le n$ we have $q \nmid k$. So, $n!_q = n!$.

Lemma 3.3. For $\delta \ge 1$, we have $(q\delta)!_q = (q\delta - 1)!_q$.

Proof. We observe that q divides $q\delta$, it means that

$$(q\delta)!_q = \prod_{\substack{k=1 \\ q\nmid k}}^{q\delta} k = \prod_{\substack{k=1 \\ q\nmid k}}^{q\delta-1} k = (q\delta - 1)!_q \,.$$

Example 3.4. In Tables 1, 2, and 3 we compute the q-adic factorials of a given positive integer, specifically for $q = 2^2, 2^3, 3^2$.

Table 1: For $q = 2^2$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$n!_{2^2}$	1	1	2	6	6	30	180	1260	1260	11340	113400	1247400

Table 2: For $q = 2^3$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$n!_{2^3}$	1	1	2	6	24	120	720	5040	5040	45360	453600	4989600

Table 3: For $q = 3^2$.

n	0	1	2	3	4	5	6	7	8	9	10	11
$n!_{3^2}$	1	1	2	6	24	120	720	5040	40320	40320	403200	4435200

A q-generalization of the Wilson congruence is given in the following theorem, which is necessary to prove the Proposition 3.6.

Theorem 3.5. Let $a \in \mathbb{Z}$ and $s > t$. Then

1. For p odd,

$$\prod_{\substack{j=a \\ q\nmid j}}^{a+p^s-1} j \equiv - \prod_{\substack{j=a \\ p\mid j}}^{a+p^s-1} j \pmod{p^s}.$$

2. For $p = 2$,

$$\prod_{\substack{j=a \\ q\nmid j}}^{a+2^s-1} j \equiv \prod_{\substack{j=a \\ j \text{ even}}}^{a+2^s-1} j \pmod{2^s}.$$

Proof. We have

$$\prod_{\substack{j=a\\ q\nmid j}}^{a+p^s-1} j = \left(\prod_{\substack{j=a\\ p\mid j}}^{a+p^s-1} j\right)\left(\prod_{\substack{j=a\\ p\nmid j}}^{a+p^s-1} j\right)$$

By generalizing the Wilson congruence, we obtain the result. For $p = 2$ is the same. From the previous, we get the following congruence.

Proposition 3.6. Let $\mathrm{n} \in \mathbb{N}$ and $\mathrm{s} > \mathrm{t}$. Then

1. For p odd, we have $\frac{(\mathrm{n}+\mathrm{p}^{\mathrm{s}})!_{\mathrm{q}}}{\mathrm{n}!_{\mathrm{q}}} \equiv -\prod_{\substack{\mathrm{j}=\mathrm{n}+1\\ \mathrm{p}\mid\mathrm{j}}}^{\mathrm{n}+\mathrm{p}^{\mathrm{s}}} \mathrm{j} \pmod{\mathrm{p}^{\mathrm{s}}}$.
2. For $\mathrm{p} = 2$, we have $\frac{(\mathrm{n}+2^{\mathrm{s}})!_{\mathrm{q}}}{\mathrm{n}!_{\mathrm{q}}} \equiv \prod_{\substack{\mathrm{j}=\mathrm{n}+1\\ \mathrm{j\ even}}}^{\mathrm{n}+2^{\mathrm{s}}} \mathrm{j} \pmod{2^{\mathrm{s}}}$.

Proof. We calculate the quotient, we get

$$\frac{(n+p^s)!_q}{n!_q} = \prod_{\substack{j=n+1\\ q\nmid j}}^{n+p^s} j\,.$$

Now, we take $a = n + 1$ in Case 1 of Theorem 3.5, we derive the congruence for p odd. Similarly, we obtain the result for $p = 2$, from case 2.

