Erzincan Üniversitesi
Fen Bilimleri Enstitüsü Dergisi
2025, 18(1), 149-170
ISSN: 1307-9085, e-ISSN: 2149-4584
Araştırma Makalesi

Erzincan University
Journal of Science and Technology
2025, 18(1), 149-170
DOI: 10.18185/erzifbed.1543499
Research Article

The Solutions of Caputo-Fabrizio Random Fractional Ordinary Differential Equations by Aboodh Transform Method

Yasin Şahin[1*] Mehmet Merdan[1]

[1]Gumushane University, Mathematical Engineering Department, Gumushane, Turkey

Received: 04/09/2024, **Revised:** 07/11/2024, **Accepted:** 07/11/2024, **Published:** 28/03/2025

Abstract

The primary objective of the paper is to employ the Aboodh transform to solve Caputo-Fabrizio ordinary fractional differential equations. Furthermore, the methodology described is used to solve some common fractional differential equations. Analyzed using the Aboodh Transform Method are random ordinary differential equations derived by randomly picking the coefficients or initial conditions of Caputo-Fabrizio fractional ordinary differential equations. The initial conditions or coefficients of the equations are converted into random variables distributed according to Normal, Uniform, and Exponential distributions. The probability characteristics, including expected value, variance, coefficient of variation, and confidence interval, of the random ordinary differential equations solved are computed using the MATLAB (2013a) package software. The acquired results are then shown and discussed.

Keywords: Caputo-Fabrizio fractional derivative, Aboodh Transform Method, Continuous probability distributions, variance

Aboodh Dönüşüm Yöntemi ile Caputo-Fabrizio Rastgele Kesirli Adi Diferansiyel Denklemlerin Çözümleri

Öz

Makalenin temel amacı, Caputo-Fabrizio adi kesirli diferansiyel denklemlerini çözmek için Aboodh dönüşümünü kullanmaktır. Ayrıca, açıklanan metodoloji bazı yaygın kesirli diferansiyel denklemleri çözmek için de kullanılır. Caputo-Fabrizio kesirli adi diferansiyel denklemlerinin katsayılarını veya başlangıç koşullarını rastgele seçerek türetilen rastgele adi diferansiyel denklemler, Aboodh Dönüşüm Yöntemi kullanılarak analiz edilmektedir. Denklemlerin başlangıç koşulları veya katsayıları, Normal, Düzgün ve Üstel dağılımlara göre seçilerek, denklemler rastgele hale dönüştürülür. Çözülen rastgele adi diferansiyel denklemlerin beklenen değer, varyans, varyasyon katsayısı ve güven aralığı gibi olasılık özellikleri, MATLAB (2013a) paket yazılımı kullanılarak hesaplanır. Daha sonra elde edilen sonuçlar grafiksel olarak gösterilir ve elde edilen sonuçlar yorumlanır.

Anahtar Kelimeler: Caputo-Fabrizio kesirli türev, Aboodh Dönüşüm Metodu, Sürekli olasılık dağılımı, varyans.

*Corresponding Author: mmerdan@gumushane.edu.tr
Yasin ŞAHİN, https://orcid.org/ 0009-0006-4349-7004
Mehmet MERDAN, https://orcid.org/ 0000-0002-8509-3044

1.Introduction

As an expanded version of the standard differential equation, fractional differential equations (FDEs) require the derivatives of a function to have fractional degrees. In order to provide a more comprehensive description of the behavior of physical events and systems, these equations are utilized. Electrical circuits, bioengineering, thermal systems, chemistry, physics, and control theory are just a few of the numerous fields that rely on fractional differential equations. It is possible to use mathematics to model many issues in these domains. These issues are first expressed as differential equations, and then approximate analytical solutions are found using a variety of approaches.

In 2013, Khalid Aboodh introduced the world to the Aboodh transform[1-2]. The solutions of both ordinary and partial differential equations have also been subjected to this transform. The usual Fourier integral equation yields the Aboodh transform. The Aboodh transform's fundamental characteristics stem from elementary mathematical procedures. Both ordinary and partial differential equations are made easier to solve with this transform. Recently, many studies using the Aboodh transform method have been published[3-8] .

Research on RDEs has recently progressed; for example, Asai and Kloeden used RDEs to find numerical solutions to random ordinary RDEs [9]. Mathematical Modeling of Dengue Fever Disease Under Random Effects was solved by Bekiryazıcı et al. [10]. In their study on the Hepatitis C model, Merdan et al. [11] looked at its deterministic stability and its random behavior. In a study conducted by Merdan et al. [12], the researchers compared stochastic and random models used to predict bacterial resistance. In a recent study by Anaç et al. [13], the novel Sumudu transform approach was employed to discover solutions to random temporal fractional partial differential equations. According to Bekiryazıcı et al.[14], compartmental models were subjected to a variant of the Random differential transform approach. Merdan et al. [15], studied the random component time fractional Klein-Gordon equation with the new Sumudu transformation method. Mathematical Modeling of of Zika virus transmission Under Random Effects was solved by Bekiryazıcı et al. [16]. Anaç et al.[17] obtained the solution of some random partial differential equations using the differential transform method and the Laplace-Padé method. A. Alkan[18-21] has recently conducted significant research on the solutions of fractional order with Advection, Fornberg-Whitham, Burgers, and Newell-Whitehead-Segel equations of Caputo-Fabrizio derivatives.

An investigation of the use of the Aboodh transform technique for solving ordinary differential equations of fractional order using the Caputo-Fabrizio fractional derivative is carried out in this paper. In order to make solving fractional order differential equations easier, the Aboodh transform reduces them to a more straightforward algebraic form. This transform will be used to find solutions to several differential equations. Applying the Aboodh transform method to fractional order differential equations with randomized coefficients and initial conditions using random variables, we will study the solutions' probabilistic properties, find their expected value, variance, coefficient of variation, and confidence interval using parameters drawn from various probability distributions, and finally, analyze the graphs' behavior in MATLAB programs.

This investigation investigates the utilization of the Aboodh transform method to resolve fractional order problems. The current document is organized as follows: Section 2 offers a concise overview of numerous pivotal definitions. A description of the research methodology is provided in Section 3. In Section 4, three examples are provided to illustrate the application of the methodology. In Section 5, the findings are presented and subsequently elaborated upon and interpreted. The principal findings are ultimately summarized in Section 6.

2.Preliminaries

Here we present some basic definitions of Caputo-Fabrizio that are important to our research.

Definition 1. Caputo-Fabrizio fractional derivative[22] of order $g(t)$

$$ {}^{CF}_{a}D^{\alpha}_{t} g(t) = \frac{K(\alpha)}{1-\alpha} \int_a^t g'(\tau) e^{\left[\frac{-\alpha(t-\tau)}{1-\alpha}\right]} d\tau \ , \quad 0 < \alpha \leq 1 \tag{1} $$

$g' \epsilon G' (a,b)$, $b > 0$ and $K(\alpha)$ is the normalization constant depending on α where $K(0) = K(1) = 1$

Definition 2. Caputo fractional derivative of $g(t)$, where $\alpha > 0$ is the order of the Caputo fractional derivative[22-24]:

$$ {}^{c}D^{\alpha}_{t} g(t) = \begin{cases} \frac{1}{\Gamma(k-\alpha)} \int_0^t (t-s)^{k-\alpha-1} g^{(p)}(s) ds \, , k-1 < \alpha \leq k \\ \frac{\partial^n}{\partial t^n} g(t) \ , \qquad \alpha = n \epsilon \mathbb{N} \end{cases} \tag{2} $$

It is defined as.

Definition 3. The functions in the set $\mathcal{A}$ described as follows are taken into consideration by a new transform for exponential functions known as the Aboodh transform [1-3].

$$ \mathcal{A} = \{ g(t) : \exists \ M, p_1 , p_2 > 0 \, , | \ g(t) | < M e^{-st} \} \tag{3} $$

Aboodh transform of function $g(t)$

$$ \mathcal{A}\{g(t)\} = \mathcal{A}\{h(s)\} = \frac{1}{s} \int_0^{\infty} g(t) \, e^{-st} \, dt \ , t \geq 0 \, , p_1 \leq s \leq p_2 \tag{4} $$

It is defined as.

