Journal of New Theory
ISSN: 2149-1402

49 (2024) 30-42
Journal of New Theory
https://dergipark.org.tr/en/pub/jnt
Open Access

On Factorization and Calculation of Determinant of Block Matrices with Triangular Submatrices

Ufuk Kaya[1], Fatma Altun[2]

Article Info
Received: 07 Sep 2024
Accepted: 06 Dec 2024
Published: 31 Dec 2024
doi:10.53570/jnt.1545032
Research Article

Abstract — In this paper, we consider some block matrices of dimension $nm \times nm$ whose components are triangular matrices of dimension $n \times n$. We prove that the determinant of such block matrices is determined only by the diagonal elements of their submatrices and that this determinant is expressed as the multiplication of some subdeterminants. If the components of dimension $n \times n$ are all diagonal matrices, then we prove that such a block matrix can be written as a product of simpler matrices. Besides, we investigate the eigenvalues, the adjoint, and the inverse of such block matrices.

Keywords *Block matrix, determinant, triangular matrix, trigonometric system, Wronskian, factorization*

Mathematics Subject Classification (2020) 15A15, 15A23

1. Introduction

The determinant is a scalar value corresponding to a square matrix and is denoted by $|A|$, $D(A)$, $\det A$, or $\det(A)$. Besides, it is a function that maps from square matrix spaces to complex numbers. The determinant has many uses in mathematics. For instance, it determines whether a square matrix is invertible, is used to solve a system of linear equations, helps to find the inverse of a matrix, is used to solve some boundary value problems, etc. The determinant of a square matrix can be calculated using Laplace expansion. In particular, the determinant of a matrix of dimension 3×3 can be calculated by the Sarrus rule. Calculating determinants becomes more difficult for square matrices of dimension 4×4 and larger. For more information about factorization and calculation of determinants of large block matrices, see [1–14]. Sometimes calculating a determinant is easier if there are many zeros in the entries of the considered matrix. For instance, it is easier to calculate the determinants of the following matrices B and C by hand than the determinant of the following matrix D.

$$B = \begin{pmatrix} 3 & 0 & 3 & 7 \\ 0 & -2 & 0 & 0 \\ 5 & 2 & 0 & 13 \\ 0 & 0 & 5 & -8 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 65 & 3 & 7 \\ 0 & -2 & 4 & -8 \\ 0 & 0 & 9 & 18 \\ 0 & 0 & 0 & -8 \end{pmatrix}, \quad \text{and} \quad D = \begin{pmatrix} 5 & 5 & -7 & 7 \\ 8 & 12 & -8 & 2 \\ 5 & -2 & 9 & 1 \\ 1 & -1 & 3 & -1 \end{pmatrix}$$

The more zeros, the easier it is to calculate the determinant. Indeed, the determinant of C is the easiest since C is an upper triangular matrix. The determinant of an upper or lower triangular square matrix is the product of the main diagonal entries. A similar calculation is provided for upper triangular block matrices. Let E be an upper triangular square block matrix as follows:

[1]mat-ufuk@hotmail.com (Corresponding Author); [2]altnfatma34@gmail.com
[1,2]Department of Mathematics, Faculty of Arts and Sciences, Bitlis Eren University, Bitlis, Türkiye

$$E = \begin{pmatrix} E_{11} & E_{12} & \cdots & E_{1m} \\ 0 & E_{22} & \cdots & E_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & E_{mm} \end{pmatrix}$$

where 0 denotes the zero matrix. Then, $|E| = \prod_{i=1}^{n} |E_{ii}|$.

Simply having many zeros does not make it easy to calculate a determinant. Both matrices B and C have six zeros. However, since matrix C is upper triangular, it is easier to calculate its determinant. That is, the location or arrangement of the zeros is also essential.

This paper considers a different arrangement of zeros in a square matrix. It presents the following type of square block matrices of dimension $nm \times nm$ whose components are upper triangular matrices of dimension $n \times n$:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}_{nm \times nm}$$

where

$$A_{ij} = \begin{pmatrix} a_{n(i-1)+1,n(j-1)+1} & a_{n(i-1)+1,n(j-1)+2} & \cdots & a_{n(i-1)+1,n(j-1)+n} \\ 0 & a_{n(i-1)+2,n(j-1)+2} & \cdots & a_{n(i-1)+2,n(j-1)+n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n(i-1)+n,n(j-1)+n} \end{pmatrix}_{n \times n} \tag{1.1}$$

Firstly, we show that the determinant of A depends only on the diagonal entries of the matrices A_{ij}. Secondly, we construct a factorization of the matrix A when all the sub-matrices A_{ij} are diagonal matrices and obtain a formula for the determinant of A. Finally, we consider the eigenvalues, adjoint, and inverse of the matrix A.