More generally, we establish the following proposition along with its immediate corollaries:

Proposition 3.7. Let $\mathrm{n}, \mathrm{m}, \mathrm{s} \in \mathbb{Z}_+$ where $\mathrm{s} > \mathrm{t}$. Then

1. For p odd, we have

$$\frac{(\mathrm{n}+\mathrm{mp}^{\mathrm{s}})!_{\mathrm{q}}}{\mathrm{n}!_{\mathrm{q}}} \equiv (-1)^{\mathrm{m}} \prod_{\substack{\mathrm{j}=\mathrm{n}+1\\ \mathrm{p}\mid\mathrm{j}}}^{\mathrm{n}+\mathrm{mp}^{\mathrm{s}}} \mathrm{j} \pmod{\mathrm{p}^{\mathrm{s}}}.$$

2. For $\mathrm{p} = 2$ and $\mathrm{s} \geq 3$, we have

$$\frac{(n+m2^s)!_q}{n!_q} \equiv \prod_{\substack{j=n+1\\ j\ \mathrm{even}}}^{n+2^s} j \pmod{2^s}.$$

Proof. This is proved by induction on m. Indeed, about the case $p \geq 3$ we have:

For $m = 1$ is true. We assume that the property is true at rank m and we demonstrate it at rank $m + 1$. We write

$$\frac{(n+(m+1)p^s)!_q}{n!_q} = \frac{(n+mp^s)!_q}{n!_q} \cdot \prod_{\substack{j=n+mp^s+1 \\ q \nmid j}}^{n+(m+1)p^s} j$$

On the other hand

$$\frac{(n+mp^s)!_q}{n!_q} \equiv (-1)^m \prod_{\substack{j=n+1 \\ p \mid j}}^{n+mp^s} j \pmod{p^s}$$

and

$$\prod_{\substack{j=n+mp^s+1 \\ q \nmid j}}^{n+(m+1)p^s} j \equiv - \prod_{\substack{j=n+mp^s+1 \\ p \mid j}}^{n+(m+1)p^s} j \pmod{p^s}$$

So

$$\frac{(n+(m+1)p^s)!_q}{n!_q} \equiv \left((-1)^m \prod_{\substack{j=n+1 \\ p \mid j}}^{n+mp^s} j \right) \left(- \prod_{\substack{j=n+mp^s+1 \\ p \mid j}}^{n+(m+1)p^s} j \right) \pmod{p^s}$$

$$\equiv (-1)^{m+1} \prod_{\substack{j=n+1 \\ p \mid j}}^{n+(m+1)p^s} j \pmod{p^s}$$

Then the property is true for all m. For $p = 2$ is the same.

The following corollaries follow from Proposition 3.6 and are presented here without proof.

Corollary 3.8. For $p \geq 3$, $n \in \mathbb{N}$ and $s \in \mathbb{Z}_+$ where $s > t$, we have

$$\left| (n+p^s)!_q + n!_q \prod_{\substack{j=n+1 \\ p \mid j}}^{n+p^s} j \right|_p \leq \frac{1}{p^s}.$$

Corollary 3.9. For p = 2, n ∈ ℕ and s ∈ $\mathbb{Z}_+$ and s ≥ 3 where s > t, we have

$$\left|(n+2^s)!_q - n!_q \prod_{\substack{j=n+1 \\ j \text{ even}}}^{n+2^s} j\right|_2 \leq \frac{1}{2^s}.$$

3.2. Generalized p-adic Gamma Function

In this important subsection, we present the definition of the generalized p-adic gamma function, which was originally introduced by Kaori Ota in [4], and we also explore in depth several of its significant properties. Moreover, we establish new properties of this function that were not covered in Ota's original work. For other references on this subject, see [12-13].

