Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Published online: 30.11.2024
DOI : 10.15672/hujms.1543634

Research Article

The categories of L-convex spaces and L-convergence spaces: extensionality and productivity of quotient maps

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Abstract

Based on a complete residuated lattice L, we show that the category of L-convex spaces is not extensional and is closed under the formation of finite products of quotient maps. Then we propose the concept of (preconcave, concave) L-convergence spaces via L-co-Scott closed sets and prove that the category of concave L-convergence spaces is isomorphic to that of L-concave spaces. Finally, we investigate the categorical properties of L-convergence spaces and show that it is extensional and closed under the formation of finite products of quotient maps.

Mathematics Subject Classification (2020). 52A01, 54A40, 54A20

Keywords. L-convex space, L-concave space, L-convergence space, extensionality, quotient map

1. Introduction

A convex structure (also called an algebraic closure system) via abstracting three basic properties of convex sets is an important mathematical structure. Explicitly, a convex structure on a set X is a subset $\mathcal{C}$ of the powerset of X satisfying: $\varnothing, X \in \mathcal{C}$; $\mathcal{C}$ is closed for any intersections; $\mathcal{C}$ is closed for any directed unions. As a topology-like structure, convex structures are closely related to many other mathematical structures [31]. Adopting the lattice-valued approach in topological structures, convex structures are also studied in a lattice-valued viewpoint, which leads to several types of lattice-valued convex structures [18, 27, 29, 30]. To date, lattice-valued convex structures have been extensively studied in a topological approach, such as closure operators [22, 28, 39], interval operators [19, 32], categorical relationship [14, 20, 33] and so on. This demonstrates the feasibility of applying the studying methods in the theory of lattice-valued topological structures to that of lattice-valued convex structures.

From a categorical aspect, extensionality and productivity of quotient maps are important categorical properties of topological categories [24]. But the category of lattice-valued topological spaces satisfies neither the extensionality nor the productivity of quotient maps. This motivates us to consider if the category of lattice-valued convex spaces

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Received: 05.09.2024; Accepted: 10.11.2024

satisfies these two kinds of categorical properties. Besides, convergence structures via filters [3, 4, 6, 15, 16, 25], or lattice-valued convergence structures via lattice-valued filters [5, 7, 12, 13, 17, 34–38] serve as an important tool of characterizing topological structures and possess better categorical properties than topological structures. This motivates us to introduce the concept of lattice-valued convergence structures in the framework of lattice-valued convex spaces and study its relationship with lattice-valued convex structures as well as its categorical properties.

The aim of this paper is to apply the lattice-valued topological methods to the theory of lattice-valued convex structures. Concretely, we will discuss the extensionality and productivity of quotient maps in the category of lattice-valued convex spaces from a categorical aspect. Then we will propose lattice-valued convergence structures via lattice-valued filter analogues in a lattice-valued concave space and study its categorical relationship with lattice-valued concave spaces as well as its extensionality and productivity of quotient maps in a categorical sense.

The content is organized as follows. In Section 2, we recall some necessary concepts and notations. In Section 3, we discuss the categorical properties of L-convex spaces. In Section 4, we introduce the concept of L-co-Scott closed sets and use L-co-Scott closed sets to define L-convergence structures and study their relationship with L-concave structures. In Section 5, we discuss the categorical properties of L-convergence spaces.

2. Preliminaries

In this paper, if not otherwise specified, $(L, *, \top)$ is always a complete residuated lattice [2]. That is, L is a complete lattice with the top element $\top$ and the bottom element $\bot$ and $*$ is a binary operation on L such that

(i) $(L, *, \top)$ is a commutative monoid;
(ii) $*$ distributes over arbitrary joins, i.e.,

$$\alpha * \Big(\bigvee_{i\in I} \beta_i\Big) = \bigvee_{i\in I} \alpha * \beta_i$$

for each $\alpha \in L$ and $\{\beta_i\}_{i\in I} \subseteq L$.

Since the binary operation $*$ distributes over arbitrary joins, the map $\alpha * (-) : L \longrightarrow L$ has a right adjoint $\alpha \to (-) : L \longrightarrow L$ given by $\alpha \to \beta = \bigvee\{\gamma \in L \mid \alpha * \gamma \leq \beta\}$. The binary operation $\to$ is called the implication with respect to $*$. Some basic properties of the binary operations $*$ and $\to$ are collected in the following proposition, which can be found in many works, for instance [2, 10].

Proposition 2.1. *Let $(L, *, \top)$ be a complete residuated lattice. Then*

(I1) $\bot * \alpha = \bot$ *and* $\top \to \alpha = \alpha$;
(I2) $\alpha \to \beta = \top \Longleftrightarrow \alpha \leq \beta$;
(I3) $\alpha * (\alpha \to \beta) \leq \beta$ *and* $(\alpha \to \beta) * (\beta \to \gamma) \leq \alpha \to \gamma$;
(I4) $\alpha \to (\beta \to \gamma) = (\alpha * \beta) \to \gamma = \beta \to (\alpha \to \gamma)$;
(I5) $(\bigvee_{j\in J} \alpha_j) \to \beta = \bigwedge_{j\in J}(\alpha_j \to \beta)$;
(I6) $\alpha \to (\bigwedge_{j\in J} \beta_j) = \bigwedge_{j\in J}(\alpha \to \beta_j)$;
(I7) $\alpha \leq \beta \Longrightarrow \alpha \to \gamma \geq \beta \to \gamma$ *and* $\gamma \to \alpha \leq \gamma \to \beta$.

For a nonempty set X, $\mathcal{P}(X)$ denotes the powerset of X and L^X denotes the set of all L-subsets on X. For each nonempty $U \in \mathcal{P}(X)$, let $\top_U$ denote the characteristic function of U. We do not distinguish between an element $\alpha \in L$ and the constant map $\alpha_X : X \longrightarrow L$ such that $\alpha_X(x) = \alpha$ for each $x \in X$. All algebraic operations on L can be extended to L^X pointwisely.

A subfamily $\{A_j\}_{j\in J}$ of L^X is called directed (resp. co-directed) if for each $A_{j_1}, A_{j_2} \in \{A_j\}_{j\in J}$, there exists $A_{j_3} \in \{A_j\}_{j\in J}$ such that $A_{j_1} \leq A_{j_3}$ and $A_{j_2} \leq A_{j_3}$ (resp. $A_{j_3} \leq A_{j_1}$

and $A_{j_3} \leq A_{j_2}$). We usually use the symbols $\{A_j\}_{j\in J} \subseteq^{dir} \mathcal{B}$ (resp. $\{A_j\}_{j\in J} \subseteq^{cdir} \mathcal{B}$) to denote that $\{A_j\}_{j\in J}$ is a directed (resp. co-directed) subset of $\mathcal{B}$. Let $f : X \longrightarrow Y$ be an ordinary map. Define $f^{\rightarrow} : L^X \longrightarrow L^Y$ and $f^{\leftarrow} : L^Y \longrightarrow L^X$ by $f^{\rightarrow}(A)(y) = \bigvee_{f(x)=y} A(x)$ for each $A \in L^X$ and $y \in Y$, and $f^{\leftarrow}(B) = B \circ f$ for each $B \in L^Y$ [26]. A complete lattice L is called join continuous if for each $\alpha \in L$, the map $\alpha \vee (\cdot) : L \longrightarrow L$ is co-Scott continuous, that is,

$$\alpha \vee \bigwedge_{j\in J} \beta_j = \bigwedge_{j\in J} \alpha \vee \beta_j$$

for each co-directed set $\{\beta_j\}_{j\in J}$.

Definition 2.2 ([5]). The map $\mathcal{S}(-,-) : L^X \times L^X \longrightarrow L$ defined by

$$\forall A, B \in L^X, \mathcal{S}(A, B) = \bigwedge_{x\in X} \Big(A(x) \rightarrow B(x)\Big),$$

is called the lattice-valued inclusion order between L-subsets on X.

Definition 2.3 ([18, 27]). A subset $\mathcal{C}$ of L^X is called an L-convex structure on X if it satisfies

(LCE1) $\bot_X, \top_X \in \mathcal{C}$;
(LCE2) $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}$ implies $\bigwedge_{\lambda\in\Lambda} A_\lambda \in \mathcal{C}$, where $\Lambda \neq \emptyset$;
(LCE3) If $\{A_j\}_{j\in J} \subseteq \mathcal{C}$ is nonempty and directed, then $\bigvee_{j\in J} A_j \in \mathcal{C}$.

For an L-convex structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an L-convex space.

A map $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{C}^Y)$ between two L-convex spaces is called L-convexity-preserving if $f^{\leftarrow}(B) \in \mathcal{C}^X$ for each $B \in \mathcal{C}^Y$.

It is easy to check that L-convex spaces and their L-convexity-preserving maps form a category, denoted by **LConvex**.

An L-convex structure $\mathcal{C}$ is called stratified if it further satisfies

(LCEs) $\alpha * A \in \mathcal{C}$ for each $\alpha \in L$ and $A \in \mathcal{C}$;

An L-convex structure $\mathcal{C}$ is called co-stratified if it further satisfies

(LCEcs) $\alpha \rightarrow A \in \mathcal{C}$ for each $\alpha \in L$ and $A \in \mathcal{C}$.

