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Generalized Pathway Fractional Integral Formulas Involving Extended Multi-Index Mittag-Leffler Function in Kernel of SUM Transform

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ABSTRACT

The generalized pathway fractional integral formulas for the newly extended multi-index Mittag-Leffler function defined by using two Fox-Wright functions as its kernel is studied. Moreover, the SUM integral transform of the composition formula for the pathway fractional integral and extended multi-index Mittag-Leffler function is also presented.

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1. Introduction

Fractional calculus deal with the study of derivative and integration of arbitrary order such as fractional, irrational and complex, it is also applicable in almost all area of real life such as epidemic model, reaction-diffusion, drying model, control theory, economics, fluid dynamics, etc (see for example, [1–4]). The following pathway fractional integral operator was introduced in [5]:

$$\left(P_{0^+}^{\tau,\rho;b}\right)(x) = x^{-\tau}\int_0^{\left[\frac{x}{b(1-\rho)}\right]}\left(1-\frac{b(1-\rho)t}{x}\right)^{\frac{\tau}{1-\rho}} f(t)dt,$$

where $f(t) \in L(a,b)$ and $\tau \in \mathbb{C}$, $x \in \mathbb{R}^+$ such that $b > 0$, $Re(\tau) > 0$ and $\rho < 1$ (ρ is a pathway parameter).

In 2013, Nair [6] introduced the following generalized pathway hypergeometric fractional integral operator:

$$\left(P_{0^+}^{\tau,\vartheta,\mu,\rho;b}\right)(x) = x^{-\tau}\int_0^{\left[\frac{x}{b(1-\rho)}\right]}\left(1-\frac{b(1-\rho)t}{x}\right)^{\frac{\tau-1}{1-\rho}} \times {}_2F_1\left(\frac{\tau-1}{1-\rho}+\vartheta+1,-\mu;\frac{\tau-1}{1-\rho}+1;1-\frac{b(1-\rho)t}{x}\right) \times f(t)dt, \quad (1)$$

where $f(t) \in \mathbb{R}^+$ and $\tau,\vartheta,\mu,\rho \in \mathbb{C}$, $x \in \mathbb{R}^+$ such that $b > 0$, $Re\left(\frac{\tau-1}{1-\rho}+1\right) > 0$, $Re(\mu-\vartheta) > 0$ and $\rho < 1$.

Remark 1: The following are also reported in [7]:

(i) If $Re(\tau) > 0$ when $\rho = 0$ and $b = 1$, Eq. (1) reduces to the following Saigo fractional integral operator:

$$\left(P_{0^+}^{\tau,\vartheta,\mu,0;1}\right)(x) = x\Gamma(\tau)\left(I_{0^+}^{\tau,\vartheta,\mu}\right)(x).$$

(ii) If $Re(\tau) > 0$ when $\rho = 0$, $b = 1$ and $\vartheta = -\tau$, Eq. (1) reduces to the following Riemann-Liouville fractional integral operator:

$$\left(P_{0^+}^{\tau,-\tau,\mu,0;1}\right)(x) = x\Gamma(\tau)\left(I_{0^+}^{\tau}\right)(x).$$

(iii) If $\rho = 0$, $\vartheta = 0$ and $b = 1$, Eq. (1) reduces to the following Kober fractional integral operator:

$$\left(P_{0^+}^{\tau,0,\mu,0;1}\right)(x) = x\Gamma(\tau)\left(I_{0^+}^{\tau,\mu}\right)(x).$$

(iv) If $\mu = 0$, $Re(\tau - 1) > 0$, Eq. (1) reduces to the following classical pathway fractional integral operator:

$$\left(P_{0^+}^{\tau-1,\vartheta,0,\rho;b}\right)(x) = x^{1-\tau-\vartheta}\Gamma(\tau)\left(P_{0^+}^{\tau-1,\rho}\right)(x).$$

(v) If $\mu = 0$, $\rho \to 1_-$, Eq. (1) reduces to the following Laplace integral transform:

$$\lim_{\rho\to 1_-}\left(P_{0^+}^{\tau,\vartheta,0,\rho;b}\right)(x) = x^{1-\vartheta}L\{f(t)\}\left(\frac{x}{b(1-\rho)}\right).$$