Definition 3.10. [4] Let $x \in \mathbb{Z}_p$ be given by its p-adic expansion $\sum_{j=0}^{+\infty} x_j p^j$, where $x_0 \neq 0$ and $x_j \in \{0, 1, \cdots, p-1\}$ for all $j \in \mathbb{N}$. Ota define a map h_ℓ for $\ell \in \mathbb{N}$ by a formula

$$h_\ell(x) = \sum_{j=\ell}^{+\infty} x_j p^{j-\ell}. \tag{4}$$

Furthermore, we have

$$x = \sum_{j=0}^{\ell-1} x_j p^j + p^\ell h_\ell(x). \tag{5}$$

Definition 3.11. [4] The generalized p-adic gamma function is defined from $\mathbb{Z}_p$ to $\mathbb{Z}_p$ by:

$$\Gamma_q(x+1) = \prod_{\ell=0}^{t-1} \Gamma_p(h_\ell(x)+1). \tag{6}$$

Remark 3.12. For $t = 1$ the function Γ_q coincides with Γ_p, i.e.

$$\Gamma_q(x+1) = \Gamma_p(h_0(x)+1) = \Gamma_p(x+1).$$

Example 3.13. For $t = 2$, we have $q = p^2$ so

$$\Gamma_q(x+1) = \Gamma_p(h_0(x)+1)\Gamma_p(h_1(x)+1)$$

$$= \Gamma_p(h_0(x)+1)\Gamma_p(h_1(x)+1).$$

according to the relationship Eqn. (5) we have $x = x_0 + p h_1(x)$, hence

$$h_1(x) = \frac{x - x_0}{p}.$$

Therefore

$$\Gamma_q(x+1) = \Gamma_p(x+1)\Gamma_p\left(\frac{x - x_0}{p} + 1\right).$$

For example, if $x = 3$, $p = 3$ and $t = 2$, so $q = 3^2$. Then, $\Gamma_9(4) = \Gamma_3(4)\Gamma_3(2) = 2$.

In his paper [4], Ota presented some properties of the generalized p-adic gamma function. Among them, the following property is especially remarkable.

Proposition 3.14. [4]

1. For a positive integer n, we have

$$\Gamma_q(n+1) = (-1)^{A_n} p^{-B_n} n!_q \tag{7}$$

where

$$A_n = t + \sum_{i=0}^{t-1} t\left[\frac{n}{p^i}\right] \quad \text{and} \quad B_n = \sum_{i=1}^{t}\left[\frac{n}{p^i}\right] - t\left[\frac{n}{p^t}\right].$$

2. For p odd and $x \in \mathbb{Z}_p$, the complement formula is given by

$$\Gamma_q(x)\Gamma_q(1-x) = (-1)^{t-1+R_t(x)}, \tag{8}$$

where $R_t(x) \in \{1,2,\ldots,q\}$ is the representative of x modulo $q\mathbb{Z}_p$.

Next, we present our results concerning the combinatorial properties of the generalized p-adic gamma function and the coefficients of its Mahler expansion.

Proposition 3.15. The following statements are verified by the function Γ_q:

1. For a positive integer n and $\xi = \left[\frac{n}{q}\right]$, we have

$$\Gamma_q(n+1) = \frac{(-1)^{A_n} p^{-B_n} n!}{\xi!\ q^{\xi}}. \tag{9}$$

2. For a positive integer m and $\lambda \in \{0, 1, \cdots, q-1\}$, we have

$$\Gamma_q(mq + \lambda + 1) = \frac{(-1)^{A_{m,\lambda}}\ p^{-B_{m,\lambda}}(mq+\lambda)!}{m!\ q^m}. \tag{10}$$

where $A_{m,\lambda} = t + \lambda + mq + v_p(\lambda!)$ and $B_{m,\lambda} = m\left(\frac{q-1}{p-1}\right) - tm - t\left[\frac{\lambda}{q}\right] + v_p(\lambda!)$.

3. For $s \in \mathbb{N}$ we have

$$\Gamma_q(q^s) = \frac{(-1)^{A_{q^{s-1}}} p^{-B_{q^{s-1}}}(q^s - 1)!}{q^{s-1}!\ q^{q^{s-1}}}. \tag{11}$$

where

$$A_{q^{s-1}} = t - 1 + \sum_{i=0}^{t-1} p^{ts-i} \quad \text{and} \quad B_{q^{s-1}} = \sum_{i=1}^{t} p^{ts-i} - tq^{s-1}.$$

Proof.