Definition 4. Laplace Transform of Caputo-Fabrizio fractional derivative[25-27]:

$$ L\{ {}^{CF}_{0}D^{\alpha+n}_{t} g(t) \}(s) = \frac{s^{n+1} \, L\{g(t)\} - g(0) s^n - g\prime(0) s^{n-1} - \ldots - g^{(n)}(0)}{\alpha + s(1-\alpha)} \ , 0 < \alpha \leq 1 \tag{5} $$

for $n = 0$

$$ L\{ {}^{CF}_{0}D^{\alpha}_{t} g(t) \}(s) = \frac{s \, L\{g(t)\} - h(0)}{\alpha + s(1-\alpha)} \tag{6} $$

Theorem 1. Aboodh transform of the Caputo-Fabrizio fractional derivative[28-29] of order α, where the Aboodh transform of the function $g(t)$ is $\mathcal{A}\{h(s)\}$;

$$\mathcal{A}\{{}^{CF}_{\ 0}D_t^{\alpha} g(t)\}(s) = \frac{s\,\mathcal{A}\{h(s)\} - \frac{h(0)}{s}}{\alpha + s(1-\alpha)} \tag{7}$$

Table 1. Aboodh Transforms of Some Functions

$f(t)$	$\mathcal{A}(f(t))$
1	$\frac{1}{s^2}$
$\frac{t^n}{n!}$	$\frac{1}{s^{n+2}}\quad (n \geq 0)$
e^{-at}	$\frac{1}{s(s+a)}$
$\frac{-1}{a}.e^{-at} + \frac{1}{a}$	$\frac{1}{s^2(s+a)}$
$\frac{1}{a^2}.e^{-at} + \frac{1}{a}t - \frac{1}{a}$	$\frac{1}{s^3(s+a)}$

3. Aboodh Transform Method

When the fractional ordinary differential equation is given as

$$D_t^{\alpha} y(t) = g(t) + Ry(t) \tag{8}$$

and $y(0) = c$ initial condition, $D_t^{\alpha} \equiv {}^{CF}D_t^{\alpha}$ a Caputo-Fabrizio fractional derivative operator $0 < \alpha \leq 1$, R a linear operator, g a function that shows the homogeneity of the differential equation and $y(t)$ Let be the solution of the equation. If the Aboodh transform[1-3] is applied to both sides of the equation (8);

$$\mathcal{A}\{{}^{CF}D_t^{\alpha} y(t)\} = \mathcal{A}\{g(t)\} + \mathcal{A}\{Ry(t)\}$$

$$\frac{\mathrm{s}\,\mathcal{A}(\mathrm{y(s)})) - \frac{y(0)}{s}}{\alpha + s(1-\alpha)} = \mathcal{A}\{g(t)\} + \mathcal{A}\{Ry(t)\}$$

$$\mathcal{A}(\mathrm{y(s)})) = \frac{y(0)}{s^2} + \left(\frac{\alpha}{s} + (1-\alpha)\right)\mathcal{A}\{g(s)\} + \left(\frac{\alpha}{s} + (1-\alpha)\right)\mathcal{A}\{Ry(s)\} \tag{9}$$

is obtained. Then, if the inverse Aboodh transform of expression (9) is taken,

$$\mathcal{A}^{-1}\left[\mathcal{A}\big(y(s)\big)\right] = \mathcal{A}^{-1}\left[\frac{y(0)}{s^2} + \left(\frac{\alpha}{s} + (1-\alpha)\right)\mathcal{A}\{g(s)\} + \left(\frac{\alpha}{s} + (1-\alpha)\right)\mathcal{A}\{Ry(s)\}\right]$$

is obtained.

4.Application

This Chapter presents compelling and insightful examples, accompanied by graphical findings, to illustrate the efficacy and straightforwardness of the approach we introduced in Chapter 3.

Example1.

$$ {}^{CF}D_t^{\alpha}\, y(t) = 2 + 5y(t)\,,\ 0 < \alpha \leq 1\,, y(0) = c,\ c \sim U(a = 1, b = 3) \quad (10)$$

Application of the Aboodh transform to solve the Caputo-Fabrizio fractional random ordinary differential equation, where c is a random variable with a uniform distribution.

Solution: (10) If the Aboodh transform of both sides of the equation is taken;

$$\mathcal{A}[{}^{CF}D_t^{\alpha}\, y(t)] = \mathcal{A}[2] + 5\mathcal{A}[y(t)]$$

$$\frac{\mathrm{s}\,\mathcal{A}(\mathrm{y(s)})) - \frac{y(0)}{s}}{\alpha + s(1-\alpha)} = \frac{2}{s^2} + 5\mathcal{A}(y(s))$$

An equation is derived.

$$s\,\mathcal{A}\big(y(s)\big) - \frac{y(0)}{s} = \frac{2(\alpha + s(1-\alpha))}{s^2} + 5(\alpha + s(1-\alpha))\mathcal{A}(y(s))$$

$$\mathcal{A}\big(y(s)\big)[s - 5(\alpha + s(1-\alpha))] = \frac{c}{s} + \frac{2(\alpha + s(1-\alpha))}{s^2}$$

$$\mathcal{A}\big(y(s)\big) = \frac{c}{s((5\alpha-4)s - 5\alpha)} + \frac{2(\alpha + s(1-\alpha))}{s^2((5\alpha-4)s - 5\alpha)}$$

$$\mathcal{A}\big(y(s)\big) = \frac{c}{(5\alpha-4)s\left(s - \frac{5\alpha}{5\alpha-4}\right)} + \frac{2\big(\alpha + s(1-\alpha)\big)}{s^2\big((5\alpha-4)s - 5\alpha\big)}$$

$$\mathcal{A}\big(y(s)\big) = \frac{c}{(5\alpha-4)s\left(s - \frac{5\alpha}{5\alpha-4}\right)} + \frac{2\alpha}{(5\alpha-4)s^2(s - \frac{5\alpha}{5\alpha-4})} + \frac{2s(1-\alpha)}{(5\alpha-4)s^2(s - \frac{5\alpha}{5\alpha-4})}$$

$$\mathcal{A}\big(y(s)\big) = \frac{c}{(5\alpha-4)s\left(s - \frac{5\alpha}{5\alpha-4}\right)} + \frac{2\alpha}{(5\alpha-4)s^2(s - \frac{5\alpha}{5\alpha-4})} + \frac{2(1-\alpha)}{(5\alpha-4)s(s - \frac{5\alpha}{5\alpha-4})} \quad (11)$$

(11) If the inverse Aboodh transform is taken on both sides of the equation;

$$\mathcal{A}^{-1}[\mathcal{A}(y(s))]$$

$$= \mathcal{A}^{-1}\left[\frac{c}{(5\alpha-4)s\left(s-\frac{5\alpha}{5\alpha-4}\right)}\right] + \mathcal{A}^{-1}\left[\frac{2\alpha}{(5\alpha-4)s^2\left(s-\frac{5\alpha}{5\alpha-4}\right)}\right]$$

$$+ \mathcal{A}^{-1}\left[\frac{2(1-\alpha)}{(5\alpha-4)s\left(s-\frac{5\alpha}{5\alpha-4}\right)}\right]$$

$$y(t) = \frac{c}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2\alpha}{5\alpha-4}\left[\frac{-1}{\frac{-5\alpha}{5\alpha-4}}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{1}{\frac{-5\alpha}{5\alpha-4}}\right] + \frac{2(1-\alpha)}{(5\alpha-4)}e^{\frac{5\alpha}{5\alpha-4}t}$$

$$y(t) = \frac{c}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5} \qquad (12)$$

the general solution is obtained.

X If the random variable has a uniform distribution, using the probability density function,

$$E(X^n) = \int_a^b x^n.\frac{1}{b-a}dx$$

from the expression $c \sim U(a,b)$ the expected value of the random variable;

$$E[c] = \frac{a+b}{2}$$

$Var(X) = E(X^2) - [E(X)]^2$ The variance from the equality is[30-31];

$$Var[c] = \frac{(b-a)^2}{12}$$

It is calculated as.

Based on these moments, one can compute the anticipated values of the independent random variables X and Y by using the formula $E[XY] = E[X]E[Y]$.

$$E[y(t)] = E\left[\frac{c}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}\right]$$

$$E[y(t)] = \frac{e^{\frac{5\alpha}{5\alpha-4}t}}{5\alpha-4}E[c] + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}$$

$$E[y(t)] = \frac{e^{\frac{5\alpha}{5\alpha-4}t}}{5\alpha-4}.\frac{a+b}{2} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}$$

The expected value for special values $a = 1, b = 3$ is

$$E[y(t)] = \frac{e^{\frac{5\alpha}{5\alpha-4}t}}{5\alpha-4}.\frac{1+3}{2} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}$$

$$E[y(t)] = \frac{2}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}$$

$$E[y(t)] = \frac{4-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}$$

can be calculated as the above expressions.