For instance, if we take $n = m = 2$, then the matrix A turns into

$$A = \begin{pmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & 0 & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & 0 & a_{4,4} \end{pmatrix}$$

and we show that the following equality is valid

$$|A| = \begin{vmatrix} a_{1,1} & a_{1,2} & a_{1,3} & a_{1,4} \\ 0 & a_{2,2} & 0 & a_{2,4} \\ a_{3,1} & a_{3,2} & a_{3,3} & a_{3,4} \\ 0 & a_{4,2} & 0 & a_{4,4} \end{vmatrix} = \begin{vmatrix} a_{1,1} & a_{1,3} \\ a_{3,1} & a_{3,3} \end{vmatrix} \begin{vmatrix} a_{2,2} & a_{2,4} \\ a_{4,2} & a_{4,4} \end{vmatrix}$$

In this paper, we consider only the upper triangular matrices since a lower triangular matrix is the transpose of an upper triangular matrix.

2. Main Results

In this section, we delve into the detailed process of reducing the determinant of block matrices whose submatrices are triangular. This reduction is crucial for simplifying the determinant calculation of such complex matrices. We begin by analyzing the specific structure of these matrices and demonstrate how the arrangement of zeros in both the submatrices and the block matrix itself plays a fundamental role. The results presented here provide a framework for factorization and determinant computation, which will be elaborated upon in the following subsections. Our approach aims to significantly reduce the computational complexity of these calculations, offering a more efficient pathway for handling large-scale block matrices.

2.1. Reduction of Determinant

Consider the matrix of dimension $nm \times nm$ as follows:

$$A = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1m} \\ A_{21} & A_{22} & \cdots & A_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mm} \end{pmatrix}$$

where the submatrices A_{ij} of dimension $n \times n$ are given by the following

$$A_{ij} = \begin{pmatrix} a_{n(i-1)+1,n(j-1)+1} & a_{n(i-1)+1,n(j-1)+2} & \cdots & a_{n(i-1)+1,n(j-1)+n} \\ 0 & a_{n(i-1)+2,n(j-1)+2} & \cdots & a_{n(i-1)+2,n(j-1)+n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{n(i-1)+n,n(j-1)+n} \end{pmatrix}$$

More precisely,

$$A = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,2n} & \cdots & a_{1,n(m-1)+1} & \cdots & a_{1,nm} \\ 0 & \cdots & a_{2,n} & 0 & \cdots & a_{2,2n} & \cdots & 0 & \cdots & a_{2,nm} \\ 0 & \cdots & a_{3,n} & 0 & \cdots & a_{3,2n} & \cdots & 0 & \cdots & a_{3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n} & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\ a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1} & \cdots & a_{n+1,2n} & \cdots & a_{n+1,n(m-1)+1} & \cdots & a_{n+1,nm} \\ 0 & \cdots & a_{n+2,n} & 0 & \cdots & a_{n+2,2n} & \cdots & 0 & \cdots & a_{n+2,nm} \\ 0 & \cdots & a_{n+3,n} & 0 & \cdots & a_{n+3,2n} & \cdots & 0 & \cdots & a_{n+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n} & \cdots & 0 & \cdots & a_{2n,nm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & \cdots & a_{n(m-1)+1,n} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,2n} & \cdots & a_{n(m-1)+1,n(m-1)+1} & \cdots & a_{n(m-1)+1,nm} \\ 0 & \cdots & a_{n(m-1)+2,n} & 0 & \cdots & a_{n(m-1)+2,2n} & \cdots & 0 & \cdots & a_{n(m-1)+2,nm} \\ 0 & \cdots & a_{n(m-1)+3,n} & 0 & \cdots & a_{n(m-1)+3,2n} & \cdots & 0 & \cdots & a_{n(m-1)+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm} \end{pmatrix} \tag{2.1}$$