Recently, Kaurangini et al., [8] studied the following multi-index Mittag-Leffler function:

$$\begin{aligned}&{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z;\Phi,\Omega)\\ &= {}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left[\begin{array}{c|c|c}(B_k,b_k)_{1,\alpha} & (E_m,e_m)_{1,\gamma} & \\ & & z;\Phi,\Omega\\ (D_j,d_j)_{1,\beta} & (G_n,g_n)_{1,\delta} & \end{array}\right]\\ &= \sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(\lambda+pr,\delta-\lambda)}{B(\lambda,\delta-\lambda)}\frac{(\delta)_{qr}z^r}{\prod_{i=1}^{\eta}\Gamma(\varrho_i r+\sigma_i)(\phi)_{\kappa r}},\end{aligned} \tag{2}$$

where $\varrho_i, \sigma_i, \phi, \kappa, \lambda, \delta, \Phi, \Omega\omega, \varpi, z \in \mathbb{C}$, $\Phi, \Omega > 0$, $Re(\delta) > Re(\lambda) > 0, Re(\sigma_i) > 0, i = 1, 2, \ldots, \eta$, $Re(\varrho_{i=0}^{\eta}\aleph_i) > \max\{0, Re(q) - 1\}, Re(\phi) > 0$.

Here denotes ${}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(x, y)$ is the extended beta function [9, 10] and defined by

$$\begin{aligned}&{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(x,y)\\ &= {}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}\left[\begin{array}{c|c|c}(B_k,b_k)_{1,\alpha} & (E_m,e_m)_{1,\gamma} & \\ & & x,y\\ (D_j,d_j)_{1,\beta} & (G_n,g_n)_{1,\delta} & \end{array}\right]\\ &= \int_0^1 t^{x-1}(1-t)^{y-1}{}_{\alpha}\Psi_{\beta}\left(-\frac{\Phi}{t^{\omega}}\right){}_{\gamma}\Psi_{\delta}\left(-\frac{\Omega}{(1-t)^{\varpi}}\right)dt,\end{aligned}$$

where ${}_{\alpha}\Psi_{\beta}(.)$ is the Fox-Wright function [11] defined by

$$\begin{aligned}{}_{\alpha}\Psi_{\beta}(z) &= {}_{\alpha}\Psi_{\beta}\left[\begin{array}{c|c}(B_k,b_k)_{1,\alpha} & \\ & z\\ (D_j,d_j)_{1,\beta} & \end{array}\right]\\ &= \sum_{r=0}^{\infty}\frac{\prod_{k=1}^{\alpha}\Gamma(B_k n+b_k)}{\prod_{j=1}^{\beta}\Gamma(D_j r+d_j)}\frac{z^r}{r!},\end{aligned} \tag{3}$$

$$(z, b_k, d_j \in \mathbb{C}, B_k, D_j \in \mathbb{R}).$$

2. The Generalized Pathway Fractional Integral Formulas involving Extended Multi-Index Mittag-Leffler Function in the Kernel

The generalized pathway fractional integral formula for the new extended multi-index Mittag-Leffler function is studied.

Lemma 1: [7] The following result holds true

$$\begin{aligned}\left(P_{0^+}^{\tau,\vartheta,\mu,\rho;b}\nu^{\wp-1}\right)(x) &= \frac{1}{[b(1-\rho)]^{\wp}}\\ &\times\frac{\Gamma(\wp)\Gamma(\wp-\vartheta+\mu)\Gamma\left(1+\frac{\tau-1}{1-\rho}\right)}{\Gamma(\wp-\vartheta)\Gamma\left(1+\wp+\mu+\frac{\tau-1}{1-\rho}\right)}x^{\wp-\vartheta},\end{aligned} \tag{4}$$

where $f(t) \in \mathbb{R}^+$ and $\tau, \vartheta, \mu, \rho \in \mathbb{C}, x \in \mathbb{R}^+$ such that $b > 0$, $Re\left(\frac{\tau-1}{1-\rho}+1\right) > 0$, $Re(\wp) > \max\{0, Re(\mu-\vartheta)\} > 0$ and $\rho < 1$.

Theorem 1: The following pathway fractional formula holds

$$\begin{aligned}&\left(P_{0^+}^{\tau,\vartheta,\mu,\rho;b}\nu^{\wp-1}{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\rho,\sigma)\right)(x)\\ &= \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{[b(1-\rho)]^{\wp}}{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^{\ell}}{[b(1-\rho)]^{\wp}};\Phi,\Omega\right)\\ &* {}_3\Psi_2\left[\begin{array}{c|c}(\wp,\ell),(\wp-\vartheta+\mu,\ell),(1,1) & \\ & \frac{zx^{\ell}}{[b(1-\rho)]^{\ell}}\\ (\wp-\vartheta,\ell),\left(1+\wp\mu+\frac{\tau-1}{1-\rho},\ell\right) & \end{array}\right],\end{aligned} \tag{5}$$

where $*$ represent Hadamard (convolution) product defined in Pohlen [12].