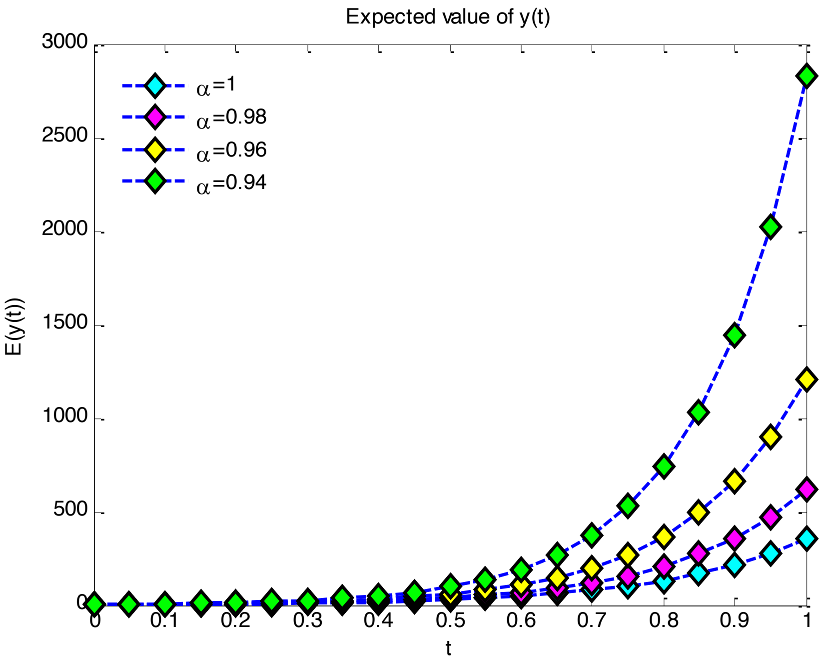

Figure 1. solution behaviour of $E(y(t))$ for $\alpha \in \{1,0.98,0.96,0.94\}$

When we look at Figure 1, it is observed that when the fractional derivative value alpha decreases from 1 to 0.94, the expected value increases.

$$Var[y(t)] = Var\left[\frac{c}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2-2\alpha}{5\alpha-4}e^{\frac{5\alpha}{5\alpha-4}t} + \frac{2}{5}e^{\frac{5\alpha}{5\alpha-4}t} - \frac{2}{5}\right]$$

$$Var[y(t)] = \frac{e^{\frac{10\alpha}{5\alpha-4}t}}{(5\alpha-4)^2}Var[c]$$

$$Var[y(t)] = \frac{e^{\frac{10\alpha}{5\alpha-4}t}}{(5\alpha-4)^2}.\frac{(b-a)^2}{12}$$

Variance for special values $a = 1, b = 3$,

$$Var[y(t)] = \frac{e^{\frac{10\alpha}{5\alpha-4}t}}{(5\alpha-4)^2}.\frac{(3-1)^2}{12}$$

$$Var[y(t)] = \frac{1}{3(5\alpha - 4)^2} e^{\frac{10\alpha}{5\alpha-4}t}$$

It is calculated as.

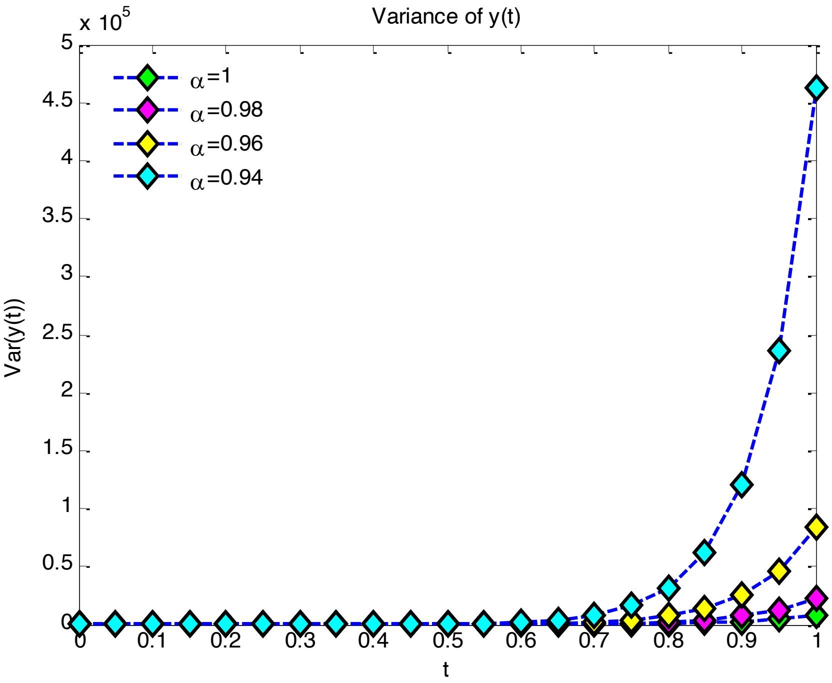

Figure 2. solution behaviour of $Var(y(t))$ for $\alpha \in \{1,0.98,0.96,0.94\}$

When we look at Figure 2, it is observed that when the fractional derivative value alpha decreases from 1 to 0.94, the variance value increases.

Confidence interval

$$[E(y(t)) - 3stddeviation(y(t)), E(y(t)) + 3stddeviation(y(t))]$$

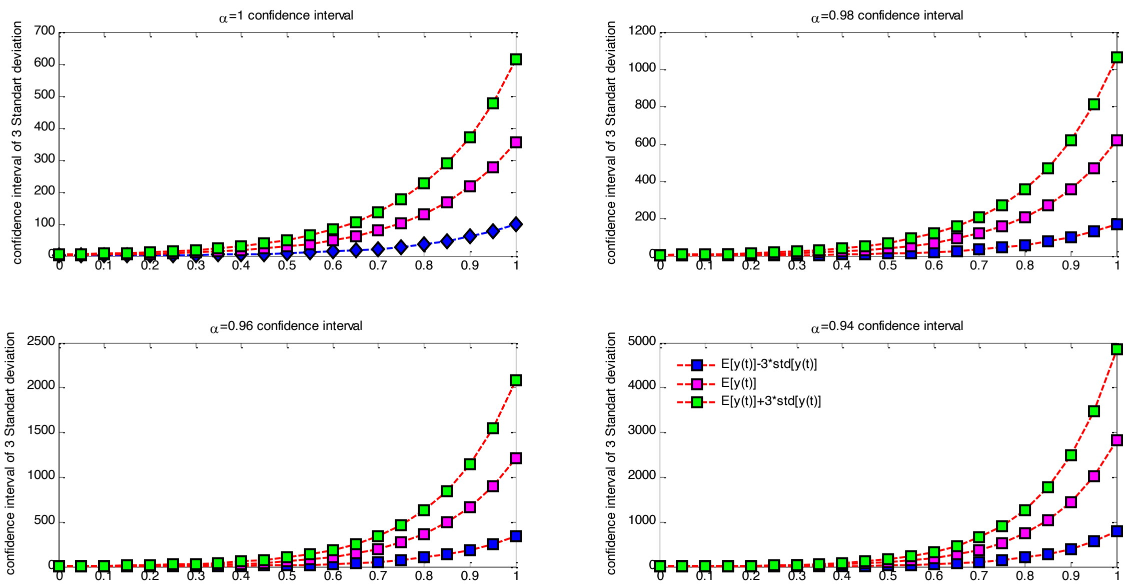

Figure 3. solution behaviour of $stddeviation(y(t))$ for $\alpha \in \{1,0.98,0.96,0.94\}$

When we look at Figure 3, confidence intervals with 3 standard deviations are drawn for the values of α∈{1,0.98,0.96,0.94}. It is seen that the confidence limits increase as alpha takes decreasing values from 1 to 0.94.