A stratified and co-stratified L-convex structure is said to be strong.

Considering a continuous lattice as the lattice background, Pang and Xiu introduced an axiomatic approach to bases and subbases in L-convex spaces in [23].

Definition 2.4 ([23]). Let $(X, \mathcal{C})$ be an L-convex space and $\mathbb{B} \subseteq \mathcal{C}$. If $\mathbb{B}$ satisfies

$$\forall\ C \in \mathcal{C},\ \exists\ \mathbb{B}_C \subseteq^{dir} \mathbb{B},\ s.t.\ C = \bigvee \mathbb{B}_C,$$

then $\mathbb{B}$ is called a base of $(X, \mathcal{C})$.

Definition 2.5 ([23]). Let $(X, \mathcal{C})$ be an L-convex space and $\mathbb{A} \subseteq \mathcal{C}$. If

$$\mathbb{B}_{\mathbb{A}} = \left\{ \bigwedge_{i\in I} A_i \;\middle|\; \{A_i \mid i \in I\} \subseteq \mathbb{A},\ I \neq \emptyset \right\}$$

is a base of $(X, \mathcal{C})$, then $\mathbb{A}$ is called a subbase of $(X, \mathcal{C})$.

Definition 2.6 ([1]). A concrete category $\mathbb{C}$ is called a topological category over **Set** with respect to the usual forgetful functor from $\mathbb{C}$ to **Set** if it satisfies the following conditions:

(TC1) Existence of final structures: For any set X, any class Λ, any family $\{(X_\lambda, \xi_\lambda)\}_{\lambda\in\Lambda}$ of $\mathbb{C}$-object and any family $\{f_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ of maps, there exists a unique $\mathbb{C}$-structure ξ on X which is final with respect to the sink $\{f_\lambda : (X_\lambda, \xi_\lambda) \longrightarrow X\}_{\lambda\in\Lambda}$, this means that for a $\mathbb{C}$-object (Y, η), a map $g : (X, \xi) \longrightarrow (Y, \eta)$ is a $\mathbb{C}$-morphism if and only if for all $\lambda \in \Lambda, g \circ f_\lambda : (X_\lambda, \xi_\lambda) \longrightarrow (Y, \eta)$ is a $\mathbb{C}$-morphism.

(TC2) Fibre-smallness: For any set X, the $\mathbb{C}$-fibre of X, i.e., the class of all $\mathbb{C}$-structures on X is a set.

Proposition 2.7 ([21]). *The category* **LConvex** *is topological over* **Set**.

Proof. We only note that for a set X, the final structure $\mathcal{C}^X$ on X with respect to a class $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$ of L-convex spaces and a family $\{f_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ of maps, is determined by

$$\mathcal{C}^X = \{A \in L^X \mid \forall\lambda \in \Lambda,\ f_\lambda^{\leftarrow}(A) \in \mathcal{C}^{X_\lambda}\}.$$

□

By Proposition 2.7, a quotient space of an L-convex space can be defined.

Definition 2.8 ([40]). Let $(X, \mathcal{C}^X)$ be an L-convex space and $f : X \longrightarrow Y$ is a surjective map. Define $\mathcal{C}^Y \subseteq L^Y$ by

$$\mathcal{C}^Y = \{B \in L^Y \mid f^{\leftarrow}(B) \in \mathcal{C}^X\}.$$

Then $(Y, \mathcal{C}^Y)$ is called a quotient space of $(X, \mathcal{C}^X)$ and f is called a quotient map.

Since **LConvex** is topological over **Set**, there are the product spaces and the subspaces of L-convex spaces in **LConvex**. Next, we recall the concepts of product spaces and subspaces of L-convex spaces.

Definition 2.9 ([23]). Let $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$ be a family of L-convex spaces, $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ be a family of projection maps. The L-convex structure $\prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}$ on $\prod_{\lambda\in\Lambda} X_\lambda$ generated by the subbase $\bigcup_{\lambda\in\Lambda} p_\lambda^{\leftarrow}(\mathcal{C}^{X_\lambda})$, is called the product structure, the pair $(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda})$ is called the product space of $\{(X_\lambda, \mathcal{C}^{X_\lambda})\}_{\lambda\in\Lambda}$.

Proposition 2.10 ([23]). *Suppose that Λ is a finite index set. Let $\{(X_\lambda, \mathcal{C}^{X_\lambda}) \mid \lambda \in \Lambda\}$ be a family of L-convex spaces. Then its product L-convex structure is defined by*

$$\prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda} = \left\{ \prod_{\lambda\in\Lambda} C_\lambda \mid \forall\ \lambda \in \Lambda,\ C_\lambda \in \mathcal{C}^{X_\lambda} \right\}.$$

Definition 2.11 ([40]). Let $(X, \mathcal{C})$ be an L-convex space and $Y \subseteq X$. The pair $(Y, \mathcal{C}|_Y)$ is called a subspace of $(X, \mathcal{C})$.

Concavity is dual to convexity. In a natural way, the concept of L-concave spaces can be defined as follows.

Definition 2.12 ([17]). A subset $\mathcal{C}$ of L^X is called an L-concave structure on X if it satisfies

(LCA1) $\bot_X, \top_X \in \mathcal{C}$;
(LCA2) $\{A_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}$ implies $\bigvee_{\lambda\in\Lambda} A_\lambda \in \mathcal{C}$, where $\Lambda \neq \varnothing$;
(LCA3) If $\{A_j\}_{j\in J} \subseteq \mathcal{C}$ is nonempty and co-directed, then $\bigwedge_{j\in J} A_j \in \mathcal{C}$.

For an L-concave structure $\mathcal{C}$ on X, the pair $(X, \mathcal{C})$ is called an L-concave space.

A map $f : (X, \mathcal{C}^X) \longrightarrow (Y, \mathcal{C}^Y)$ between two L-concave spaces is called L-concavity-preserving provided that $f^{\leftarrow}(B) \in \mathcal{C}^X$ for each $B \in \mathcal{C}^Y$.

It is easy to check that L-concave spaces and their L-concavity-preserving maps form a category, denoted by **LConcave**.

When L is a complete MV-algebra, L-convex structures and L-concave structures are dual. So **LConvex** and **LConcave** are isomorphic in a categorical sense when L is a complete MV-algebra. Hence, we will not distinguish them when it comes to categorical properties in the sequel.

3. Categorical properties of L-convex spaces

In this section, we will discuss the categorical properties of **LConvex**, including extensionality and productivity of quotients maps. We first recall the concept of partial morphisms in a topological category.

In a topological category $\mathbb{C}$, a partial morphism from X to Y is a $\mathbb{C}$-morphism $f : Z \longrightarrow Y$ whose domain is a subobject of X.

Definition 3.1 ([24])**.** A topological category $\mathbb{C}$ is called extensional if every $\mathbb{C}$-object X has a one-point extension $\overline{X}$, in the sense that every $\mathbb{C}$-object X can be embedded via the addition of a single point ∞ into a $\mathbb{C}$-object $\overline{X}$ such that for every partial morphism $f : Z \longrightarrow X$ from Y to X , the map $\overline{f} : Y \longrightarrow \overline{X}$ defined by

$$\overline{f}(x) = \begin{cases} f(x), & if\ x \in Z, \\ \infty, & if\ x \notin Z \end{cases}$$

is a $\mathbb{C}$-morphism.

It is well known that if a category is extensional, then quotient maps in this category are hereditary. Next, we will show quotient maps in **LConvex** are not necessarily hereditary via the following example.

Example 3.2. Let $X = \{a, b, c, d\}$, $Y = \{a, b, c\}$, $L = \{\bot, \top\}$, $\mathcal{C}^X = \{\bot_X, \top_{\{a,c\}}, \top_{\{b,d\}}, \top_X\}$ and $\mathcal{C}^Y = \{\bot_Y, \top_Y\}$. Then $(X, \mathcal{C}^X)$ and $(Y, \mathcal{C}^Y)$ are L-convex spaces. Define $f : X \longrightarrow Y$ by

$$f(x) = \begin{cases} a, & if\ x = a, \\ b, & if\ x = b, \\ c, & if\ x = c, d. \end{cases}$$

Then f is a surjective map and $D \in \mathcal{C}^Y$ if and only if $f^{\leftarrow}(D) \in \mathcal{C}^X$ for each $D \in L^Y$. So f is a quotient map.

Let $A = B = \{a, b\}$ and let $(A, \mathcal{C}^X|_A)$ and $(B, \mathcal{C}^Y|_B)$ be the subspaces of $(X, \mathcal{C}^X)$ and $(Y, \mathcal{C}^Y)$, respectively. Then $\mathcal{C}^X|_A = \{\bot_A, \top_{\{a\}}, \top_{\{b\}}, \top_A\}$ and $\mathcal{C}^Y|_B = \{\bot_B, \top_B\}$. The restriction of f on A, denoted by $f|_A : A \longrightarrow B$, is defined by

$$f|_A(x) = \begin{cases} a, & if\ x = a, \\ b, & if\ x = b. \end{cases}$$

Take $\top_{\{a\}} \in L^B$. Then it is easy to check that $f|_A^{\leftarrow}(\top_{\{a\}}) = \top_{\{a\}} \in \mathcal{C}^X|_A$ and $\top_{\{a\}} \notin \mathcal{C}^Y|_B$. This shows that $f|_A : (A, \mathcal{C}^X|_A) \longrightarrow (B, \mathcal{C}^Y|_B)$ is not a quotient map.