Proof: Letting the right hand side of Eq. (6) be P and using (2), changing the order of summation and pathway fractional integral operator, we have

$$\begin{aligned}P &= \sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(\lambda+pr,\delta-\lambda)}{B(\lambda,\delta-\lambda)}\frac{(\delta)_{qr}z^r}{\prod_{i=1}^{\eta}\Gamma(\varrho_i r+\sigma_i)(\phi)_{\kappa r}}\\ &\times\left(P_{0^+}^{\tau,\vartheta,\mu,\rho;b}\nu^{\wp+\ell r-1}\right)(x).\end{aligned}$$

Applying Eq. (4) to the above equation, gives

$$\begin{aligned}P &= \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{[b(1-\rho)]^{\wp}}\sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(\lambda+pr,\delta-\lambda)}{B(\lambda,\delta-\lambda)}\\ &\times\frac{(\delta)_{qr}z^r}{\prod_{i=1}^{\eta}\Gamma(\varrho_i r+\sigma_i)(\phi)_{\kappa r}}\frac{\Gamma(\wp+\ell r)}{\Gamma(\wp-\vartheta+\ell r)}\\ &\times\frac{\Gamma(\wp-\vartheta+\mu+\ell r)}{\Gamma\left(1+\wp+\mu+\frac{\tau-1}{1-\rho}+\ell r\right)}\left(\frac{zx^{\ell}}{[b(1-\rho)]^{\ell}}\right)^r.\end{aligned} \tag{6}$$

Using the multi-index Mittag-Leffler function in Eq. (2) and the Wright function in (3) to (6), the required result in (5) is obtained.

As a consequence of Theorem 1 the following corollaries are obtained:

Corollary 1: The following equility is true

$$\left(P_{0^+}^{\tau,\vartheta,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\Phi,\Omega)\right)(x) = x^{\wp-\vartheta}\Gamma(\tau)\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(z\nu^{\ell};\Phi,\Omega\right) * {}_3\Psi_2\left[\begin{array}{c|c} (\wp,\ell),(\wp-\vartheta+\mu,\ell),(1,1) & \\ & zx^{\ell} \\ (\wp-\vartheta,\ell),(\wp+\tau\mu+\tau,\ell) & \end{array}\right].$$

Corollary 2: The following result holds

$$\left(P_{0^+}^{\tau,-\tau,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\Phi,\Omega)\right)(x) = x^{\wp+\tau}\Gamma(\tau)\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(z\nu^{\ell};\Phi,\Omega\right) * {}_2\Psi_1\left[\begin{array}{c|c} (\wp,\ell),(1,1) & \\ & zx^{\ell} \\ (\wp+\tau,\ell) & \end{array}\right].$$

Corollary 3: The following formula is true

$$\left(P_{0^+}^{\tau,0,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\Phi,\Omega)\right)(x) = x^{\wp}\Gamma(\tau)\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(z\nu^{\ell};\Phi,\Omega\right) * {}_2\Psi_1\left[\begin{array}{c|c} (\wp,\ell),(1,1) & \\ & zx^{\ell} \\ (\wp+\tau+\mu,\ell) & \end{array}\right].$$

Corollary 4: The following result holds true

$$\left(P_{0^+}^{\tau,\vartheta,0,\rho;b}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\Phi,\Omega)\right)(x) = \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{[b(1-\rho)]^{\wp}} \times {}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^{\ell}}{[b(1-\rho)]^{\wp}};\Phi,\Omega\right) * {}_2\Psi_1\left[\begin{array}{c|c} (\wp,\ell),(1,1) & \\ & \frac{zx^{\ell}}{[b(1-\rho)]^{\ell}} \\ \left(1+\wp\mu+\frac{\tau-1}{1-\rho},\ell\right) & \end{array}\right].$$

Corollary 5: The following formula holds true

$$\left(P_{0^+}^{\tau,\vartheta,0,1;b}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(z\nu^{\ell};\Phi,\Omega)\right)(x) = \frac{x^{\wp-\vartheta}}{[b(1-\rho)]^{\wp}}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^{\ell}}{[b(1-\rho)]^{\wp}};\Phi,\Omega\right) * {}_2\Psi_0\left[\begin{array}{c|c} (\wp,\ell),(1,1) & \\ & \frac{zx^{\ell}}{[b(1-\rho)]^{\ell}} \\ - & \end{array}\right].$$