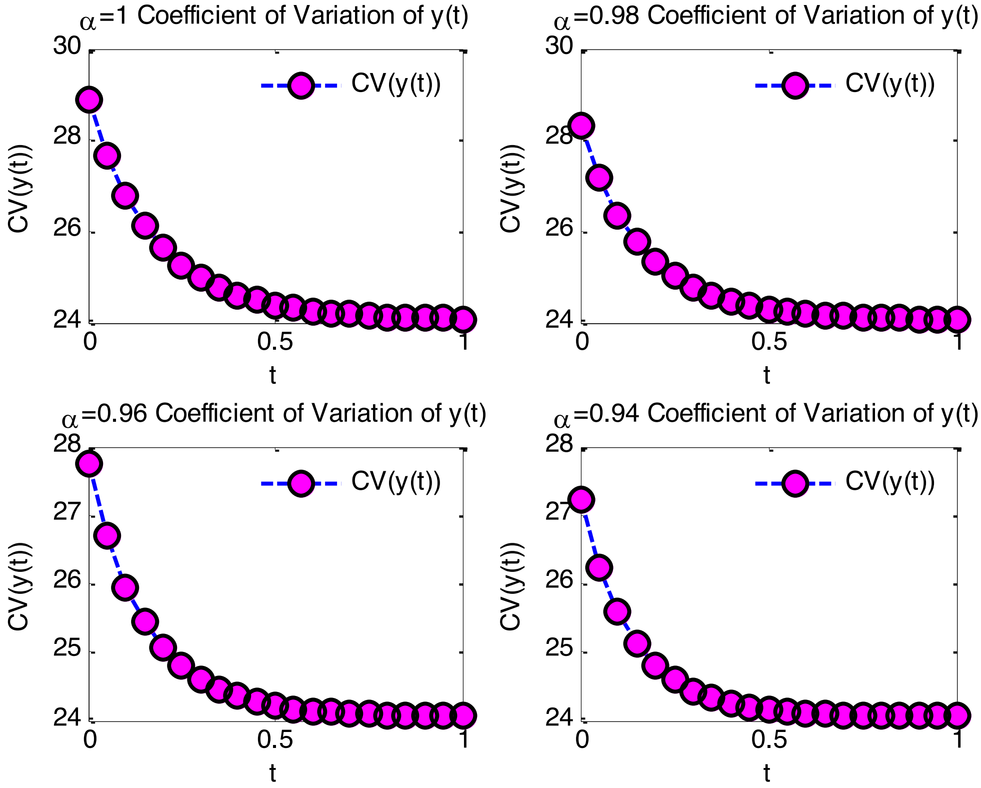

Figure 4. solution behaviour of $CV(y(t))$ coefficient variation for $\alpha \in \{1,0.98,0.96,0.94\}$

Table 2. Table for the 99% confidence interval, variance, coefficient of variation, and expectation value

t	$E[y(t)]$	$Var[y(t)]$	$CV[y(t)]$	$E[y(t)] - 3Std[y(t)]$	$E[y(t)] + 3Std[y(t)]$
0	2.126315789	0.3693444137	28.58169650	0.3031044130	3.949527165
0.1	3.853808345	1.047157723	26.55314316	0.7838866040	6.923730086
0.2	6.762558821	2.968880145	25.47917002	1.593427242	11.93169040
0.3	11.66031036	8.417308226	24.88149694	2.956531067	20.36408965
0.4	19.90714019	23.86458002	24.53963072	5.251724120	34.56255626
0.5	33.79314514	67.66036887	24.34100844	9.116368220	58.46992206
0.6	57.17438825	191.8292931	24.22456217	15.62365256	98.72512394
0.7	96.54370522	543.8704861	24.15593117	26.58061229	166.5067982
0.8	162.8337273	1541.970471	24.11535527	45.02993180	280.6375228

0.9	274.4528093	4371.763121	24.09132189	76.09488020	472.8107384
1	462.3969229	12394.73333	24.07707127	128.4020129	796.3918329

Table 2 provides the expected value, variance, coefficient of variation, and confidence interval for $K = 3$ and $\alpha = 0.99$ values. As can be seen, the results for $t \in [0,1]$ values are obtained for the parameter selected from the uniform distribution.

Example 2.

$$ {}^{CF}D_t^{\alpha}\, y(t) = 1 - y(t),\ 0 < \alpha \leq 1\,,\ y(0) = B \quad ,B \sim N(\mu = 5, \sigma^2 = 1) \quad (13)$$

Determine the solution to the Caputo-Fabrizio fractional random ordinary differential equation using the Aboodh transform, where B is a random variable following a normal distribution.

Solution: (13) If the Aboodh transform of both sides of the equation is taken;

$$\mathcal{A}\{{}^{CF}D_t^{\alpha} y(t)\} = \mathcal{A}(1) - \mathcal{A}\{y(t)\}$$

$$\frac{s\,\mathcal{A}(y(s))) - \frac{y(0)}{s}}{\alpha + s(1-\alpha)} = \frac{1}{s^2} - \mathcal{A}(y(s))$$

$$s\,\mathcal{A}(y(s)) - \frac{y(0)}{s} = \frac{\alpha + s(1-\alpha)}{s^2} - (\alpha + s(1-\alpha))\mathcal{A}(y(s))$$

$$\mathcal{A}(y(s))[s + \alpha + s(1-\alpha)] = \frac{B}{s} + \frac{\alpha + s(1-\alpha)}{s^2}$$

$$\mathcal{A}(y(s)) = \frac{B}{s((2-\alpha)s + \alpha)} + \frac{\alpha + s(1-\alpha)}{s^2((2-\alpha)s + \alpha)}$$

$$\mathcal{A}(y(s)) = \frac{B}{(2-\alpha)(s^2 + \frac{\alpha}{2-\alpha}s)} + \frac{\alpha}{2-\alpha}\cdot\frac{1}{s^2\left(s + \frac{\alpha}{2-\alpha}\right)} + \frac{1-\alpha}{2-\alpha}\cdot\frac{1}{s(s + \frac{\alpha}{2-\alpha})} \quad (14)$$

(14) If the inverse Aboodh transform is taken on both sides of the equation;

$$\mathcal{A}^{-1}\left[\mathcal{A}\big(y(s)\big)\right] = \mathcal{A}^{-1}\left[\frac{B}{(2-\alpha)s\left(s+\frac{\alpha}{2-\alpha}\right)}\right] + \mathcal{A}^{-1}\left[\frac{\alpha}{2-\alpha}.\frac{1}{s^2\left(s+\frac{\alpha}{2-\alpha}\right)}\right] + \mathcal{A}^{-1}\left[\frac{1-\alpha}{2-\alpha}.\frac{1}{s(s+\frac{\alpha}{2-\alpha})}\right]$$

$$y(t) = \frac{B}{2-\alpha}.e^{\frac{-\alpha}{2-\alpha}t} + \frac{\alpha}{2-\alpha}.\left[\left(\frac{-1}{\frac{\alpha}{2-\alpha}}\right).e^{\frac{-\alpha}{2-\alpha}t} + \frac{1}{\frac{\alpha}{2-\alpha}}\right] + \frac{1-\alpha}{2-\alpha}.e^{\frac{-\alpha}{2-\alpha}t}$$

$$y(t) = \left(\frac{B+1-\alpha}{2-\alpha}\right).e^{\frac{\alpha}{\alpha-2}t} - e^{\frac{\alpha}{\alpha-2}t} + 1 \tag{15}$$

the general solution is obtained.

for $\alpha = 1$;

$y(t) = Be^{-t} - e^{-t} + 1$ is obtained.

Let X be a random variable with a normal distribution[30-31]. By using the moment generating function of the normal distribution $M_x(t) = E[e^{tX}] = e^{\left(\mu t + \frac{1}{2}\sigma^2 t^2\right)}$, the first and second moments of the random variable $X \sim N(\mu, \sigma^2)$ are;

$$M_x'(0) = E[X] = \mu, \quad E[X^2] = \mu^2 + \sigma^2, \quad Var[X] = \sigma^2$$

$$E[y(t)] = E\left[\frac{B}{2-\alpha}.e^{\frac{\alpha}{\alpha-2}t} + \left(\frac{1-\alpha}{2-\alpha} - 1\right).e^{\frac{\alpha}{\alpha-2}t} + 1\right]$$

$$E[y(t)] = E\left[\frac{B}{2-\alpha}.e^{\frac{\alpha}{\alpha-2}t}\right] + E\left[\left(\frac{-1}{2-\alpha}\right).e^{\frac{\alpha}{\alpha-2}t}\right] + E[1]$$

$$E[y(t)] = \frac{1}{2-\alpha}\, e^{\frac{\alpha}{\alpha-2}t}\, E[B] + \frac{1}{\alpha-2}\,.e^{\frac{\alpha}{\alpha-2}t} + 1$$

$$E[y(t)] = \frac{1}{2-\alpha}\, e^{\frac{\alpha}{\alpha-2}t}\, \mu + \frac{1}{\alpha-2}\,.e^{\frac{\alpha}{\alpha-2}t} + 1$$

For the special value $\mu = 5$, it is calculated as

$$E[y(t)] = \frac{5}{2-\alpha}\, e^{\frac{\alpha}{\alpha-2}t} + \frac{1}{\alpha-2}\,.e^{\frac{\alpha}{\alpha-2}t} + 1.$$