Let $f_n : \mathbb{Z}^+ \to \{1, 2, \ldots, n\}$ be a function defined by

$$f_n(k) = \begin{cases} k \pmod n, & k \nmid m \\ n, & k \mid m \end{cases}$$

If we denote the number $f_n(k)$ by $\overline{k}$, then the following notation is true for a matrix $A = (a_{i,j})_{nm\times nm}$ of dimension $nm \times nm$:

$$\overline{i} > \overline{j} \Rightarrow a_{i,j} = 0 \tag{2.2}$$

Theorem 2.1. Let $A_{nm\times nm}$ be a matrix satisfying (2.2) and a_{i_0,j_0} be an entry of the matrix A with $\overline{i_0} < \overline{j_0}$. Then, any product including the number a_{i_0,j_0} of the following determinant formula is zero:

$$|A| = \sum_{\sigma\in S_{nm}} \operatorname{sgn}(\sigma)\, a_{1,\sigma(1)}a_{2,\sigma(2)}\ldots a_{n,\sigma(n)}$$

Proof. Let σ in S_{nm} be a permutation with $\sigma(i_0) = j_0$. Then, $\sigma^{nm!} = e$ where e is the identity function in S_{nm}, i.e., $e(i) = i$, for all $1 \leq i \leq nm$. Assume that $\overline{\sigma^k(i_0)} \leq \overline{\sigma^{k+1}(i_0)}$, for all $k \in \mathbb{Z}^+$ with $1 \leq k < (nm)!$. Then,

$$\overline{i_0} = \overline{e(i_0)} = \overline{\sigma^{nm!}(i_0)} \geq \overline{\sigma^{nm!-1}(i_0)} \geq \overline{\sigma^{nm!-2}(i_0)} \geq \cdots \geq \overline{\sigma(i_0)} = \overline{j_0}$$

This contradicts the assumption $\overline{i_0} < \overline{j_0}$, i.e., there exists a number k_0 such that the relations $1 \leq k_0 < (nm)!$ and $\overline{\sigma^{k_0}(i_0)} > \overline{\sigma^{k_0+1}(i_0)}$ hold. If $\sigma^{k_0}(i_0)$ is denoted by α_0, then $\overline{\alpha_0} > \overline{\sigma(\alpha_0)}$ since $\sigma(\alpha_0) = \sigma\left(\sigma^{k_0}(i_0)\right) = \sigma^{k_0+1}(i_0)$. By (2.2), $a_{\alpha_0,\sigma(\alpha_0)} = 0$. Consequently,

$$a_{1,\sigma(1)}\ldots a_{i_0,\sigma(i_0)}\ldots a_{\sigma_0,\sigma(\sigma_0)}\ldots a_{n,\sigma(n)} = 0$$

□

Corollary 2.2. Let $A_{nm\times nm}$ be a matrix satisfying the condition (2.2). Then, the determinant of A depends only on the entries $a_{i,j}$ with $\overline{i} = \overline{j}$, i.e., the determinant depends only on the entries on the main diagonal in the submatrices A_{ij} of A in (1.1). The entries $a_{i,j}$ with $\overline{i} < \overline{j}$ in the submatrices A_{ij} do not affect the determinant. Therefore, when calculating the determinant of matrix A, for convenience, the entries $a_{i,j}$ with $\overline{i} < \overline{j}$ can be taken as 0. Consequently, the determinant of a matrix A as in (2.1) and the determinant of the following matrix are equal:

$$\tilde{A} = \begin{pmatrix}
a_{1,1} & \cdots & 0 & a_{1,n+1} & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{n,n} & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\
a_{n+1,1} & \cdots & 0 & a_{n+1,n+1} & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n} & \cdots & 0 & \cdots & a_{2n,nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & \cdots & 0 & a_{n(m-1)+1,n+1} & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\
0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm}
\end{pmatrix} \tag{2.3}$$

2.2. Factorization

We provide in this section a method for a factorization of the matrix $\tilde{A}$ in (2.3). Note that $\tilde{A}$ can be taken as an arbitrary matrix of dimension $nm \times nm$ with the condition $\bar{i} \neq \bar{j} \Rightarrow a_{i,j} = 0$. A factorization method for matrix $\tilde{A}$ is as follows:

$$\tilde{A} = \tilde{A}_1 \tilde{A}_2 \cdots \tilde{A}_n \tag{2.4}$$

where

$$\tilde{A}_1 = \begin{pmatrix}
a_{1,1} & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{n+1,1} & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & 0 & \cdots & 0 & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