3. Generalized Pathway Fractional Integral Formulas involving Extended Multi-Index Mittag-Leffler Function in the Kernel of the SUM transform

The SUM (Sameer-Umar-Muhammad) integral transform is defined by the following formula [13, 14]:

$$S_{\Lambda}\{f(t)\}(p) = \frac{1}{p^{\hbar}}\int_0^{\infty} f(t)\Lambda^{pt}\,dt, \tag{7}$$

where $t \geq 0, \hbar \in \mathbb{N}$, $\Lambda > 0$, $m_1 \leq p \leq m_2$, $m_1, m_2 > 0$ and $f(t)$ is piecewise continuous and exponential order. The SUM transform of power function is given by

$$S_{\Lambda}\{t^{\wp}\}(p) = \frac{\Gamma(\wp+1)}{p^{\hbar}[plog(\Lambda)]^{\wp+1}}, \quad (\wp\in\mathbb{C}). \tag{8}$$

Theorem 2: The following result holds

$$S_{\Lambda}\left\{\left(t\cdot P_{0^+}^{\tau,\vartheta,\mu,\rho;b}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(t\nu^{\ell};\Phi,\Omega)\right)(x)\right\}(p) = \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{p^{\hbar}[b(1-\rho)]^{\wp}[plog(\Lambda)]^2} \times {}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^{\ell}}{[b(1-\rho)]^{\wp}[plog(\Lambda)]};\Phi,\Omega\right) * {}_4\Psi_2\left[\begin{array}{c|c} (\wp,\ell),(\wp-\vartheta+\mu,\ell),(1,1),(2,1) & \\ & \frac{x^{\ell}}{[b(1-\rho)]^{\ell}[plog(\Lambda)]} \\ (\wp-\vartheta,\ell),\left(1+\wp\mu+\frac{\tau-1}{1-\rho},\ell\right) & \end{array}\right]. \tag{9}$$

Proof: Letting the right hand side of Eq. (8) be S and using (6) and (7), changing the order of summation and pathway fractional integral operator, we have

$$S = \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{[b(1-\rho)]^{\wp}}\sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(\lambda+pr,\delta-\lambda)}{B(\lambda,\delta-\lambda)} \times \frac{(\delta)_{qr}z^r}{\prod_{i=1}^{\eta}\Gamma(\varrho_i r+\sigma_i)(\phi)_{\kappa r}}\frac{\Gamma(\wp+\ell r)}{\Gamma(\wp-\vartheta+\ell r)} \times \frac{\Gamma(\wp-\vartheta+\mu+\ell r)}{\Gamma\left(1+\wp+\mu+\frac{\tau-1}{1-\rho}+\ell r\right)}\left(\frac{zx^{\ell}}{[b(1-\rho)]^{\ell}}\right)^r S_{\Lambda}\{t^{r+1}\}(p).$$

Using Eq. (8) to the above equation, we obtain

$$S = \frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{p^{\hbar}[b(1-\rho)]^{\wp}[plog(\Lambda)]^2}\sum_{r=0}^{\infty}\frac{{}^{\Psi}B_{\Phi,\Omega}^{\omega,\varpi}(\lambda+pr,\delta-\lambda)}{B(\lambda,\delta-\lambda)} \times \frac{(\delta)_{qr}z^r}{\prod_{i=1}^{\eta}\Gamma(\varrho_i r+\sigma_i)(\phi)_{\kappa r}}\frac{\Gamma(\wp+\ell r)}{\Gamma(\wp-\vartheta+\ell r)} \times \frac{\Gamma(\wp-\vartheta+\mu+\ell r)}{\Gamma\left(1+\wp+\mu+\frac{\tau-1}{1-\rho}+\ell r\right)}\left(\frac{zx^{\ell}}{[b(1-\rho)]^{\ell}[plog(\Lambda)]}\right)^r. \tag{10}$$

Using the multi-index Mittag-Leffler function in Eq. (2) and the Wright function in (3) to (10) , the required result in (9) is obtained.