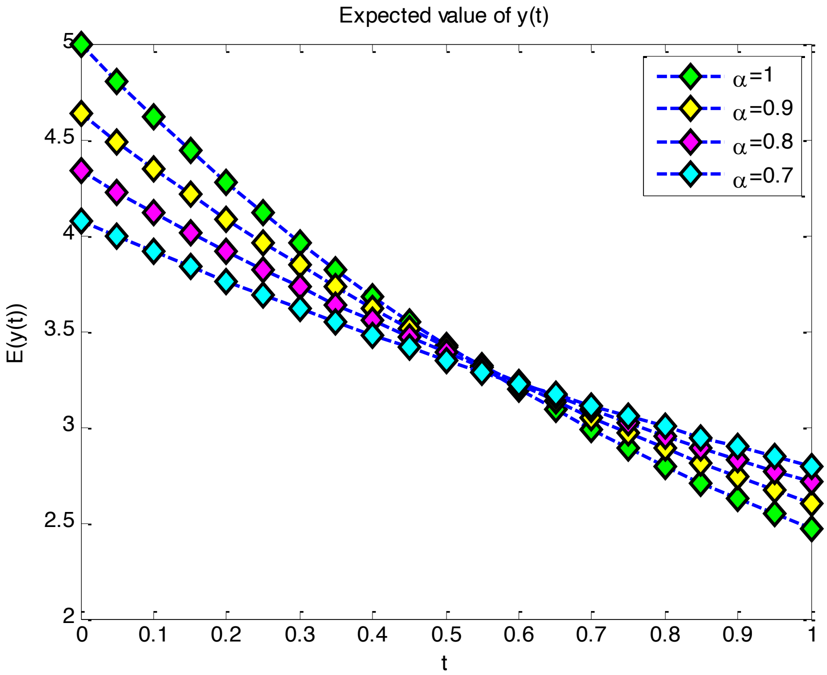

Figure 5. solution behaviour of $E(y(t))$ for $\alpha \in \{1,0.9,0.8,0.7\}$

When we look at Figure 5 carefully, we observe that the expected value solutions decrease when the alpha fractional derivative decreases from 1 to 0.7 in the range $t \in [0,0.5]$, but increase in the range $t \in [0.5\ 1]$.

$$Var[y(t)] = Var\left[\left(\frac{B+1-\alpha}{2-\alpha}\right).e^{\frac{\alpha}{\alpha-2}t} - e^{\frac{\alpha}{\alpha-2}t} + 1\right]$$

$$Var[y(t)] = Var\left[\frac{B}{2-\alpha}.e^{\frac{\alpha}{\alpha-2}t}\right] + Var\left[\left(\frac{1-\alpha}{2-\alpha} - 1\right).e^{\frac{\alpha}{\alpha-2}t} + 1\right]$$

$$Var[y(t)] = \frac{1}{(2-\alpha)^2}.e^{\frac{2\alpha}{\alpha-2}t}.Var[B]$$

$$Var[y(t)] = \frac{1}{(2-\alpha)^2}.e^{\frac{2\alpha}{\alpha-2}t}.\sigma^2$$

and for the special value $\sigma^2 = 1$, it is calculated as

$$Var[y(t)] = \frac{1}{(2-\alpha)^2}\ e^{\frac{2\alpha}{\alpha-2}t}$$

It is calculated as

$$Var[y(t)] = \frac{1}{(2-\alpha)^2}\ e^{\frac{2\alpha}{\alpha-2}t}.$$

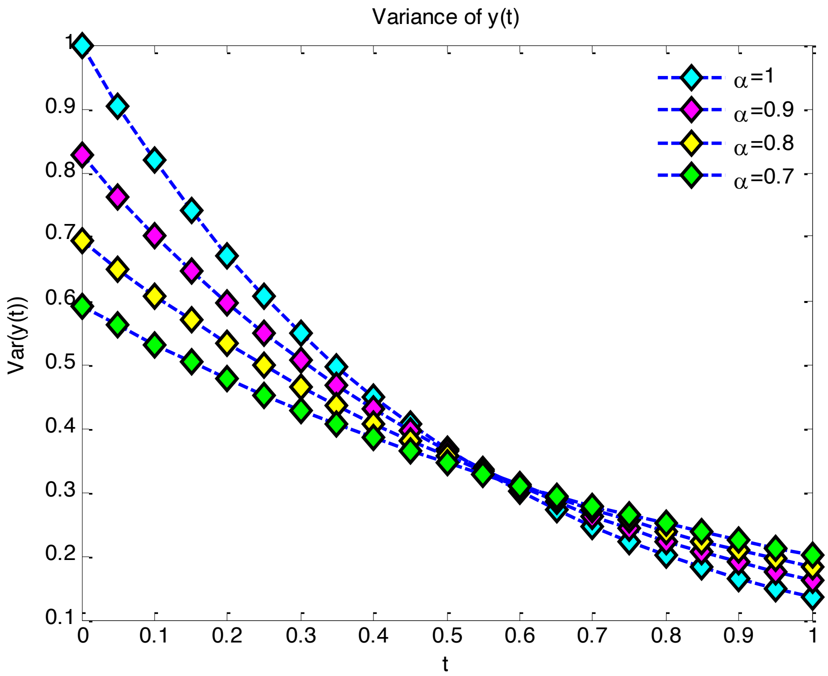

Figure 6. solution behaviour of $Var(y(t))$ for $\alpha \in \{1,0.9,0.8,0.7\}$

When we look at Figure 6 carefully, it is observed that the variance solutions decrease when the alpha fractional derivative decreases from 1 to 0.7 in the range of $t \in [0,0.5]$, but increases in the range of $t \in [0.5\ 1]$.

Confidence interval

$$[E(y(t)) - 3stddeviation(y(t)), E(y(t)) + 3stddeviation(y(t))]$$

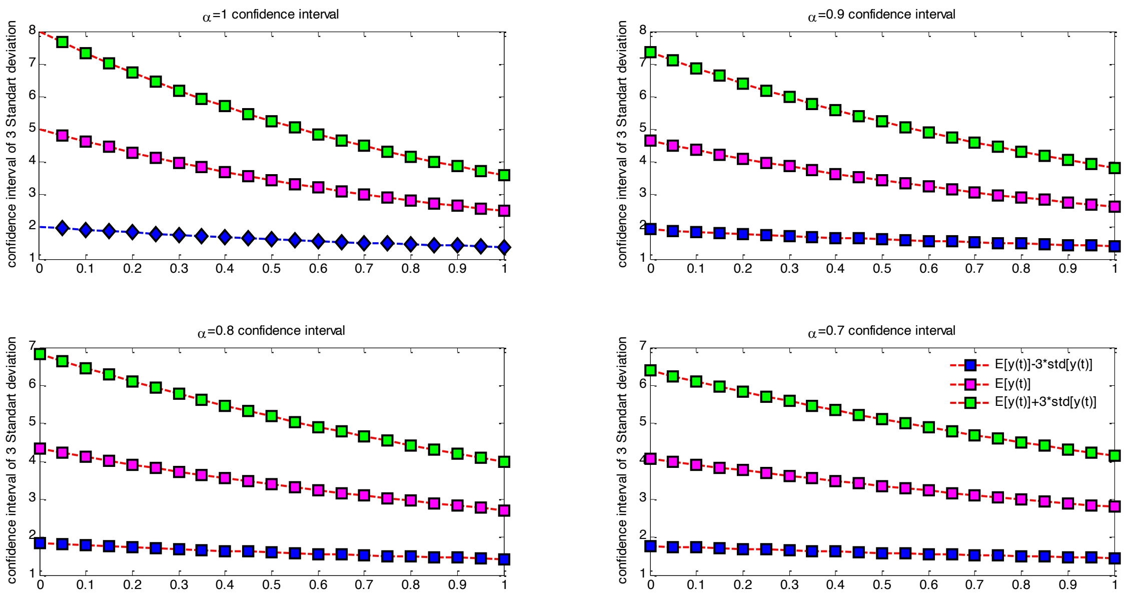

Figure 7. solution behaviour of $stddeviation(y(t))$ for $\alpha \in \{1,0.9,0.8,0.7\}$

When we look at Figure 7, confidence intervals with 3 standard deviations are drawn for the values of α∈{1,0.9,0.8,0.7}. It is seen that the confidence limits decrease as alpha takes decreasing values from 1 to 0.7.