By Example 3.2, we can obtain the following proposition.

Proposition 3.3. *In* **LConvex** *quotient maps are not hereditary.*

Since quotient maps in an extensional category must be hereditary, we have

Theorem 3.4. *The category* **LConvex** *is not extensional.*

In the following, we will go on exploring the productivity of quotient maps in **LConvex**. The following theorem illustrates that **LConvex** is closed under the formation of finite products of quotient maps.

Theorem 3.5. *Suppose that Λ is a finite index set. Let $\{(X_\lambda, \mathcal{C}^{X_\lambda}) \mid \lambda \in \Lambda\}$ be a family of L-convex spaces. If $\{f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in* **LConvex***, then the product map*

$$\prod_{\lambda\in\Lambda} f_\lambda : \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}\Big) \longrightarrow \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{Y_\lambda}\Big)$$

is a quotient map in **LConvex**.

Proof. Define

$$f := \prod_{\lambda\in\Lambda} f_\lambda,\ (X, \mathcal{C}^X) := \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda}\Big),\ (Y, \mathcal{C}^Y) := \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \mathcal{C}^{Y_\lambda}\Big).$$

Let

$$\begin{array}{ccc} (X, \mathcal{C}^X) & \xrightarrow{\ f\ } & (Y, \mathcal{C}^Y) \\ {\scriptstyle p_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle q_\lambda} \\ (X_\lambda, \mathcal{C}^{X_\lambda}) & \xrightarrow[\ f_\lambda\ ]{} & (Y_\lambda, \mathcal{C}^{Y_\lambda}) \end{array}$$

be the product communication diagram with respect to sets. Since $\{f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in **LConvex**, for each $B_\lambda \in L^{Y_\lambda}$, we have

$$B_\lambda \in \mathcal{C}^{Y_\lambda} \Longleftrightarrow f_\lambda^{\leftarrow}(B_\lambda) \in \mathcal{C}^{X_\lambda}.$$

Let $\mathcal{C}^Y_*$ be the quotient structure of $(X, \mathcal{C}^X)$ with respect to f. Then

$$\mathcal{C}^Y_* = \{B \in L^Y \mid f^{\leftarrow}(B) \in \mathcal{C}^X\}.$$

It suffices to verify that $\mathcal{C}^Y = \mathcal{C}^Y_*$.

On the one hand, take any $B \in L^Y$. Then

$$\begin{aligned} B \in \mathcal{C}^Y \quad &\Longleftrightarrow\quad \exists\ B_\lambda \in \mathcal{C}^{Y_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } B = \prod_{\lambda\in\Lambda} B_\lambda \\ &\Longleftrightarrow\quad \exists\ B_\lambda \in \mathcal{C}^{Y_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } f^{\leftarrow}(B) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(\prod_{\lambda\in\Lambda} B_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(B_\lambda) \\ &\Longrightarrow\quad \exists\ B_\lambda \in \mathcal{C}^{Y_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(B_\lambda) \in \prod_{\lambda\in\Lambda} \mathcal{C}^{X_\lambda} = \mathcal{C}^X. \end{aligned}$$

This shows that $\mathcal{C}^Y \subseteq \mathcal{C}^Y_*$.

On the other hand, take any $B \in L^Y$. Then

$$\begin{aligned} B \in \mathcal{C}^Y_* \quad &\Longleftrightarrow\quad f^{\leftarrow}(B) \in \mathcal{C}^X \\ &\Longleftrightarrow\quad \exists\ A_\lambda \in \mathcal{C}^{X_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} A_\lambda \\ &\Longleftrightarrow\quad \exists\ A_\lambda \in \mathcal{C}^{X_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } B = f^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} A_\lambda\Big) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(\prod_{\lambda\in\Lambda} A_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\rightarrow}(A_\lambda) \\ &\Longleftrightarrow\quad \exists\ A_\lambda \in \mathcal{C}^{X_\lambda} \text{ for each } \lambda \in \Lambda, \text{ s.t. } f^{\leftarrow}(B) = \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}\Big(\prod_{\lambda\in\Lambda} f_\lambda^{\rightarrow}(A_\lambda)\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda)). \end{aligned}$$

This implies that

$$f^{\leftarrow}(B) = \prod_{\lambda\in\Lambda} A_\lambda = \prod_{\lambda\in\Lambda} f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda)).$$

Then it follows that $f_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(A_\lambda)) = A_\lambda \in \mathcal{C}^{X_\lambda}$ for each $\lambda \in \Lambda$. Since $f_\lambda : (X_\lambda, \mathcal{C}^{X_\lambda}) \longrightarrow (Y_\lambda, \mathcal{C}^{Y_\lambda})$ is a quotient map, we have $f_\lambda^{\rightarrow}(A_\lambda) \in \mathcal{C}^{Y_\lambda}$. This implies that $B = \prod_{\lambda\in\Lambda} f_\lambda^{\rightarrow}(A_\lambda) \in \mathcal{C}^Y$. By the arbitrariness of B, we have $\mathcal{C}^Y_* \subseteq \mathcal{C}^Y$. □

Extensionality is an important categorical property. Regretly, **LConvex** is not extensional. This motivates us to find an extensional structure that is closely related to L-convex or L-concave structures. Inspired by L-filter convergence structures in L-topological spaces [12], we will consider convergence structures in L-convex spaces or L-concave spaces. To this end, we need to determine the filter analogues as the tools to define a convergence structure in an L-convex or L-concave space, which is exactly the L-co-Scott closed sets in the following section.

4. L-convergence space and its relationship with L-concave space

In this section, we will first propose L-co-Scott closed sets and study its basic properties. Then we will use L-co-Scott closed sets to define L-convergence structures and study their relationship with L-concave structures.

Note that many results in this section parallel to that in [8], where L-convergence structures were defined via L-ordered co-Scott closed sets. So we only give some necessary proofs herein.

4.1. L-co-Scott closed sets

In this subsection, we will focus on L-co-Scott closed sets on L^X.

Definition 4.1. A map $\mathcal{F} : L^X \longrightarrow L$ is called an L-co-Scott closed set on L^X if it satisfies

(LCSC1) $\mathcal{F}(\top_X) = \top$;
(LCSC2) $\mathcal{S}(A, B) * \mathcal{F}(A) \le \mathcal{F}(B)$ for each $A, B \in L^X$;
(LCSC3) $\bigwedge_{j\in J} \mathcal{F}(A_j) \le \mathcal{F}(\bigwedge_{j\in J} A_j)$ for each $\{A_j\}_{j\in J} \subseteq^{cdir} L^X$.

Remark 4.2.

(1) If $L = \{\bot, \top\}$, then an L-co-Scott closed set on L^X reduces to a co-Scott closed set on the powerset of X in the classical case [9].
(2) An L-co-Scott closed set $\mathcal{F}$ is called stratified if it further satisfies (LCSCs): $\alpha * \mathcal{F}(A) \le \mathcal{F}(\alpha * A)$ for each $\alpha \in L$ and $A \in \mathcal{C}$; an L-co-Scott closed set $\mathcal{F}$ is called co-stratified if it further satisfies (LCSCcs): $\alpha \to \mathcal{F}(A) \le \mathcal{F}(\alpha \to A)$ for each $\alpha \in L$ and $A \in \mathcal{C}$. Hence, an L-co-Scott closed set in Definition 4.1 is a little different from an L-ordered co-Scott closed set in [8] by relaxing the stratified and co-stratified conditions with respect to $*$ and $\to$ on L.

Let $\mathcal{C}_L(X)$ denote all L-co-Scott closed sets on L^X. For an L-co-Scott closed set $\mathcal{F}$ on L^X, the pair $(X, \mathcal{F})$ is called an L-co-Scott closed set space. An order on $\mathcal{C}_L(X)$ can be defined by $\mathcal{F} \le \mathcal{G}$ if and only if $\mathcal{F}(A) \le \mathcal{G}(A)$ for each $A \in L^X$.

Example 4.3. Let X be a nonempty set.

(1) Define a map $[x] : X \longrightarrow L$ by $[x](A) = A(x)$ for each $A \in L^X$ and $x \in X$. Then $[x] \in \mathcal{C}_L(X)$.
(2) Let $f : X \longrightarrow Y$ be a map and $\mathcal{F} \in \mathcal{C}_L(X)$. Then the map $f^{\Rightarrow}(\mathcal{F}) : L^Y \longrightarrow L$ defined by $f^{\Rightarrow}(\mathcal{F})(B) = \mathcal{F}(f^{\leftarrow}(B))$ for each $B \in L^Y$, is an L-co-Scott closed set, which is called the image of $\mathcal{F}$ under f in [11].
(3) For a family of L-co-Scott closed sets $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda} \subseteq \mathcal{C}_L(X)$, define $\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda : L^X \longrightarrow L$ by

$$\forall A \in L^X, \Big(\bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big)(A) = \bigwedge_{\lambda\in\Lambda} \mathcal{F}_\lambda(A).$$

Obviously, $\bigwedge_{\lambda\in\Lambda}\mathcal{F}_\lambda \in \mathcal{C}_L(X)$.