Corollary 6: The following result is also true

$$S_\Lambda\left\{\left(t\cdot P_{0^+}^{\tau,\vartheta,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(t\nu^\ell;\Phi,\Omega)\right)(x)\right\}(p)$$
$$=\frac{x^{\wp-\vartheta}\Gamma(\tau)}{p^{\hbar}[plog(\Lambda)]^2}$$
$$\times\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^\ell}{[b(1-\rho)]^{\wp}[plog(\Lambda)]};\Phi,\Omega\right)$$
$$*\,{}_4\Psi_2\left[\begin{array}{c|c}(\wp,\ell),(\wp-\vartheta+\mu,\ell),(1,1),(2,1) & \\ & \frac{x^\ell}{[plog(\Lambda)]}\\ (\wp+\tau+\mu,\ell),(\wp-\vartheta,\ell) & \end{array}\right].$$

Corollary 7: The following formula holds

$$S_\Lambda\left\{\left(t\cdot P_{0^+}^{\tau,-\tau,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(t\nu^\ell;\Phi,\Omega)\right)(x)\right\}(p)$$
$$=\frac{x^{\wp+\tau}\Gamma(\tau)}{p^{\hbar}[plog(\Lambda)]^2}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{\nu^\ell}{[plog(\Lambda)]};\Phi,\Omega\right)$$
$$*\,{}_3\Psi_1\left[\begin{array}{c|c}(\wp,\ell),(1,1),(2,1) & \\ & \frac{x^\ell}{[plog(\Lambda)]}\\ (\wp+\tau,\ell) & \end{array}\right].$$

Corollary 8: The following results equation is true

$$S_\Lambda\left\{\left(t\cdot P_{0^+}^{\tau,0,\mu,0;1}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(t\nu^\ell;\Phi,\Omega)\right)(x)\right\}(p)$$
$$=\frac{x^{\wp}\Gamma(\tau)}{p^{\hbar}[plog(\Lambda)]^2}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{\nu^\ell}{[plog(\Lambda)]};\Phi,\Omega\right)$$
$$*\,{}_3\Psi_1\left[\begin{array}{c|c}(\wp,\ell),(1,1),(2,1) & \\ & \frac{x^\ell}{[plog(\Lambda)]}\\ (\wp+\mu,\ell) & \end{array}\right].$$

Corollary 9: The following equility holds

$$S_\Lambda\left\{\left(t\cdot P_{0^+}^{\tau,\vartheta,0,\rho;b}\nu^{\wp-1}\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}(t\nu^\ell;\Phi,\Omega)\right)(x)\right\}(p)$$
$$=\frac{x^{\wp-\vartheta}\Gamma\left(1+\frac{\tau-1}{\rho-1}\right)}{p^{\hbar}[b(1-\rho)]^{\wp}[plog(\Lambda)]^2}$$
$$\times\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^\ell}{[b(1-\rho)]^{\wp}[plog(\Lambda)]};\Phi,\Omega\right)$$
$$*\,{}_3\Psi_1\left[\begin{array}{c|c}(\wp,\ell),(1,1),(2,1) & \\ & \frac{x^\ell}{[b(1-\rho)]^\ell[plog(\Lambda)]}\\ \left(1+\wp\mu+\frac{\tau-1}{1-\rho},\ell\right) & \end{array}\right].$$

Corollary 10: The following result is true

$$S_\Lambda\left\{\left(\left(t\cdot P_{0^+}^{\tau,\vartheta,0,1;b}\nu^{\wp-1}\right.\right.\right.$$
$$\left.\left.\left.\times\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{\nu^\ell}{[b(1-\rho)]^\ell[plog(\Lambda)]};\Phi,\Omega\right)\right)(x)\right\}(p)$$
$$=\frac{x^{\wp-\vartheta}}{[b(1-\rho)]^{\wp}[plog(\Lambda)]^2}$$
$$\times\,{}^{\Psi}E_{(\varrho_i,\sigma_i)_\eta,\ell,\varrho}^{\lambda,\delta,p,q;\omega,\varpi}\left(\frac{z\nu^\ell}{[b(1-\rho)]^{\wp}[plog(\Lambda)]};\Phi,\Omega\right)$$
$$*\,{}_3\Psi_0\left[\begin{array}{c|c}(\wp,\ell),(1,1),(2,1) & \\ & \frac{x^\ell}{[b(1-\rho)]^\ell}\\ - & \end{array}\right].$$

4. Conclusion

The SUM integral transform is applied on generalized pathway fractional integral operator with extended multi-index Mittag-Leffler function in the kernel. Our result is general in nature which in some of it special cases include the recent results obtained by in [15, 16] follows. It is hope that the result obtained here will have potential applications in science, technology and engineering.

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