$$\tilde{A}_2 = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & a_{2,2} & \cdots & 0 & 0 & a_{2,n+2} & \cdots & 0 & \cdots & 0 & a_{2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & a_{n+2,2} & \cdots & 0 & 0 & a_{n+2,n+2} & \cdots & 0 & \cdots & 0 & a_{n+2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & a_{n(m-1)+2,2} & \cdots & 0 & 0 & a_{n(m-1)+2,n+2} & \cdots & 0 & \cdots & 0 & a_{n(m-1)+2,n(m-1)+2} & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{pmatrix}$$

$$\vdots$$

$$\tilde{A}_n = \begin{pmatrix}
1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{n,n} & 0 & 0 & \cdots & a_{n,2n} & \cdots & 0 & 0 & \cdots & a_{n,nm} \\
0 & 0 & \cdots & 0 & 1 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{2n,n} & 0 & 0 & \cdots & a_{2n,2n} & \cdots & 0 & 0 & \cdots & a_{2n,nm} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 1 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nm,n} & 0 & 0 & \cdots & a_{nm,2n} & \cdots & 0 & 0 & \cdots & a_{nm,nm}
\end{pmatrix}$$

(2.4) can be observed by matrix multiplication.

2.3. Determinant Formula

We have proved that the determinant of A and the determinant of $\tilde{A}$ are equal in Corollary 2.2. We have factored the matrix $\tilde{A}$ in the former section. In this section, we give a formula for the determinant of the matrix A by calculating the determinants of the factors of the matrix $\tilde{A}$.

Theorem 2.3. Consider a block matrix A whose submatrices are triangular as in (2.1). Then, the following determinant formula is valid:

$$|A| = \prod_{k=1}^{n} \begin{vmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{vmatrix} \tag{2.5}$$

Proof. Consider the matrices $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ in (2.4). Hence,

$$|A| = \left|\tilde{A}_1\right| \left|\tilde{A}_2\right| \ldots \left|\tilde{A}_n\right|$$

The determinant of the matrix $\tilde{A}_1$ is calculated as follows: If we expand the determinant along the 2nd, 3rd, ..., nth rows, respectively, then

$$
\left|\tilde{A}_1\right| = \left|\begin{array}{ccccccccccccc}
a_{1,1} & 0 & \cdots & 0 & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 1 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
a_{n+1,1} & 0 & \cdots & 0 & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & 0 & \cdots & 0 & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}\right|
$$

$$
= \left|\begin{array}{cccccccccc}
a_{1,1} & a_{1,n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\
a_{n+1,1} & a_{n+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 1
\end{array}\right|
$$

We expand the last determinant along the 3rd, 4th, ..., $(n+1)$th rows, respectively:

$$
\left|\tilde{A}_1\right| = \left|\begin{array}{ccccccccccc}
a_{1,1} & a_{1,n+1} & a_{1,2n+1} & 0 & \cdots & 0 & \cdots & a_{1,n(m-1)+1} & 0 & \cdots & 0 \\
a_{n+1,1} & a_{n+1,n+1} & a_{n+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{n+1,n(m-1)+1} & 0 & \cdots & 0 \\
a_{2n+1,1} & a_{2n+1,n+1} & a_{2n+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{2n+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & 0 & 1 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & a_{n(m-1)+1,2n+1} & 0 & \cdots & 0 & \cdots & a_{n(m-1)+1,n(m-1)+1} & 0 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 1 & \cdots & 0 \\
0 & 0 & 0 & 0 & \cdots & 0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & 0 & \cdots & 1 & \cdots & 0 & 0 & \cdots & 1
\end{array}\right|
$$

If we continue this procedure, then

$$\left|\tilde{A}_1\right| = \begin{vmatrix} a_{1,1} & a_{1,n+1} & \cdots & a_{1,n(m-1)+1} \\ a_{n+1,1} & a_{n+1,n+1} & \cdots & a_{n+1,n(m-1)+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+1,1} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,n(m-1)+1} \end{vmatrix}$$

Similarly,

$$\left|\tilde{A}_k\right| = \begin{vmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{vmatrix}$$

□

We denote the matrix of dimension $m \times m$ on the right-hand side of the above relation by

$$\tilde{A}_k^* = \begin{pmatrix} a_{k,k} & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k} & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k} \end{pmatrix} \tag{2.6}$$

Theorem 2.3 shows that the determinants of the matrices $\tilde{A}_k$ and $\tilde{A}_k^*$ are equal.