Proposition 4.4. *Suppose that L is join continuous. Let $\mathcal{F}, \mathcal{G}$ be two L-co-Scott closed sets on L^X. Define $\mathcal{F} \vee \mathcal{G} : L^X \longrightarrow L$ by $(\mathcal{F} \vee \mathcal{G})(A) = \mathcal{F}(A) \vee \mathcal{G}(A)$ for each $A \in L^X$. Then $\mathcal{F} \vee \mathcal{G}$ is the supremum of $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{C}_L(X)$.*

Proof. By the definition of $\mathcal{F} \vee \mathcal{G}$, we only need to verify that $\mathcal{F} \vee \mathcal{G}$ satisfies (LCSC1)–(LCSC3). (LCSC1) and (LCSC2) are straightforward, so we prove (LCSC3).

For (LCSC3), take any $\{A_j\}_{j\in J}\subseteq^{cdir} L^X$. Then

$$\begin{aligned}
\bigwedge_{j\in J}(\mathcal{F}\vee\mathcal{G})(A_j) &= \bigwedge_{j\in J}\Big(\mathcal{F}(A_j)\vee\mathcal{G}(A_j)\Big)\\
&\leq \bigwedge_{j_1\in J}\bigwedge_{j_2\in J}\Big(\mathcal{F}(A_{j_1})\vee\mathcal{G}(A_{j_2})\Big) \quad \text{(by the co-directedness of } \{A_j\}_{j\in J})\\
&= \bigwedge_{j_1\in J}\mathcal{F}(A_{j_1})\vee\bigwedge_{j_2\in J}\mathcal{G}(A_{j_2})\\
&\leq \mathcal{F}\Big(\bigwedge_{j_1\in J}A_{j_1}\Big)\vee\mathcal{G}\Big(\bigwedge_{j_2\in J}A_{j_2}\Big)\\
&= (\mathcal{F}\vee\mathcal{G})\Big(\bigwedge_{j\in J}A_j\Big).
\end{aligned}$$

□

Proposition 4.5. *Let $f: X\longrightarrow Y$ be a map and $\mathcal{G}\in\mathcal{C}_L(Y)$. Define $f^{\Leftarrow}(\mathcal{G}): L^X\longrightarrow L$ by*

$$\forall\ A\in L^X,\ f^{\Leftarrow}(\mathcal{G})(A)=\bigvee_{f^{\leftarrow}(B)\leq A}\mathcal{G}(B).$$

Then $f^{\Leftarrow}(\mathcal{G})\in\mathcal{C}_L(X)$.

Proof. Adopting the proof of Proposition 3.4 in [8]. □

The L-co-Scott closed set $f^{\Leftarrow}(\mathcal{G})$ is called the inverse image of $\mathcal{G}$ under f.

Proposition 4.6. *Let $(X,\mathcal{C})$ be an L-concave space. Define $\mathcal{N}^x_{\mathcal{C}}: L^X\longrightarrow L$ by*

$$\forall\ A\in L^X,\ \mathcal{N}^x_{\mathcal{C}}(A)=\bigvee_{B\in\mathcal{C},B\leq A}B(x).$$

Then $\mathcal{N}^x_{\mathcal{C}}\in\mathcal{C}_L(X)$.

Proof. Adopting the proof of Proposition 3.5 in [8]. □

By Proposition 4.6, we have $A\in\mathcal{C}$ if and only if $\mathcal{N}^x_{\mathcal{C}}(A)=A(x)$ for each $x\in X$. For an L-concave space $(X,\mathcal{C})$, define $\widehat{\mathcal{N}_{\mathcal{C}}}: L^X\longrightarrow L^X$ by

$$\widehat{\mathcal{N}_{\mathcal{C}}}(A)(x)=\mathcal{N}^x_{\mathcal{C}}(A)$$

for each $A\in L^X$ and $x\in X$. Then we have

Lemma 4.7. *Let $(X,\mathcal{C})$ be an L-concave space and $x\in X$. Then*

$$\mathcal{N}^x_{\mathcal{C}}(A)=\mathcal{N}^x_{\mathcal{C}}(\widehat{\mathcal{N}_{\mathcal{C}}}(A))$$

for each $A\in L^X$.

Proof. Adopting the proof of Lemma 4.10 in [8]. □

Proposition 4.8. *Let $f: X\longrightarrow Y$ be a map, $\mathcal{F}\in\mathcal{C}_L(X)$ and $\mathcal{G}\in\mathcal{C}_L(Y)$. Then*

(1) *$f^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F}))\leq\mathcal{F}$. If f is injective, then $f^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F}))=\mathcal{F}$;*
(2) *$\mathcal{G}\leq f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{G}))$. If f is surjective, then $\mathcal{G}=f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{G}))$.*

Proof. (1) Take any $A\in L^X$. Then

$$\begin{aligned}
f^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F}))(A) &= \bigvee_{f^{\leftarrow}(B)\leq A}f^{\Rightarrow}(\mathcal{F})(B)\\
&= \bigvee_{f^{\leftarrow}(B)\leq A}\mathcal{F}(f^{\leftarrow}(B))\\
&\leq \mathcal{F}(A).
\end{aligned}$$

This shows that $f^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F})) \leq \mathcal{F}$. If f is injective, then $A = f^{\leftarrow}(f^{\rightarrow}(A))$. This implies that $\mathcal{F} \leq f^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F}))$.

(2) Take any $B \in L^X$. Then

$$\begin{aligned} f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{G}))(B) &= f^{\Leftarrow}(\mathcal{G})(f^{\leftarrow}(B)) \\ &= \bigvee_{f^{\leftarrow}(C) \leq f^{\leftarrow}(B)} \mathcal{G}(C) \\ &\geq \mathcal{G}(B). \end{aligned}$$

This shows that $\mathcal{G} \leq f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{G}))$. If f is surjective, then $C = f^{\rightarrow}(f^{\leftarrow}(C)) \leq f^{\rightarrow}(f^{\leftarrow}(B)) = B$. This implies that $f^{\Rightarrow}(f^{\Leftarrow}(\mathcal{G})) \leq \mathcal{G}$. □

Remark 4.9. By Proposition 4.8, we know $(f^{\Leftarrow}, f^{\Rightarrow}) : \mathcal{C}_L(Y) \longrightarrow \mathcal{C}_L(X)$ is a Galois correspondence between $\mathcal{C}_L(Y)$ and $\mathcal{C}_L(X)$. Moreover, $f^{\Leftarrow}$ is the left adjoint and $f^{\Rightarrow}$ is the right adjoint.

Definition 4.10. A map $f : (X, \mathcal{F}) \longrightarrow (Y, \mathcal{G})$ between L-co-Scott closed set spaces is called continuous if $f^{\Leftarrow}(\mathcal{G}) \leq \mathcal{F}$.

It is easy to check that L-co-Scott closed set spaces and their continuous maps form a category, denoted by **LCSC**.

For $\mathcal{F} \in \mathcal{C}_L(X)$ and $\mathcal{G} \in \mathcal{C}_L(Y)$, by Propositions 4.4 and 4.5, we can obtain an L-co-Scott closed set $\mathcal{F} \times \mathcal{G}$ on $L^{X \times Y}$ in the following way:

$$\mathcal{F} \times \mathcal{G} = p_X^{\Leftarrow}(\mathcal{F}) \vee p_Y^{\Leftarrow}(\mathcal{G}),$$

where $p_X : X \times Y \longrightarrow X$ and $p_Y : X \times Y \longrightarrow Y$ are the projection maps.

Definition 4.11. Suppose that L is join continuous. For $\mathcal{F} \in \mathcal{C}_L(X)$ and $\mathcal{G} \in \mathcal{C}_L(Y)$, $\mathcal{F} \times \mathcal{G}$ is called the product of $\mathcal{F}$ and $\mathcal{G}$.

Definition 4.12. For two L-co-Scott closed sets $\mathcal{F}$ and $\mathcal{G}$ on L^X, $(X, \mathcal{G})$ is called coarser than $(X, \mathcal{F})$ if $id_X : (X, \mathcal{F}) \longrightarrow (X, \mathcal{G})$ is continuous.

It is easy to verify that $(X \times Y, \mathcal{F} \times \mathcal{G})$ is the coarsest L-co-Scott closed set space on $L^{X \times Y}$ such that $p_X : (X \times Y, \mathcal{F} \times \mathcal{G}) \longrightarrow (X, \mathcal{F})$ and $p_Y : (X \times Y, \mathcal{F} \times \mathcal{G}) \longrightarrow (Y, \mathcal{G})$ are continuous. The next proposition shows that $(X \times Y, \mathcal{F} \times \mathcal{G})$ is exactly the product object in the category **LCSC**.

Proposition 4.13. *Suppose that L is join continuous. Let $(X, \mathcal{F})$, $(Y, \mathcal{G})$ be two L-co-Scott closed set spaces. Then the pair $(X \times Y, \mathcal{F} \times \mathcal{G})$ is the product object of $(X, \mathcal{F})$ and $(Y, \mathcal{G})$ in* **LCSC**.