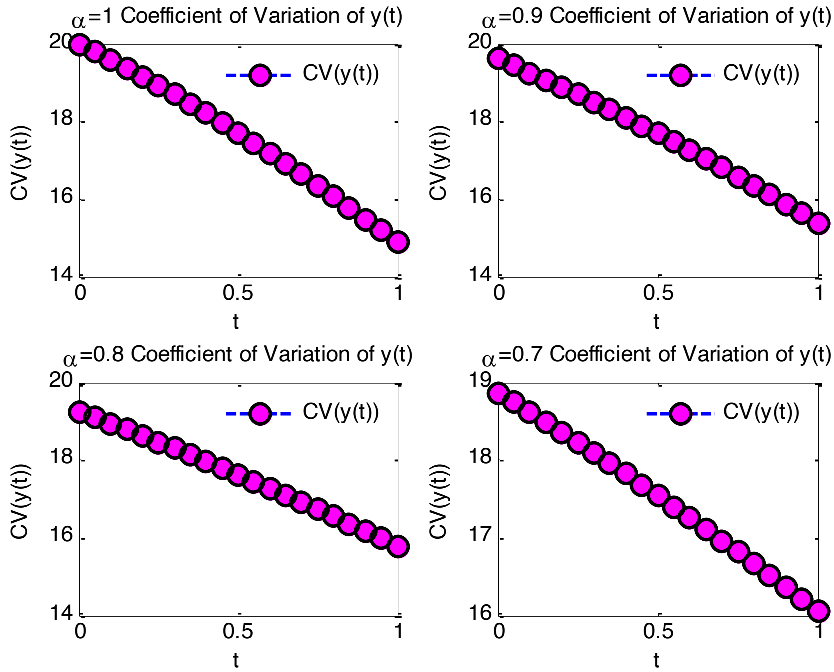

Figure 8. solution behaviour of $CV(y(t))$ coefficient variation for $\alpha \in \{1,0.9,0.8,0.7\}$

Table 3. Statistical table displaying the expected value, variance, coefficient of variation, and 99% confidence range.

t	$E[y(t)]$	$Var[y(t)]$	$CV[y(t)]$	$E[y(t)] - 3Std[y(t)]$	$E[y(t)] + 3Std[y(t)]$
0	4.921568627	0.9611687812	19.92031872	1.980392157	7.862745097
0.1	4.562324575	0.7931347739	19.52033726	1.890581144	7.234068006
0.2	4.235989877	0.6544769055	19.09819176	1.808997469	6.662982285
0.3	3.939549798	0.5400595635	18.65409722	1.734887450	6.144212146
0.4	3.670265774	0.4456449566	18.18850417	1.667566444	5.672965104
0.5	3.425650115	0.3677361550	17.70211517	1.606412529	5.244887701
0.6	3.203443019	0.3034475712	17.19589678	1.550860755	4.856025283
0.7	3.001591700	0.2503980835	16.67108572	1.500397925	4.502785475
0.8	2.818231423	0.2066228441	16.12918839	1.454557857	4.181904989

0.9	2.651668273	0.1705005053	15.57197303	1.412917068	3.890419478
1	2.500363512	0.1406931669	15.00145384	1.375090878	3.625636146

Table 3 provides the expected value, variance, coefficient of variation, and confidence interval for $K = 3$ and $\alpha = 0.98$ values. As can be seen, the results for $t \in [0,1]$ values are obtained for the parameter selected from the normal distribution

Example 3.

$$^{CF}D_t^{\alpha}\, y(t) = ny(t),\ 0 < \alpha \leq 1\,,\ y(0) = m \quad ,m\sim Exp(\lambda = 2) \qquad (16)$$

Solve the Caputo-Fabrizio fractional random differential equation with the help of the Aboodh transform, where m is a random variable with exponential distribution.

Solution: (16) If the Aboodh transform of both sides of the equation is taken;

$$\mathcal{A}\{^{CF}D_t^{\alpha} y(t)\} = \mathcal{A}\{ny(t)\}$$

$$\frac{\mathrm{s}\,\mathcal{A}(\mathrm{y(s)})) - \frac{y(0)}{s}}{\alpha + s(1-\alpha)} = n\mathcal{A}(y(s))$$

$$s\,\mathcal{A}\big(y(s)\big) - \frac{m}{s} = n(\alpha + s(1-\alpha))\mathcal{A}(y(s))$$

$$\mathcal{A}\big(y(s)\big)[s - n(\alpha + s(1-\alpha))] = \frac{m}{s}$$

$$\mathcal{A}\big(y(s)\big) = \frac{m}{s((1-n+n\alpha)s - n\alpha)} = \frac{m}{(1-n+n\alpha)s(s - \frac{n\alpha}{1-n+n\alpha})} \qquad (17)$$

(17) If the inverse Aboodh transform is taken on both sides of the equation;

$$\mathcal{A}^{-1}\big[\mathcal{A}\big(y(s)\big)\big] = \mathcal{A}^{-1}\left[\frac{m}{(1-n+n\alpha)s(s-\frac{n\alpha}{1-n+n\alpha})}\right]$$

$$y(t) = \frac{m}{1-n+n\alpha} e^{\frac{n\alpha}{1-n+n\alpha}t} \qquad (18)$$

the general solution is obtained.

For α=1;

$y(t) = me^t$ is obtained.

If the random variable X has an exponential distribution, using the moment generating function[30-31],

$$M_x(t) = E[e^{tX}] = \frac{\lambda}{\lambda - t}; t < \lambda$$

$$E[X] = \frac{1}{\lambda}, Var[X] = \frac{1}{\lambda^2}$$

$$E[y(t)] = E\left[\frac{m}{1-n+n\alpha} e^{\frac{n\alpha}{1-n+n\alpha}t}\right] = \frac{e^{\frac{n\alpha}{1-n+n\alpha}t}}{1-n+n\alpha} E[m] = \frac{e^{\frac{n\alpha}{1-n+n\alpha}t}}{1-n+n\alpha}\frac{1}{\lambda}$$

Expected value for special value $\lambda = 2$

$$E[y(t)] = \frac{e^{\frac{n\alpha}{1-n+n\alpha}t}}{2(1-n+n\alpha)}$$

It is calculated as.

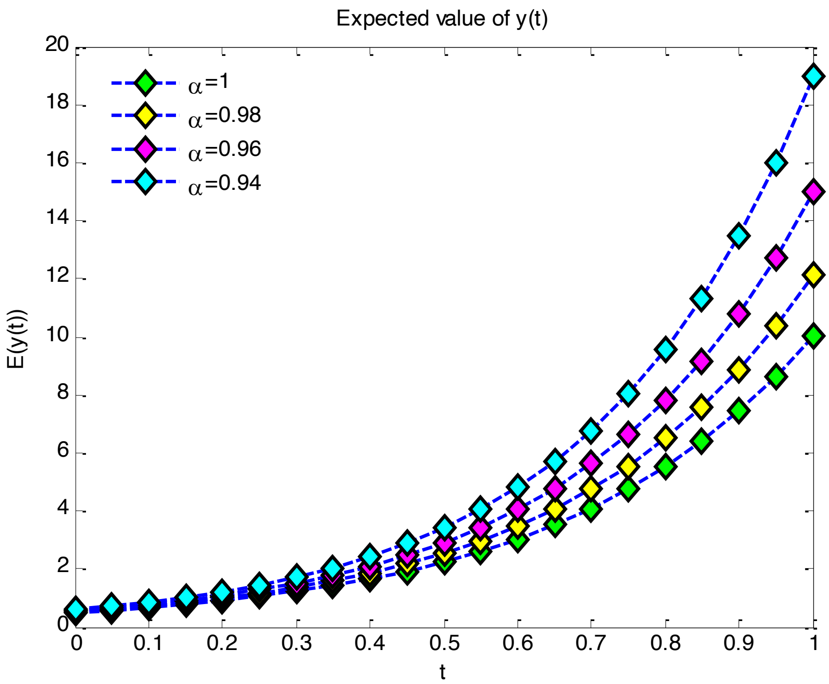

Figure 9. solution behaviour of $E(y(t))$ for $\alpha \in \{1, 0.98, 0.96, 0.94\}$ and $n = 3$

$$Var[y(t)] = Var\left[\frac{m}{1-n+n\alpha} e^{\frac{n\alpha}{1-n+n\alpha}t}\right] = \left[\frac{e^{\frac{2n\alpha}{1-n+n\alpha}t}}{(1-n+n\alpha)^2}\right] Var[m] = \frac{e^{\frac{2n\alpha}{1-n+n\alpha}t}}{(1-n+n\alpha)^2}\frac{1}{\lambda^2}$$

$$Var[y(t)] = \frac{e^{\frac{2n\alpha}{1-n+n\alpha}t}}{4(1-n+n\alpha)^2}$$

It is calculated as.