1. We know that

$$n! = \prod_{k=1}^{n} k = \left(\prod_{\substack{k=1\\ q|k}}^{n} k\right)\left(\prod_{\substack{k=1\\ q\nmid k}}^{n} k\right) = \left(\prod_{\substack{k=1\\ q|k}}^{n} k\right) n!_q. \tag{12}$$

On the other hand, we know that card $\{1 \leq k \leq n/\ q|k\} = \left[\frac{n}{q}\right]$. Thus,

$$\prod_{\substack{k=1\\ q|k}}^{n} k = \prod_{j=1}^{\left[\frac{n}{q}\right]} (jq) = \left[\frac{n}{q}\right]!\ q^{\left[\frac{n}{q}\right]}.$$

Therefore, by the Proposition 3.14, we get

$$\Gamma_q(n+1) = (-1)^{A_n} p^{-B_n} n!_q = \frac{(-1)^{A_n} p^{-B_n} n!}{\xi!\ q^{\xi}}.$$

2. The Euclidean division gives $n = mq + \lambda$, where $\lambda \in \{0, 1, \cdots, q-1\}$. So,

$$\left[\frac{n}{q}\right] = m + \left[\frac{\lambda}{q}\right] = m.$$

Substituting this into Eqn. (9), we obtain

$$\Gamma_q(mq + \lambda + 1) = \frac{(-1)^{A_{mq+\lambda}} p^{-B_{mq+\lambda}} n!}{m!\ q^m}. \tag{13}$$

Such that $A_{mq+\lambda}$ and $B_{mq+\lambda}$ are calculated as follows:

- For $A_{mq+\lambda}$, we have

$$A_{mq+\lambda} = t + \sum_{i=0}^{t-1} \left[\frac{mq+\lambda}{p^i}\right] = t + mq + \lambda + mq \sum_{i=1}^{t-1} \left(\frac{1}{p}\right)^i + \sum_{i=1}^{t-1} \left[\frac{\lambda}{p^i}\right]. \tag{14}$$

We know that $v_p(\lambda!) = \sum_{i=0}^{t-1} \left[\frac{\lambda}{p^i}\right]$. Since the sum $\sum_{i=0}^{t-1} \left(\frac{1}{p}\right)^i$ is even, it comes that

$$(-1)^{\sum_{i=0}^{t-1}\left(\frac{1}{p}\right)^i} = 1.$$

Then, $(-1)^{A_{mq+\lambda}} = (-1)^{t+\lambda+mq+v_p(\lambda!)}$. Finally, we obtain $A_{m,\lambda} = t + \lambda + mq + v_p(\lambda!)$.

- For $B_{mq+\lambda}$, we have

$$B_{mq+\lambda} = \sum_{i=1}^{t} \left[\frac{mq+\lambda}{p^i}\right] - t\left[\frac{mq+\lambda}{q}\right] = mq \sum_{i=1}^{t} \left(\frac{1}{p}\right)^i + \sum_{i=1}^{t-1} \left[\frac{\lambda}{p^i}\right] + \left[\frac{\lambda}{p^t}\right] - tm - t\left[\frac{\lambda}{q}\right]. \tag{15}$$

So $B_{m,\lambda} = m\left(\frac{q-1}{p-1}\right) - tm - t\left[\frac{\lambda}{q}\right] + v_p(\lambda!)$.

3. We simply need to substitute $n = q^s - 1$ into Eqn. (9) to determine the value of $\Gamma_q(q^s)$.

Proposition 3.16.

1. We have $\Gamma_q(0) = 1$ and $\left|\Gamma_q(x)\right|_p = 1$, for all $x \in \mathbb{Z}_p$.

2. For $p \geq 3$, let $x, y \in \mathbb{Z}_p$. We have

   - If $|x-y|_p = 1$, then $\left|\Gamma_q(x) - \Gamma_q(y)\right|_p \leq |x-y|_p$.
   - If $|x-y|_p = \frac{1}{p^s}$, with $\geq t$, then $\left|\Gamma_q(x) - \Gamma_q(y)\right|_p \geq |x-y|_p$.

Proof.