Proof. It suffices to verify that for each L-co-Scott closed set space $(Z, \mathcal{H})$ and two continuous maps $f : (Z, \mathcal{H}) \longrightarrow (X, \mathcal{F})$ and $g : (Z, \mathcal{H}) \longrightarrow (Y, \mathcal{G})$, there exists a unique continuous map $h : (Z, \mathcal{H}) \longrightarrow (X \times Y, \mathcal{F} \times \mathcal{G})$ such that $p_X \circ h = f$ and $p_Y \circ h = g$. Let $h = f \times g$, where $(f \times g)(z) = (f(z), g(z))$ for each $z \in Z$. By Definition 4.10, we need to show $h^{\Leftarrow}(\mathcal{F} \times \mathcal{G}) \leq \mathcal{H}$.

Since $f^{\Leftarrow}(\mathcal{F}) \leq \mathcal{H}$ and $g^{\Leftarrow}(\mathcal{G}) \leq \mathcal{H}$, we have

$$\begin{aligned} h^{\Leftarrow}(\mathcal{F} \times \mathcal{G}) &= h^{\Leftarrow}(p_X^{\Leftarrow}(\mathcal{F}) \vee p_Y^{\Leftarrow}(\mathcal{G})) \\ &= h^{\Leftarrow}(p_X^{\Leftarrow}(\mathcal{F})) \vee h^{\Leftarrow}(p_Y^{\Leftarrow}(\mathcal{G})) \quad \text{(by Remark 4.9)} \\ &= (p_X \circ h)^{\Leftarrow}(\mathcal{F}) \vee (p_Y \circ h)^{\Leftarrow}(\mathcal{G}) \\ &= f^{\Leftarrow}(\mathcal{F}) \vee g^{\Leftarrow}(\mathcal{G}) \\ &\leq \mathcal{H}. \end{aligned}$$

This shows that $h^{\Leftarrow}(\mathcal{F} \times \mathcal{G}) \leq \mathcal{H}$, as desired. □

Adopting Definition 4.11, the product of arbitrary finite L-co-Scott closed sets can be defined.

Definition 4.14. Suppose that Λ is a finite index set and L is join continuous. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ be the projection maps, $\mathcal{F}_\lambda \in \mathcal{C}_L(X_\lambda)$ $(\lambda \in \Lambda)$. Then $\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda = \bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda)$ is an L-co-Scott closed set on $L^{\prod_{\lambda\in\Lambda} X_\lambda}$, which is called the product of $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$.

Proposition 4.15. *Suppose that Λ is a finite index set and L is join continuous. Let $\{X_\lambda\}_{\lambda\in\Lambda}$ be a family of nonempty sets, $p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda$ be the projection maps, $\mathcal{F}_\lambda \in \mathcal{C}_L(X_\lambda)$ $(\lambda \in \Lambda)$ and $\mathcal{F} \in \mathcal{C}_L(\prod_{\lambda\in\Lambda} X_\lambda)$. Then the following statements hold:*

(1) $\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathcal{F}) \leq \mathcal{F}$;
(2) $\mathcal{F}_\mu \leq p_\mu^{\Rightarrow}(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda)$;
(3) $p_\mu^{\Rightarrow}(\prod_{\lambda\in\Lambda} p_\lambda^{\Rightarrow}(\mathcal{F})) = p_\mu^{\Rightarrow}(\mathcal{F})$.

Proof. By Proposition 4.8 and Remark 4.9, it is straightforward and is omitted. □

4.2. L-convergence spaces

In this subsection, adopting the approach in [8], we will use L-co-Scott closed sets instead of L-ordered co-Scott closed sets to define L-convergence structures.

Definition 4.16. A map $\lim : \mathcal{C}_L(X) \longrightarrow L^X$ is called an L-convergence structure on X if it satisfies

(LCS1) $\forall x \in X, \lim[x](x) = \top$;
(LCS2) $\mathcal{S}(\mathcal{F},\mathcal{G}) * \lim \mathcal{F}(x) \leq \lim \mathcal{G}(x)$ for each $\mathcal{F},\mathcal{G} \in \mathcal{C}_L(X)$.

For an L-convergence structure lim on X, the pair $(X, \lim)$ is called an L-convergence space.

A map $f : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ between two L-convergence spaces is called continuous provided that $\lim^X \mathcal{F}(x) \leq \lim^Y f^{\Rightarrow}(\mathcal{F})(f(x))$ for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$.

It is easy to check that L-convergence spaces and their continuous maps form a category, denoted by **LCS**.

Theorem 4.17. *The category* **LCS** *is a topological category over* **Set**.

Proof. We only note that for a set X, the initial structure $\lim^X$ on X with respect to a class $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$ of L-convergence spaces and a family $\{f_\lambda : X \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ of maps, is determined by

$$\lim^X \mathcal{F}(x) = \bigwedge_{\lambda\in\Lambda} \lim^{X_\lambda} f_\lambda^{\Rightarrow}(\mathcal{F})(f_\lambda(x))$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. □

Remark 4.18. For a set X, the final structure $\lim^X$ on X with respect to a class $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$ of L-convergence spaces and a family $\{f_\lambda : X_\lambda \longrightarrow X\}_{\lambda\in\Lambda}$ of maps, is determined by

$$\lim^X \mathcal{F}(x) = \begin{cases} \top, & \textit{if } \mathcal{F} \geq [x], \\ \bigvee\limits_{\lambda\in\Lambda} \bigvee\limits_{f_\lambda(x_\lambda)=x} \bigvee\limits_{f_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\leq\mathcal{F}} \lim^{X_\lambda} \mathcal{F}_\lambda(x_\lambda), & \text{otherwise} \end{cases}$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. In particular, the definition of quotient maps is available in **LCS**. Concretely, let $f : X \longrightarrow Y$ be a surjective map with $(X, \lim^X) \in |\mathbf{LCS}|$. If the structure $\lim^Y$ on Y is final with respect to $f : (X, \lim^X) \longrightarrow Y$ in the sense that

$$\lim^Y \mathcal{G}(y) = \bigvee_{f(x)=y} \bigvee_{f^{\Rightarrow}(\mathcal{F})\leq\mathcal{G}} \lim^X \mathcal{F}(x)$$

for each $\mathcal{G} \in \mathcal{C}_L(Y)$ and $y \in Y$, then the map $f : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ is called a quotient map.

Since **LCS** is topological over **Set**, there are the product and subspace of L-convergence spaces in **LCS**. We now introduce the concepts of the product and subspace of L-convergence spaces.

Definition 4.19. Let $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$ be a family of L-convergence spaces and $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ be the source formed by the family of the projection maps $\{p_\lambda\}_{\lambda\in\Lambda}$. The initial structure with respect to $\{p_\lambda : \prod_{\mu\in\Lambda} X_\mu \longrightarrow X_\lambda\}_{\lambda\in\Lambda}$ is called the product of $\{\lim^{X_\lambda}\}_{\lambda\in\Lambda}$, denoted by $\prod_{\lambda\in\Lambda} \lim^{X_\lambda}$. The pair $(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \lim^{X_\lambda})$ is called the product space of $\{(X_\lambda, \lim^{X_\lambda})\}_{\lambda\in\Lambda}$. Hence, for each $\mathcal{F} \in \mathcal{C}_L(\prod_{\lambda\in\Lambda} X_\lambda)$ and $x \in \prod_{\lambda\in\Lambda} X_\lambda$, we have

$$\Big(\prod_{\lambda\in\Lambda} \lim^{X_\lambda}\Big)\mathcal{F}(x) = \bigwedge_{\lambda\in\Lambda} \lim^{X_\lambda} p_\lambda^{\Rightarrow}(\mathcal{F})(p_\lambda(x)).$$

Definition 4.20. Let $(X, \lim^X)$ be an L-convergence space, $Y \subseteq X$ and $i_Y : Y \longrightarrow X$ be the source. The initial structure with respect to $i_Y : Y \longrightarrow X$ is called the subspace convergence structure, denoted by $\lim^X|_Y$. The pair $(Y, \lim^X|_Y)$ is called the subspace of $(X, \lim^X)$. Hence, we have

$$\lim^X|_Y \mathcal{F}(y) = \lim^X i_Y^{\Rightarrow}(\mathcal{F})(y).$$

In an L-convergence space $(X, \lim)$, a special L-co-Scott closed set can be defined in the following way.

Proposition 4.21. *Let $(X, \lim)$ be an L-convergence space and $x \in X$. Define $\mathcal{N}^x_{\lim} : L^X \longrightarrow L$ by*

$$\mathcal{N}^x_{\lim}(A) = \bigwedge_{\mathcal{F}\in\mathcal{C}_L(X)} \Big(\lim \mathcal{F}(x) \to \mathcal{F}(A)\Big)$$

for each $A \in L^X$. Then $\mathcal{N}^x_{\lim} \in \mathcal{C}_L(X)$.

Proof. It is straightforward and is omitted. □

Definition 4.22. An L-convergence space $(X, \lim)$ is called preconcave if it satisfies

$$\textbf{(Lcp)} \quad \lim \mathcal{F}(x) = \mathcal{S}(\mathcal{N}^x_{\lim}, \mathcal{F})$$

for each $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$.