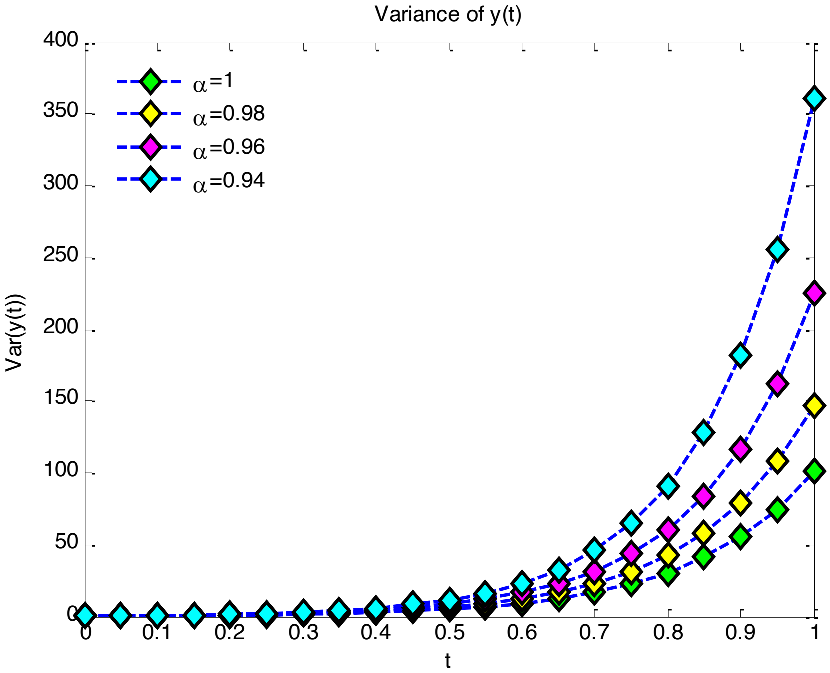

Figure 10. solution behaviour of $Var(y(t))$ for $\alpha \in \{1,0.98,0.96,0.94\}$ and $n = 3$

Confidence interval

$$[E(y(t)) - 3stddeviation(y(t)), E(y(t)) + 3stddeviation(y(t))]$$

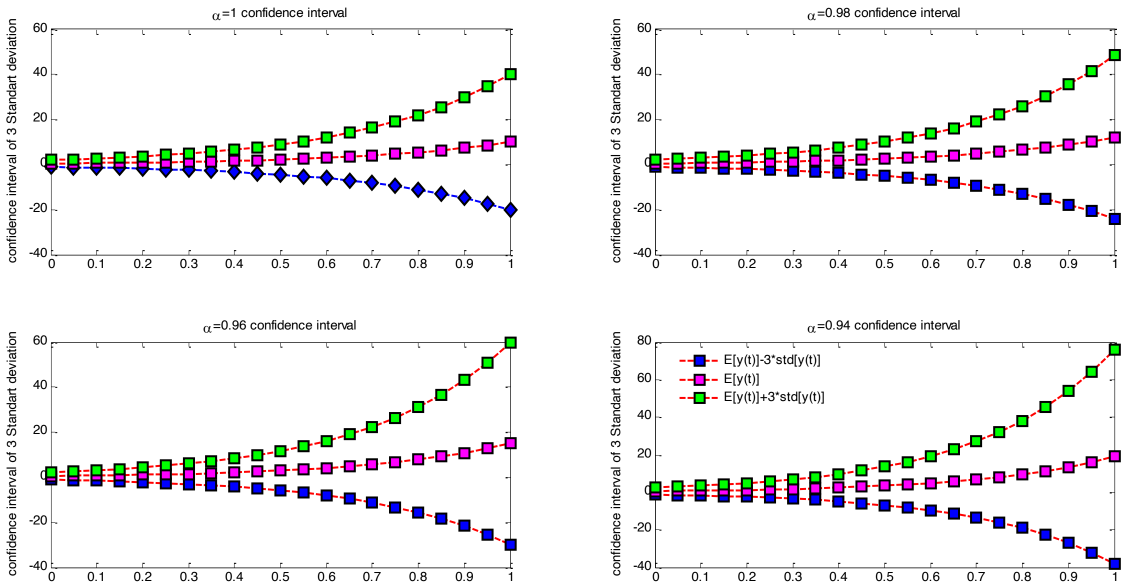

Figure 11. solution behaviour of $stddeviation(y(t))$ for $\alpha \in \{1,0.98,0.96,0.94\}$ and $n = 3$

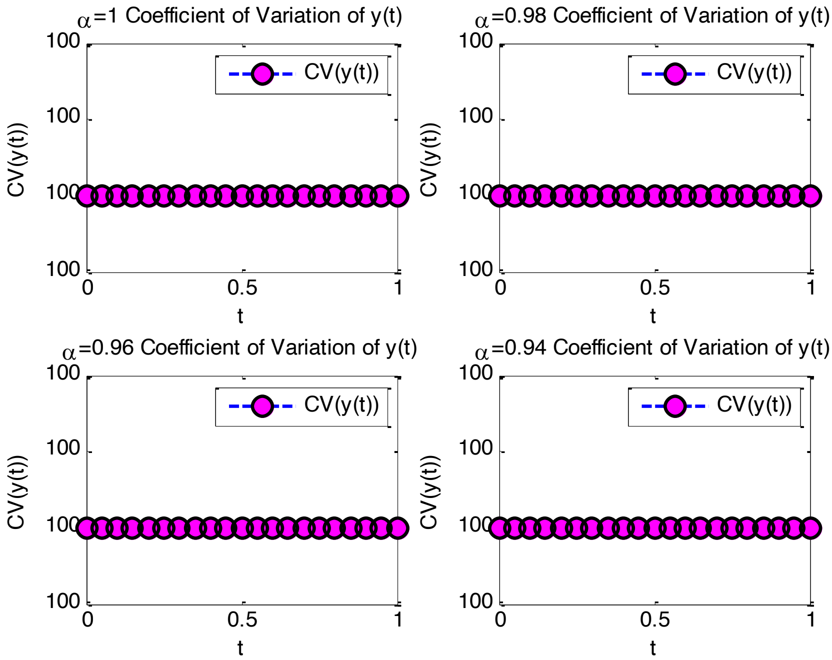

Figure 12. solution behaviour of $CV(y(t))$ coefficient variation for $\alpha \in \{1,0.98,0.96,0.94\}$ and $n = 3$

Table 4. The table shows the expected value, variance, coefficient of variation, and 99% confidence interval.

t	$E[y(t)]$	$Var[y(t)]$	$CV[y(t)]$	$E[y(t)] - 3Std[y(t)]$	$E[y(t)] + 3Std[y(t)]$
0	0.5494505494	0.3018959062	100.	-1.098901099	2.197802197
0.1	0.7564973252	0.5722882032	100.	-1.512994650	3.025989300
0.2	1.041564530	1.084856669	99.99999994	-2.083129058	4.166258118
0.3	1.434052221	2.056505774	100.	-2.868104444	5.736208886
0.4	1.974439139	3.898409913	100.	-3.948878277	7.897756555
0.5	2.718457429	7.390010794	99.99999999	-5.436914858	10.87382972
0.6	3.742840510	14.00885508	100.	-7.485681020	14.97136204
0.7	5.153236881	26.55585032	99.99999996	-10.30647375	20.61294751
0.8	7.095106043	50.34052981	100.0000001	-14.19021210	28.38042418
0.9	9.768720318	95.42789666	99.99999999	-19.53744063	39.07488127
1	13.44981965	180.8976485	100.	-26.89963929	53.79927859

Table 4 provides the expected value, variance, coefficient of variation, and confidence interval for $K = 3$ and $\alpha = 0.97$ values. Evidently, the t∈[0,1] values are derived for the parameter chosen from the exponential distribution.