1. By the definition of Γ_q, we have $\Gamma_q(0) = \prod_{\ell=0}^{t-1} \Gamma_p(h_\ell(-1)+1)$. Furthermore, we have

$$-1 = \sum_{i=0}^{\ell-1} (p-1)p^i + p^\ell \sum_{i=0}^{+\infty} (p-1)p^i. \tag{16}$$

So, $h_\ell(-1) = -1$. Then, $\Gamma_q(0) = \prod_{\ell=0}^{t-1} \Gamma_p(0) = 1$.

For the second part $\left|\Gamma_q(x)\right|_p = 1$, we simply use the fact that $\left|\Gamma_p(x)\right|_p = 1$.

2. First, we prove the property for positive integers and then extend the result to p-adic

integers by taking the limit. For $m, n \in \mathbb{N}$, we consider the following cases:

- If $|n-m|_p = 1$. Then,

$$\left|\Gamma_q(n) - \Gamma_q(m)\right|_p \leq \max\left(\left|\Gamma_q(n)\right|_p, \left|\Gamma_q(m)\right|_p\right) \leq 1 = |n-m|_p. \tag{17}$$

- If $n > m$ and $|n-m|_p = \frac{1}{p^s}$, with $s \geq t$. Then, there exists $\mu \in \mathbb{N}$ such that $n = m + \mu p^s$ and $p \nmid \mu$. Therefore, by applying the Eqn. (7), we get

$$\Gamma_q(n) = \Gamma_q(m + \mu p^s) = (-1)^{A_n} p^{-B_n} \prod_{\substack{k=1 \\ q \nmid k}}^{m+\mu p^s - 1} k$$

$$= (-1)^{A_n} p^{-B_n} \left(\prod_{\substack{k=1 \\ q \nmid k}}^{m-1} k \right) \left(\prod_{\substack{k=m \\ q \nmid k}}^{m+\mu p^s - 1} k \right). \tag{18}$$

Now, since $s \geq t$ we must have $A_n = A_1 + A_2$ and $B_n = B_1 + B_2$, with

$$A_1 = t + \sum_{i=0}^{t-1} \left[\frac{m}{p^i}\right] \quad , \quad A_2 = \sum_{i=0}^{t-1} \mu p^{s-i}.$$

$$B_1 = \sum_{i=1}^{t} \left[\frac{m}{p^i}\right] - t\left[\frac{m}{q}\right] \quad , \quad B_2 = \sum_{i=1}^{t} \mu p^{s-i} - t\mu p^{s-t}.$$

Therefore

$$\Gamma_q(n) = \left((-1)^{A_1} p^{-B_1} \prod_{\substack{k=1 \\ q \nmid k}}^{m-1} k \right) \left((-1)^{A_2} p^{-B_2} \prod_{\substack{k=m \\ q \nmid k}}^{m+\mu p^s - 1} k \right)$$

$$= \Gamma_q(m) \left((-1)^{A_2} p^{-B_2} \prod_{\substack{k=m \\ q \nmid k}}^{m+\mu p^s - 1} k \right). \tag{19}$$

Hence

$$\left|\Gamma_q(n)-\Gamma_q(m)\right|_p=\left|\Gamma_q(m)\right|_p\left|\left((-1)^{A_2}p^{-B_2}\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k\right)-1\right|_p. \tag{20}$$

Now, rewriting the product as follows

$$\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k=\left(\prod_{\substack{k=m\\ q\nmid k}}^{m+p^s-1}k\right)\left(\prod_{\substack{k=m+p^s\\ q\nmid k}}^{m+2p^s-1}k\right)\cdots\left(\prod_{\substack{k=m+(\mu-1)p^s\\ q\nmid k}}^{m+\mu p^s-1}k\right). \tag{21}$$

By Theorem 3.5, we have for all $\theta\in\{1,2,\dots,\mu\}$

$$\prod_{\substack{k=m+(\theta-1)p^s\\ q\nmid k}}^{m+\theta p^s-1}k\equiv-\prod_{\substack{j=m+(\theta-1)p^s\\ p|j}}^{m+\theta p^s-1}k\pmod{p^s}.$$