Lemma 4.23. *Let $\mathcal{F} \in \mathcal{C}_L(X)$ and $\alpha \in L$. Then $\alpha \to \mathcal{F} \in \mathcal{C}_L(X)$.*

Proof. It is straightforward and is omitted. □

For an L-convergence space $(X, \lim)$, we consider the following axioms:

- **(Lcn)** For each $x \in X$, $\lim \mathcal{N}^x_{\lim}(x) = \top$;
- **(Lcq)** For each $\{\mathcal{F}_j\}_{j\in J} \subseteq \mathcal{C}_L(X)$, $\bigwedge_{j\in J} \lim \mathcal{F}_j = \lim(\bigwedge_{j\in J} \mathcal{F}_j)$ and $\lim(\alpha \to \mathcal{F}) = \alpha \to \lim \mathcal{F}$.

Proposition 4.24. *Let $(X, \lim)$ be an L-convergence space. Then* **(Lcn)** $\Longleftrightarrow$ **(Lcp)** $\Longleftrightarrow$ **(Lcq)**.

Proof. Adopting the proof of Proposition 4.6 in [8]. □

For an L-convergence space $(X, \lim)$, define $\widehat{\mathcal{N}_{\lim}} : L^X \longrightarrow L^X$ by

$$\widehat{\mathcal{N}_{\lim}}(A)(x) = \mathcal{N}^x_{\lim}(A)$$

for each $A \in L^X$ and $x \in X$. Then we have

Proposition 4.25. *Let $(X, \lim)$ be an L-convergence space and $x \in X$. Then $\mathcal{N}^x_{\lim} \circ \widehat{\mathcal{N}_{\lim}} \in \mathcal{C}_L(X)$.*

Proof. Adopting the proof of Proposition 4.7 in [8]. □

Definition 4.26. A preconcave L-convergence space $(X, \lim)$ is called concave if it satisfies

$$\textbf{(Lct)} \quad \mathcal{N}^x_{\lim} \leq \mathcal{N}^x_{\lim} \circ \widehat{\mathcal{N}_{\lim}}.$$

The full subcategory of **LCS** consisting of concave L-convergence spaces is denoted by **CLCS**.

Theorem 4.27. **CLCS** *is isomorphic to* **LConcave**.

Proof. Adopting the proof of Propositions 4.9, 4.12, 4.13 and 4.15 in [8]. □

Remark 4.28. In [8], the authors showed concave L-convergence spaces via L-ordered co-Scott closed sets are categorically isomorphic to strong L-concave spaces. Herein, we relax L-ordered co-Scott closed sets and strong L-concave spaces. Then we obtain the isomorphism between concave L-convergence space via L-co-Scott closed sets and L-concave spaces. Since most of the proofs can be adopted from the corresponding ones in [8], we only presented some necessary proofs in this subsection.

5. Categorical properties of L-convergence spaces

In this section, we will discuss the categorical properties of **LCS**, including extensionality and productivity of quotients maps.

Firstly, let us explore the extensionality of the category of L-convergence spaces.

For convenience, let $(X, \lim^X)$ be an L-convergence space, $\overline{X} = X \cup \{\infty\}$ with $\infty \notin X$ and $i_X : X \longrightarrow \overline{X}$ denote the inclusion map.

Proposition 5.1. *Let $(X, \lim^X)$ be an L-convergence space. Define $\lim^{\overline{X}} : \mathcal{C}_L(\overline{X}) \longrightarrow L^{\overline{X}}$ by*

$$\forall\ \mathcal{F} \in \mathcal{C}_L(\overline{X}), \forall\ x \in \overline{X},\ \lim^{\overline{X}} \mathcal{F}(x) = \lim^X i^{\Leftarrow}_X(\mathcal{F})(x) \vee \top_{\{\infty\}}(x).$$

Then $(\overline{X}, \lim^{\overline{X}})$ is an L-convergence space.

Proof. It suffices to verify that $\lim^{\overline{X}}$ satisfies (LCS1) and (LCS2).

For (LCS1), if $x = \infty$, then $\lim^{\overline{X}}[\infty](\infty) = \top$. If $x \in X$, then $i^{\Leftarrow}_X([x]) = [x]$ and $\lim^X[x](x) = \top$. So $\lim^{\overline{X}}[x](x) = \top$.

For (LCS2), take any $\mathcal{F}, \mathcal{G} \in \mathcal{C}_L(\overline{X})$. If $x = \infty$, then the conclusion holds. If $x \in X$, then

$$\begin{aligned} \mathcal{S}(\mathcal{F}, \mathcal{G}) * \lim^{\overline{X}} \mathcal{F}(x) &= \mathcal{S}(\mathcal{F}, \mathcal{G}) * \lim^X i^{\Leftarrow}_X(\mathcal{F})(x) \\ &\leq \mathcal{S}(i^{\Leftarrow}_X(\mathcal{F}), i^{\Leftarrow}_X(\mathcal{G})) * \lim^X i^{\Leftarrow}_X(\mathcal{F})(x) \\ &\leq \lim^X i^{\Leftarrow}_X(\mathcal{G})(x) \\ &= \lim^{\overline{X}} \mathcal{G}(x). \end{aligned}$$

□

Theorem 5.2. *The category* **LCS** *is extensional.*

Proof. Let $(X, \lim^X)$ be an L-convergence space. By Proposition 5.1, we obtain an L-convergence structure $\lim^{\overline{X}}$ on $\overline{X}$. It suffices to show that $(\overline{X}, \lim^{\overline{X}})$ is a one-point extension of $(X, \lim^X)$.

Firstly, we show that $(X, \lim^X)$ is a subspace of $(\overline{X}, \lim^{\overline{X}})$, that is, $\lim^X = \lim^{\overline{X}}|_X$. Take any $\mathcal{F} \in \mathcal{C}_L(X)$ and $x \in X$. Since $i^{\Leftarrow}_X(i^{\Rightarrow}_X(\mathcal{F})) = \mathcal{F}$, we have

$$\lim^{\overline{X}}|_X \mathcal{F}(x) = \lim^{\overline{X}} i^{\Rightarrow}_X(\mathcal{F})(x) = \lim^X i^{\Leftarrow}_X(i^{\Rightarrow}_X(\mathcal{F}))(x) = \lim^X \mathcal{F}(x).$$

Next, let $(Y,\lim^Y)$ be an L-convergence space, $(Z,\lim^Z)$ be a subspace of $(Y,\lim^Y)$ and $f:(Z,\lim^Z)\longrightarrow(X,\lim^X)$ be continuous. For the inclusion map $i_Z:Z\longrightarrow Y$ and the extensional map $\overline{f}:Y\longrightarrow\overline{X}$ of f defined by $\overline{f}(y)=f(y)$ for each $y\in Z$, and $\overline{f}(y)=\infty$ otherwise, there exists a commutative diagram in the category **Set** of sets as follows:

$$\begin{array}{ccc} Z & \xrightarrow{\quad f\quad} & X \\ {\scriptstyle i_Z}\big\downarrow & & \big\downarrow{\scriptstyle i_X} \\ Y & \xrightarrow[\quad\overline{f}\quad]{} & \overline{X} \end{array}$$

In order to prove $\overline{f}:(Y,\lim^Y)\longrightarrow(\overline{X},\lim^{\overline{X}})$ is continuous, it suffices to verify that $\lim^Y\mathcal{G}(y)\le\lim^{\overline{X}}\overline{f}^{\Rightarrow}(\mathcal{G})(\overline{f}(y))$ for each $\mathcal{G}\in\mathcal{C}_L(Y)$ and $y\in Y$. Now we divide into two cases:

Case 1: $\overline{f}(y)=\infty$, i.e., $y\in Y/Z$;
Case 2: $\overline{f}(y)\neq\infty$, i.e., $y\in Z$.

For case 1, by the definition of $\lim^{\overline{X}}$, we have $\lim^Y\mathcal{G}(y)\le\lim^{\overline{X}}\overline{f}^{\Rightarrow}(\mathcal{G})(\overline{f}(y))$.

For case 2, take any $B\in L^Y$ and $x\in X$. Then

$$\begin{aligned} f^{\rightarrow}(i_Z^{\leftarrow}(B))(x) &= \bigvee_{f(z)=x} i_Z^{\leftarrow}(B)(z) \\ &= \bigvee_{f(z)=x} B(z) \\ &= \bigvee_{\overline{f}(y)=x} B(y) \\ &= i_X^{\leftarrow}(\overline{f}^{\rightarrow}(B))(x). \end{aligned}$$

It follows that $f^{\rightarrow}(i_Z^{\leftarrow}(B))=i_X^{\leftarrow}(\overline{f}^{\rightarrow}(B))$. Take any $A\in L^X$. Then

$$\begin{aligned} f^{\Rightarrow}(i_Z^{\Leftarrow}(\mathcal{G}))(A) &= i_Z^{\Leftarrow}(\mathcal{G})(f^{\leftarrow}(A)) \\ &= \bigvee_{i_Z^{\leftarrow}(B)\le f^{\leftarrow}(A)} \mathcal{G}(B) \\ &= \bigvee_{f^{\rightarrow}(i_Z^{\leftarrow}(B))\le A} \mathcal{G}(B) \\ &= \bigvee_{i_X^{\leftarrow}(\overline{f}^{\rightarrow}(B))\le A} \mathcal{G}(B) \\ &\le \bigvee_{i_X^{\leftarrow}(D)\le A} \mathcal{G}(\overline{f}^{\leftarrow}(D)) \\ &= i_X^{\Leftarrow}(\overline{f}^{\Rightarrow}(\mathcal{G}))(A). \end{aligned}$$