5. Results and Discussions

This work presents Tables II-III-IV displaying the probabilistic properties of fractional order random equations in the simulation results produced using the prescribed approach for three distinct situations. The 2D plots of the predicted value of the approximation precise solution of Example 1 for values $\alpha \in \{1,0.98,0.96,0.94\}$ are depicted schematically in Figure 1. Figure 2 analyzes the two-dimensional characteristics of the variance of the approximation solution for various values of $\alpha \in \{1,0.98,0.96,0.94\}$. Figure 3 analyzes the two-dimensional characteristics of the confidence intervals for the approximation solution for various values of $\alpha \in \{1,0.98,0.96,0.94\}$. Figure 4 analyzes the two-dimensional characteristics of the coefficient of variation of the approximation solution for various values of $\alpha \in \{1,0.98,0.96,0.94\}$. Figure 5 displays the characteristics of the two-dimensional plots representing the anticipated value of the approximation precise solution to Example 2 for values of $\alpha \in \{1,0.9,0.8,0.7\}$. Figure 6 explores the two-dimensional structure of the variance of the approximation solution for various values of $\alpha \in \{1,0.9,0.8,0.7\}$. For various values of $\alpha \in \{1,0.9,0.8,0.7\}$, Figure 7 analyzes the two-dimensionality of the confidence intervals of the approximate solution. For various values of $\alpha \in \{1,0.9,0.8,0.7\}$, Figure 8 analyzes the two-dimensional character of the coefficient of variation of the approximation solution. Figure 9 displays the characteristics of the two-dimensional plots representing the anticipated value of the approximation precise solution to Example 3 for values of $\alpha \in \{1,0.98,0.96,0.94\}$. The 2D variance of the approximation solution for various values of $\alpha \in \{1,0.98,0.96,0.94\}$ is shown in Figure 10. Figure 11 shows the estimated solution's confidence intervals for various values of $\alpha \in \{1,0.98,0.96,0.94\}$, illustrating its 2D character. The 2D character of the coefficient of variation of the approximation solution is examined in Figure 12 for various values of $\alpha \in \{1,0.98,0.96,0.94\}$.

6. Conclusions

In this research, we solved a system of random fractional order ordinary differential equations using Caputo-Fabrizio derivatives by applying a technique called the Aboodh Transform Method (ATM). This study examined three examples where the parameters were chosen from normal, uniform, and exponential distributions. For each example, the most popular probability features of the equations expected value, variance, confidence intervals, and coefficient of variation were computed for varying values of α. We used the Matlab package program to plot the results that we found. The outcomes demonstrate the calculated reliability and accuracy of the suggested method.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally.

Acknowledgments

We would like to thank the referees for guiding us to provide important this publication and for the stimulating questions.

References

[1] Aboodh, K.S. (2013). The New Integral Transform "Aboodh Transform". Global Journal of Pure and Applied Mathematics, 9(1), 35-43

[2] Aboodh, K.S. (2014). Application of New Transform "Aboodh Transform" to Partial Differential Equations. Global Journal of Pure and Applied Mathematics, 10(2), 249-254

[3] Benattıa, M.E., Kacem Belghaba, K. (2019). A Varıational Iteration Method for Solving Ordinary Differential Equations Using the Aboodh Transform, Global Journal of Pure and Applied Scıences, 25, 55-61

[4] Şahin Y., (2024). Solutions of Random Fractional Differential Equations Using Aboodh and Aboodh-Adomian Decomposition Method. Master's Thesis, Gümüşhane University, Postgraduate Education Institute

[5] Abdelilah K., Sedeeg, H., and Abdelrahim Mahgoub, Mohand M., (2016). Aboodh Transform Homotopy Perturbation Method For Solving System Of Nonlinear Partial Differential Equations, Mathematical Theory and Modeling 6,8

[6] Abdelbagy, A., Alshikhand Mohand M., Abdelrahim Mahgoub, (2016). A Comparative Study Between Laplace Transform and Two New Integrals "ELzaki" Transform and "Aboodh" Transform, Pure and Applied Mathematics Journal, 5(5), 145-150

[7] Verma, D., Alam, A., Singh,A. P. (2020). Applications of Aboodh Transform, International Research Journal of Innovations in Engineering and Technology, 4(6), 47-51

[8] Almardy, I. A., Farah, R. A., Alkeer, M. A., Aboodh, K. S., Osman, A. K., Mohammed, M. A. (2023). Aboodh Adomian Decomposition Method Applied To Logistic Differential Model, Journal of Positive School Psychology, 7(2), 482-488

[9] Asai, Y., Kloeden, P.E. (2013). Numerical schemes for random ODEs via stochastic differential equations. Commun. Appl. Anal, 17(3 & 4), 511–528

[10] Bekiryazıcı, Z., Merdan, M., Kesemen, T., Najmuldeen, M. (2016). Mathematical Modeling of Dengue Disease under Random Effects Mathematıcal Scıences and Applıcatıons E-Notes, 4 (2), 58-70.

[11] Merdan, M., Bekiryazıcı, Z., Kesemen, T., Khaniyev, T. (2017a). Deterministic stability and random behavior of a Hepatitis C model, PLoS ONE 12(7): e0181571

[12] Merdan, M., Bekiryazıcı, Z., Kesemen, T., Khaniyev, T. (2017b). Comparison of stochastic and random models for bacterial resistance Advances in Difference Equations, 133.

[13] Anaç, H., Merdan, M., Kesemen, T. (2020). Solving for the random component time-fractional partial differential equations with the new Sumudu transform iterative method SN Applied Sciences, , 2,1112

[14] Bekiryazıcı, Z., Merdan, M., Kesemen, T. (2021). Modification of the random differential transformation method and its applications to compartmental models, Communications in Statistics - Theory and Methods, 50:18, 4271-4292

[15] Merdan, M., Anaç, H., Kesemen, T. (2019). The new Sumudu transform iterative method for studying the random component time-fractional Klein-Gordon equation. Sigma J Eng Nat Sci 10(3), 343–354

[16] Bekiryazici Z., Kesemen T., Merdan M., Khaniyev T. (2022). A modification of approximate random characteristics for a model of Zika virus transmission, Thermal Science , 26(4A),3067-3077

[17] Anaç, H., Merdan, M., Bekiryazıcı, Z., Kesemen, T.(2019). Solution of Some Random Partial Differential Equations Using Differential Transform Method and Laplace-Padé Method. Gümüşhane University Science Institute Journal. 9(1), 108–118

[18] Alkan, A.,(2024). Analysis of Fractional Advection Equation with Improved Homotopy Analysis Method. Osmaniye Korkut Ata University Journal of Science Institute, 7(3), 1215-1229

[19] Alkan, A., Anaç, H.(2024). A new study on the Newell-Whitehead-Segel equation with Caputo-Fabrizio fractional derivative. AIMS Mathematics, 9(10), 27979-27997

[20] Alkan, A., Anaç, H.(2024). The novel numerical solutions for time-fractional Fornberg-Whitham equation by using fractional natural transform decomposition method. AIMS Mathematics, 9(9), 25333-25359

[21] Alkan, A., (2022). Improving homotopy analysis method with an optimal parameter for time-fractional Burgers equation. Karamanoğlu Mehmetbey University Journal of Engineering and Natural Sciences, 4(2), 117-134

[22] Al-Refai, M., Pal, K. (2019). New Aspects of Caputo-Fabrizio Fractional Derivative, Progress in Fractional Differentiation and Applications an International Journal, Progr. Fract. Differ. Appl. 5(2), 157-166

[23] Sania Qureshi, S., A. Rangaig, N., Baleanu, D. (2019). New Numerical Aspects of Caputo-Fabrizio Fractional Derivative Operator, Mathematics, 7(4), 374

[24] Owolabi, M., Atangana, K. A.(2017). Analysis and Application of New Fractional Adams–Bashforth Scheme With Caputo–Fabrizio Derivative, Chaos, Solitons & Fractals, 111-119

[25] Bhangale, N., B.Kachhia, K., Gómez-Aguilar, J. F.(2022). A New Iterative Method with ρ Laplace Transform for Solving Fractional Differential Equations with Caputo Generalized Fractional Derivative. Engineering with Computers, 38(3) ,2125–2138

[26] Caputo, M., Fabrizio, M. (2016). Applications of New Time and Spatial Fractional Derivatives with Exponential Kernels. Progr. Fract. Differ. Appl. 2, 1-11

[27] Mboro Nchama, G. A. (2020). Properties of Caputo-Fabrizio Fractional Operators. New Trends in Mathematical Sciences. 1(8), 001-025

[28] Benattia, M.E., Belghaba, K. (2020). Application of The Aboodh Transform for Solving Fractional Delay Differential Equations. Universal Journal of Mathematics and Applications. 3(3), 93-101

[29] Ziane, D., Belgacem, R., Bokhari, A. (2019). A New Modified Adomian Decomposition Method for Nonlinear Partial Differential Equations. Open Journal of Mathematical Analysis.3(2),81-90

[30] Khaniyev, T., Ünver, İ., Küçük, Z., Kesemen T. (2017). Olasılık Kuramında Çözümlü Problemler, Nobel Akademik Yayıncılık.
[31] Akdeniz, F. (2014). Olasılık ve İstatistik, Akedemisyen Kitabevi, Ankara, 602s.