We know that the fact $p|k$ implies that $k=\lambda_\theta p^{\ell_\theta}$ with $p\nmid\lambda_\theta$ and $\ell_\theta\in\mathbb{N}$. Thus

$$\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k\equiv(-1)^\mu\left(\prod_{\substack{\theta=1\\ p\nmid\lambda_\theta}}^{\mu}\lambda_\theta\right)\left(\prod_{\theta=1}^{\mu}p^{\ell_\theta}\right)\pmod{p^s}. \tag{22}$$

which implies

$$(-1)^{A_2}p^{-B_2}\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k\equiv(-1)^{\mu+A_2}\,p^{\sum_{\theta=1}^{\mu}\ell_\theta-B_2}\prod_{\substack{\theta=1\\ p\nmid\lambda_\theta}}^{\mu}\lambda_\theta\pmod{p^s}. \tag{23}$$

On the other hand, we have

$$\mu+A_2=\mu\left(1+p^{s+1-t}\frac{p^t-1}{(p-1)}\right)\qquad,\qquad B_2=\mu p^s\left(\frac{p^t-1}{(p-1)p^{t-1}}-tp^{-t}\right).$$

So, we get

$$\left((-1)^{A_2}p^{-B_2}\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k\right)\equiv\ p^{\sum_{\theta=1}^{\mu}\ell_\theta-B_2}\prod_{\substack{\theta=1\\ p\nmid\lambda_\theta}}^{\mu}\lambda_\theta\pmod{p^s}. \tag{24}$$

Thus

$$\left|\left((-1)^{A_2}p^{-B_2}\prod_{\substack{k=m\\ q\nmid k}}^{m+\mu p^s-1}k\right)-1\right|_p=\max\left\{p^{B_2-\sum_{\theta=1}^{\mu}\ell_\theta},p^{-s},1\right\}\geq\frac{1}{p^s}\,. \tag{25}$$

Which yields the result $\left|\Gamma_q(n)-\Gamma_q(m)\right|_p\geq|n-m|_p$.

By passing to the limit, we conclude that the property holds for all elements of $\mathbb{Z}_p$.

Corollary 3.17. For p odd, we have

$$\Gamma_q{}^2\left(\frac{1}{2}\right)=(-1)^t.$$

Proof. According to the complements Eqn. (8), for p odd and $x=\frac{1}{2}$ we have

$$\Gamma_q\left(\frac{1}{2}\right)\Gamma_q\left(\frac{1}{2}\right)=\Gamma_q{}^2\left(\frac{1}{2}\right)=(-1)^{t-1+R_t\left(\frac{1}{2}\right)}, \tag{26}$$

with $R_t\left(\frac{1}{2}\right)\in\{1,2,\dots,q\}$ is the representative of $\frac{1}{2}$ modulo $q\mathbb{Z}_p$. In fact, we know that

$$\frac{1}{2}=\frac{p+1}{2}+\sum_{i=1}^{+\infty}\frac{p-1}{2}p^i=\frac{p+1}{2}+\frac{p-1}{2}\sum_{i=1}^{t-1}p^i+\sum_{i=t}^{+\infty}\frac{p-1}{2}p^i, \tag{27}$$

which implies that

$$\frac{1}{2}=\frac{q+1}{2}+q\sum_{i=1}^{+\infty}\frac{p-1}{2}p^i, \tag{28}$$

thus $R_t\left(\frac{1}{2}\right)=\frac{q+1}{2}$, which is even, so $(-1)^{-1+R_t\left(\frac{1}{2}\right)}=1$. Hence, the result follows.