This shows that $f^{\Rightarrow}(i_Z^{\Leftarrow}(\mathcal{G}))\le i_X^{\Leftarrow}(\overline{f}^{\Rightarrow}(\mathcal{G}))$. Then by $\mathcal{G}\le i_Z^{\Rightarrow}(i_Z^{\Leftarrow}(\mathcal{G}))$, we have

$$\begin{aligned} \lim^Y\mathcal{G}(y) &\le \lim^Y i_Z^{\Rightarrow}(i_Z^{\Leftarrow}(\mathcal{G}))(y) \\ &= \lim^Z i_Z^{\Leftarrow}(\mathcal{G})(y) \\ &\le \lim^X f^{\Rightarrow}(i_Z^{\Leftarrow}(\mathcal{G}))(f(y)) \\ &\le \lim^X i_X^{\Leftarrow}(\overline{f}^{\Rightarrow}(\mathcal{G}))(f(y)) \\ &= \lim^{\overline{X}}\overline{f}^{\Rightarrow}(\mathcal{G})(\overline{f}(y)). \end{aligned}$$

Hence, we obtain that $\overline{f}:(Y,\lim^Y)\longrightarrow(\overline{X},\lim^{\overline{X}})$ is continuous. □

Next, we will show that finite products of quotients maps are quotient maps in **LCS**. To this end, we first give an important property of L-co-Scott closed sets.

Lemma 5.3. *Suppose that Λ is a finite index set and L is join continuous. Let $\{f_\lambda : X_\lambda \longrightarrow Y_\lambda\}_{\lambda\in\Lambda}$ be a family of surjective maps and $\{\mathcal{F}_\lambda\}_{\lambda\in\Lambda}$ be a family of L-co-Scott closed sets with $\mathcal{F}_\lambda \in \mathcal{C}_L(X_\lambda)$ for each $\lambda \in \Lambda$. Then*

$$\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(\mathcal{F}_\lambda).$$

Proof. Let

$$\begin{array}{ccc} \prod\limits_{\lambda\in\Lambda} X_\lambda & \xrightarrow{\prod_{\lambda\in\Lambda} f_\lambda} & \prod\limits_{\lambda\in\Lambda} Y_\lambda \\ {\scriptstyle p_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle q_\lambda} \\ X_\lambda & \xrightarrow[f_\lambda]{} & Y_\lambda \end{array}$$

be the product commutation diagram with respect to sets, where p_λ and q_λ denote the corresponding projective maps.

On the one hand, take any $B \in L^{Y_\lambda}$ and $A \in L^{\prod_{\lambda\in\Lambda} Y_\lambda}$ such that $q_\lambda^{\leftarrow}(B) \le A$. Then

$$\begin{aligned} \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}(p_\lambda^{\leftarrow}(f_\lambda^{\leftarrow}(B))) &= \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}((f_\lambda\circ p_\lambda)^{\leftarrow}(B)) \\ &= \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\Big(\Big(q_\lambda\circ\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}(B)\Big) \\ &= \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}\circ\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}(q_\lambda^{\leftarrow}(B)) \\ &= q_\lambda^{\leftarrow}(B) \\ &\le A. \end{aligned}$$

It follows that

$$\bigvee_{q_\lambda^{\leftarrow}(B)\le A} \mathcal{F}_\lambda(f_\lambda^{\leftarrow}(B)) \le \bigvee_{(\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C))\le A} \mathcal{F}_\lambda(C).$$

On the other hand, assume $(\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C)) \le A$. Take any $y \in Y$. Then

$$\begin{aligned} q_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(C))(y) &= f_\lambda^{\rightarrow}(C)(q_\lambda(y)) \\ &= \bigvee_{f_\lambda(x_\lambda)=q_\lambda(y)} C(x_\lambda) \\ &= \bigvee_{(\prod_{\lambda\in\Lambda} f_\lambda)(x)=y} p_\lambda^{\leftarrow}(C)(x) \\ &\quad \text{(since } \{f_\lambda : X_\lambda \longrightarrow Y_\lambda\}_{\lambda\in\Lambda} \textit{ are surjective maps}) \\ &= \Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\rightarrow}(p_\lambda^{\leftarrow}(C))(y). \end{aligned}$$

This implies that $q_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(C)) = (\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C))$. Then

$$\begin{aligned} \bigvee_{(\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C))\le A} \mathcal{F}_\lambda(C) &= \bigvee_{q_\lambda^{\leftarrow}(f_\lambda^{\rightarrow}(C))\le A} \mathcal{F}_\lambda(C) \\ &\le \bigvee_{q_\lambda^{\leftarrow}(B)\le A} \mathcal{F}_\lambda(f_\lambda^{\leftarrow}(B)). \end{aligned}$$

So

$$\bigvee_{(\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C))\le A} \mathcal{F}_\lambda(C) = \bigvee_{q_\lambda^{\leftarrow}(B)\le A} \mathcal{F}_\lambda(f_\lambda^{\leftarrow}(B)).$$

Take any $A \in L^{\prod_{\lambda\in\Lambda} Y_\lambda}$. Then

$$\begin{aligned}
\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big)(A) &= \Big(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big)\Big(\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}(A)\Big) \\
&= \Big(\bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda)\Big)\Big(\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}(A)\Big) \\
&= \bigvee_{\lambda\in\Lambda} p_\lambda^{\Leftarrow}(\mathcal{F}_\lambda)\Big(\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\leftarrow}(A)\Big) \\
&= \bigvee_{\lambda\in\Lambda} \bigvee_{p_\lambda^{\leftarrow}(C)\le(\prod_{\lambda\in\Lambda} f_\lambda)^{\leftarrow}(A)} \mathcal{F}_\lambda(C) \\
&= \bigvee_{\lambda\in\Lambda} \bigvee_{(\prod_{\lambda\in\Lambda} f_\lambda)^{\rightarrow}(p_\lambda^{\leftarrow}(C))\le A} \mathcal{F}_\lambda(C) \\
&= \bigvee_{\lambda\in\Lambda} \bigvee_{q_\lambda^{\leftarrow}(B)\le A} \mathcal{F}_\lambda(f_\lambda^{\leftarrow}(B)) \\
&= \bigvee_{\lambda\in\Lambda} \bigvee_{q_\lambda^{\leftarrow}(B)\le A} f_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)(B) \\
&= \bigvee_{\lambda\in\Lambda} q_\lambda^{\Leftarrow}(f^{\Rightarrow}(\mathcal{F}_\lambda))(A) \\
&= \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(\mathbb{F}_\lambda)(A).
\end{aligned}$$

This implies that

$$\Big(\prod_{\lambda\in\Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} \mathcal{F}_\lambda\Big) = \prod_{\lambda\in\Lambda} f_\lambda^{\Rightarrow}(\mathcal{F}_\lambda).$$

□

Theorem 5.4. *Suppose that Λ is a finite index set and L is a completely distributive lattice. If $\{f_\lambda : (X_\lambda, \lim^{X_\lambda}) \longrightarrow (Y_\lambda, \lim^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in* **LCS**, *then the product map*

$$\prod_{\lambda\in\Lambda} f_\lambda : \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \lim^{X_\lambda}\Big) \longrightarrow \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \lim^{Y_\lambda}\Big)$$

is a quotient map in **LCS**.

Proof. Define

$$f := \prod_{\lambda\in\Lambda} f_\lambda,\ (X, \lim^X) := \Big(\prod_{\lambda\in\Lambda} X_\lambda, \prod_{\lambda\in\Lambda} \lim^{X_\lambda}\Big),\ (Y, \lim^Y) := \Big(\prod_{\lambda\in\Lambda} Y_\lambda, \prod_{\lambda\in\Lambda} \lim^{Y_\lambda}\Big).$$

Let

$$\begin{array}{ccc}
(X, \lim^X) & \xrightarrow{\quad f \quad} & (Y, \lim^Y) \\
{\scriptstyle p_\lambda}\big\downarrow & & \big\downarrow{\scriptstyle q_\lambda} \\
(X_\lambda, \lim^{X_\lambda}) & \xrightarrow[\quad f_\lambda \quad]{} & (Y_\lambda, \lim^{Y_\lambda})
\end{array}$$

be the product communication diagram with respect to sets. Since $\{f_\lambda : (X_\lambda, \lim^{X_\lambda}) \longrightarrow (Y_\lambda, \lim^{Y_\lambda})\}_{\lambda\in\Lambda}$ is a family of quotient maps in **LCS**, for each $\mathcal{H}_\lambda \in \mathcal{C}_L(Y_\lambda)$ and $y_\lambda \in Y_\lambda$, we have

$$\lim^{Y_\lambda}\mathcal{H}_\lambda(y_\lambda) = \bigvee_{f_\lambda(x_\lambda)=y_\lambda} \bigvee_{f_\lambda^{\Rightarrow}(\mathcal{F}_\lambda)\le\mathcal{H}_\lambda} \lim^{X_\lambda}\mathcal{F}_\lambda(x_\lambda).$$

Suppose that $\lim_*^Y$ is the quotient structure with respect to f. Then

$$\lim_*^Y \mathcal{H}(y) = \bigvee_{f(x)=y} \bigvee_{f^{\Rightarrow}(\mathcal{G})\le\mathcal{H}} \lim^X \mathcal{G}(x).$$

It suffices to verify that $\lim_*^Y = \lim^Y$.