Proposition 3.18. (Mahler expansion of Γ_q)

Let the Mahler expansion of $\Gamma_q(x+1)$ be given by

$$\Gamma_q(x+1)=\sum_{\eta=0}^{+\infty}a_\eta\binom{x}{\eta}.$$

For $x\in\mathbb{Z}_p$. So, the coefficients a_η in this expansion satisfy the following relation

$$\exp_p\left(p^{\frac{q-1}{p-1}}x^q + x\right) = \frac{1}{\delta_q}\sum_{\eta=0}^{+\infty}(-1)^\eta \alpha_\eta \frac{x^\eta}{\eta!}\,, \tag{29}$$

where

$$\delta_q = \sum_{\lambda=0}^{q-1}(-1)^{t+v_p(\lambda!)}\, p^{t\left[\frac{\lambda}{q}\right]-v_p(\lambda!)}(-x)^\lambda.$$

Proof. The coefficients α_η satisfy the generating function relation

$$\sum_{\eta=0}^{+\infty}\alpha_\eta \frac{x^\eta}{\eta!} = e^{-x}\sum_{\eta=0}^{+\infty}\Gamma_q(\eta+1)\frac{x^\eta}{\eta!}\,. \tag{30}$$

The aim is to find an expression for the generating series of $\left(\Gamma_q(n+1)\right)_{n\in\mathbb{N}}$ given by

$$f(x) = \sum_{\eta=0}^{+\infty}\Gamma_q(\eta+1)\frac{x^\eta}{\eta!}\,.$$

To achieve this, we'll first decompose the index η in the sum according to its residue modulo q. Let $\eta = mq + \lambda$ where $m \geq 0$ and $\lambda \in \{0, 1, \dots, q-1\}$.

By separating the sum based on the possible values of λ, we can express $f(x)$ as

$$f(x) = \sum_{\lambda=0}^{q-1}\sum_{m=0}^{+\infty}\frac{\Gamma_q(mq+\lambda+1)}{(mq+\lambda)!}x^{mq+\lambda}. \tag{31}$$

We now use relation (10) to derive an explicit form of the generating series f

$$\begin{aligned} f(x) &= \sum_{\lambda=0}^{q-1}\sum_{m=0}^{+\infty}\frac{(-1)^{t+\lambda+mq+v_p(\lambda!)}\, p^{-m\left(\frac{q-1}{p-1}\right)+tm+t\left[\frac{\lambda}{q}\right]-v_p(\lambda!)}}{m!\; q^m}x^{mq+\lambda} \\ &= \sum_{\lambda=0}^{q-1}(-1)^{t+v_p(\lambda!)}\, p^{t\left[\frac{\lambda}{q}\right]-v_p(\lambda!)}(-x)^\lambda \sum_{m=0}^{+\infty}\left(\frac{(-1)^q\, x^q}{q}\right)^m\frac{1}{m!} \\ &= \delta_q \exp\left(p^{\frac{q-1}{p-1}}(-x)^q\right), \end{aligned} \tag{32}$$

where

$$\delta_q = \sum_{\lambda=0}^{q-1} (-1)^{t+v_p(\lambda!)}\, p^{t\left[\frac{\lambda}{q}\right]-v_p(\lambda!)} (-x)^{\lambda}$$

Hence

$$e^{-x} f(x) = \delta_q \exp\left(p^{\frac{q-1}{p-1}} (-x)^q - x\right). \tag{33}$$

Then

$$\sum_{\eta=0}^{+\infty} \alpha_\eta \frac{x^\eta}{\eta!} = \delta_q \exp\left(p^{\frac{q-1}{p-1}} (-x)^q - x\right). \tag{34}$$

By replacing $-\boldsymbol{x}$ with $\boldsymbol{x}$, we obtain

$$\sum_{\eta=0}^{+\infty} (-1)^\eta \alpha_\eta \frac{x^\eta}{\eta!} = \delta_q \exp\left(p^{\frac{q-1}{p-1}} x^q + x\right).$$

This completes the proof.

4. Conclusion

In this work, we introduced and systematically studied the q-adic factorial $n!_q$, establishing its fundamental properties, including several novel congruences and inequalities. Building on this foundation, we reformulated the generalized p-adic gamma function Γ_q given by Ota through the q-adic factorial, revealing new combinatorial identities. In addition, we proposed the Mahler expansion of Γ_q and proved the relationships between its coefficients. These results collectively advance the theory of p-adic special functions and open new avenues for further research in p-adic analysis and its applications.

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