On the one hand, for each $\mathcal{G} \in \mathcal{C}_L(X), \mathcal{H} \in \mathcal{C}_L(Y)$ with $f^{\Rightarrow}(\mathcal{G}) \leq \mathcal{H}$ and for each $y \in Y$ with $f(x) = y$, since $f_\lambda \circ p_\lambda = q_\lambda \circ f$, we have

$$f_\lambda^{\Rightarrow} \circ p_\lambda^{\Rightarrow}(\mathcal{G}) = q_\lambda^{\Rightarrow} \circ f^{\Rightarrow}(\mathcal{G}) \leq q_\lambda^{\Rightarrow}(\mathcal{H})$$

and

$$f_\lambda \circ p_\lambda(x) = q_\lambda \circ f(x) = q_\lambda(y)$$

for each $\lambda \in \Lambda$. It follows from the continuity of $f_\lambda \circ p_\lambda$ that

$$\lim^X \mathcal{G}(x) \leq \lim^{Y_\lambda}(f_\lambda \circ p_\lambda)^{\Rightarrow}(\mathcal{G})(f_\lambda \circ p_\lambda(x)) \leq \lim^{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{H})(q_\lambda(y)).$$

Hence, we have

$$\lim_*^Y \mathcal{H}(y) = \bigvee_{f(x)=y} \bigvee_{f^{\Rightarrow}(\mathcal{G}) \leq \mathcal{H}} \lim^X \mathcal{G}(x) \leq \lim^{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{H})(q_\lambda(y))$$

for each $\lambda \in \Lambda$. This implies that $\lim_*^Y \mathcal{H}(y) \leq \bigwedge_{\lambda \in \Lambda} \lim^{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{H})(q_\lambda(y)) = \lim^Y \mathcal{H}(y)$.

On the other hand, let

$$\mathbb{G}_\lambda = \left\{ \mathcal{G}_\lambda \in \mathcal{C}_L(X_\lambda) \mid f_\lambda^{\Rightarrow}(\mathcal{G}_\lambda) \leq q_\lambda^{\Rightarrow}(\mathcal{H}) \right\}$$

for each $\lambda \in \Lambda$ and let

$$\prod_{\lambda \in \Lambda} \mathbb{G}_\lambda = \left\{ g : \Lambda \longrightarrow \coprod \mathbb{G}_\lambda \;\middle|\; \forall \lambda \in \Lambda, g(\lambda) \in \mathbb{G}_\lambda \right\}$$

be the set of choice functions. Then

$$\forall\ \lambda \in \Lambda,\ \exists\ \mathcal{G}_\lambda \in \mathcal{C}_L(X_\lambda),\ s.t.\ f_\lambda^{\Rightarrow}(\mathcal{G}_\lambda) \leq q_\lambda^{\Rightarrow}(\mathcal{H}) \Longleftrightarrow \exists\ g \in \prod_{\lambda \in \Lambda} \mathbb{G}_\lambda,\ s.t.\ \forall\ \lambda \in \Lambda,\ f_\lambda^{\Rightarrow}(g(\lambda)) \leq q_\lambda^{\Rightarrow}(\mathcal{H}).$$

Furthermore, we have

$$\prod_{\lambda \in \Lambda} f_\lambda^{\Rightarrow}(g(\lambda)) \leq \prod_{\lambda \in \Lambda} q_\lambda^{\Rightarrow}(\mathcal{H}) \leq \mathcal{H},$$

which implies

$$f^{\Rightarrow}\Big(\prod_{\lambda \in \Lambda} g(\lambda)\Big) = \Big(\prod_{\lambda \in \Lambda} f_\lambda\Big)^{\Rightarrow}\Big(\prod_{\lambda \in \Lambda} g(\lambda)\Big) = \prod_{\lambda \in \Lambda} f_\lambda^{\Rightarrow}(g(\lambda)) \leq \mathcal{H}.$$

Let

$$H_\lambda = \{x_\lambda \in X_\lambda \mid f_\lambda(x_\lambda) = q_\lambda(y)\}$$

for each $\lambda \in \Lambda$ and let

$$\prod_{\lambda \in \Lambda} H_\lambda = \left\{ h : \Lambda \longrightarrow \coprod H_\lambda \;\middle|\; \forall\ \lambda \in \Lambda,\ f_\lambda(h(\lambda)) = q_\lambda(y) \right\}$$

be the set of choice functions. Then

$$\forall\ \lambda \in \Lambda,\ \exists\ x_\lambda \in X_\lambda,\ s.t.\ f_\lambda(x_\lambda) = q_\lambda(y) \Longleftrightarrow \exists\ h \in \coprod_{\lambda \in \Lambda} H_\lambda,\ s.t.\ \forall\ \lambda \in \Lambda,\ f_\lambda(h(\lambda)) = q_\lambda(y).$$

Furthermore, we have

$$f\Big((h(\lambda))_{\lambda \in \Lambda}\Big) = \Big(\prod_{\lambda \in \Lambda} f_\lambda\Big)\Big((h(\lambda))_{\lambda \in \Lambda}\Big) = \Big(f_\lambda(h(\lambda))\Big)_{\lambda \in \Lambda} = \Big(q_\lambda(y)\Big)_{\lambda \in \Lambda} = y.$$

Then for each $\mathcal{H} \in \mathcal{C}_L(Y)$ and $y \in Y$, we have

$$\begin{aligned}
\lim^Y \mathcal{H}(y) &= \bigwedge_{\lambda\in\Lambda} \lim^{Y_\lambda} q_\lambda^{\Rightarrow}(\mathcal{H})(q_\lambda(y)) \\
&= \bigwedge_{\lambda\in\Lambda} \bigvee_{f_\lambda(x_\lambda)=q_\lambda(y)} \bigvee_{f_\lambda^{\Rightarrow}(\mathcal{G}_\lambda)\le q_\lambda^{\Rightarrow}(\mathcal{H})} \lim^{X_\lambda}\mathcal{G}_\lambda(x_\lambda) \\
&= \bigvee_{h\in\prod\limits_{\lambda\in\Lambda} H_\lambda} \bigwedge_{\lambda\in\Lambda} \bigvee_{f_\lambda^{\Rightarrow}(\mathcal{G}_\lambda)\le q_\lambda^{\Rightarrow}(\mathcal{H})} \lim^{X_\lambda}\mathcal{G}_\lambda(h(\lambda)) \\
&= \bigvee_{h\in\prod\limits_{\lambda\in\Lambda} H_\lambda} \bigvee_{g\in\prod\limits_{\lambda\in\Lambda} \mathbb{G}_\lambda} \bigwedge_{\lambda\in\Lambda} \lim^{X_\lambda} g(\lambda)(h(\lambda)) \\
&\le \bigvee_{h\in\prod\limits_{\lambda\in\Lambda} H_\lambda} \bigvee_{g\in\prod\limits_{\lambda\in\Lambda} \mathbb{G}_\lambda} \bigwedge_{\lambda\in\Lambda} \lim^{X_\lambda} p_\lambda^{\Rightarrow}\Big(\prod_{\lambda\in\Lambda} g(\lambda)\Big)\Big(p_\lambda\Big(\prod_{\lambda\in\Lambda} h(\lambda)\Big)\Big) \\
&= \bigvee_{h\in\prod\limits_{\lambda\in\Lambda} H_\lambda} \bigvee_{g\in\prod\limits_{\lambda\in\Lambda} \mathbb{G}_\lambda} \lim^X\Big(\prod_{\lambda\in\Lambda} g(\lambda)\Big)\Big(\prod_{\lambda\in\Lambda} h(\lambda)\Big) \\
&\le \bigvee_{f(x)=y} \bigvee_{f^{\Rightarrow}(\mathcal{G})\le\mathcal{H}} \lim^X \mathcal{G}(x) \\
&= \lim_*^Y \mathcal{H}(y)
\end{aligned}$$

This shows that $\lim^Y \mathcal{H}(y) \le \lim_*^Y \mathcal{H}(y)$. As a consequence, we obtain $\lim^Y = \lim_*^Y$. □

6. Conclusions

In this paper, we first studied that the categorical properties of L-convex spaces and its corresponding convergence spaces and showed that: (i) the category of L-convex spaces is not extensional and is closed under the formation of finite products of quotient maps; (ii) the category of concave L-convergence spaces is isomorphic to that of L-concave spaces; (iii) the category of L-convergence spaces is extensional and closed under the formation of finite products of quotient maps.

Next we list some of our future work related to this paper.

(1) Whether the conclusion of Theorems 3.5 and 5.4 can be extended to the case of infinite product.
(2) It is well known that Cartesian closedness is an important categorical property. We will consider the Cartesian closedness of the category of L-convergence spaces.
(3) Introducing the concept of L-co-Scott closed set spaces, considering its categorical properties and establishing its categorical relationship with L-convergence spaces.

Acknowledgements

The authors would like to express their sincere thanks to the anonymous reviewers and the editor for their careful reading and constructive comments. This work is supported by the National Natural Science Foundation of China (Nos. 12471428, 12271036, 12071033).

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