Corollary 2.4. The matrix A of dimension $nm \times nm$ in (2.1) is invertible if and only if the matrix $\tilde{A}_k^*$, for all $1 \leq k \leq n$, of dimension $m \times m$ is invertible.

Example 2.5. Calculate the determinant of the matrices

$$\begin{pmatrix} 1 & -19 & 32 & -1 & 13 & 21 \\ 0 & -1 & -7 & 0 & -2 & 12 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ -2 & 22 & -24 & 3 & 5 & -9 \\ 0 & 2 & 17 & 0 & -1 & -23 \\ 0 & 0 & -1 & 0 & 0 & 3 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 3 & 20 & -1 & 12 & 2 & 52 \\ 0 & 1 & 0 & -1 & 0 & 5 \\ 3 & 25 & -3 & 32 & 1 & 78 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 9 & 5 & 74 & 6 & 10 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{pmatrix}$$

By Theorem 2.3,

$$\begin{vmatrix} 1 & -19 & 32 & -1 & 13 & 21 \\ 0 & -1 & -7 & 0 & -2 & 12 \\ 0 & 0 & 2 & 0 & 0 & 3 \\ -2 & 22 & -24 & 3 & 5 & -9 \\ 0 & 2 & 17 & 0 & -1 & -23 \\ 0 & 0 & -1 & 0 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ -2 & 3 \end{vmatrix} \begin{vmatrix} -1 & -2 \\ 2 & -1 \end{vmatrix} \begin{vmatrix} 2 & 3 \\ -1 & 3 \end{vmatrix} = 45$$

and

$$\begin{vmatrix} 3 & 20 & -1 & 12 & 2 & 52 \\ 0 & 1 & 0 & -1 & 0 & 5 \\ 3 & 25 & -3 & 32 & 1 & 78 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ -1 & 9 & 5 & 74 & 6 & 10 \\ 0 & 1 & 0 & -2 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 3 & -1 & 2 \\ 3 & -3 & 1 \\ -1 & 5 & 6 \end{vmatrix} \begin{vmatrix} 1 & -1 & 5 \\ 1 & 1 & -1 \\ 1 & -2 & 2 \end{vmatrix} = 312$$

2.4. Wronskian of the Trigonometric System

In this section, we calculate the trigonometric system $\cos p_1x$, $\sin p_1x$, $\cos p_2x$, $\sin p_2x$, $\cdots$, $\cos p_mx$, and $\sin p_mx$ where $p_1, p_2, \cdots, p_m$ are arbitrary real constants. These $2m$ functions are the fundamental solutions of the differential equation of order $2m$ corresponding to the characteristic equation

$$\left(t^2+p_1^2\right)\left(t^2+p_2^2\right)\cdots\left(t^2+p_m^2\right)=0$$

This polynomial contains no odd terms. Therefore, the Wronskian of any fundamental solutions of the corresponding differential equation is a constant, see [15]. Then, the Wronskian can be calculated at point 0:

$$W = W\left[\cos p_1x, \sin p_1x, \cos p_2x, \sin p_2x, \ldots, \cos p_mx, \sin p_mx\right]$$

$$= \begin{vmatrix} \cos p_1x & \sin p_1x & \cdots & \cos p_mx & \sin p_mx \\ -p_1\sin p_1x & p_1\cos p_1x & \cdots & -p_m\sin p_mx & p_m\cos p_mx \\ -p_1^2\cos p_1x & -p_1^2\sin p_1x & \cdots & -p_m^2\cos p_mx & -p_m^2\sin p_mx \\ p_1^3\sin p_1x & -p_1^3\cos p_1x & \cdots & p_m^3\sin p_mx & -p_m^3\cos p_mx \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1}p_1^{2m-2}\cos p_1x & (-1)^{m-1}p_1^{2m-2}\sin p_1x & \cdots & (-1)^{m-1}p_m^{2m-2}\cos p_mx & (-1)^{m-1}p_m^{2m-2}\sin p_mx \\ (-1)^{m}p_1^{2m-1}\sin p_1x & (-1)^{m-1}p_1^{2m-1}\cos p_1x & \cdots & (-1)^{m}p_m^{2m-1}\sin p_mx & (-1)^{m-1}p_m^{2m-1}\cos p_mx \end{vmatrix}$$

$$= \begin{vmatrix} \cos 0 & \sin 0 & \cdots & \cos 0 & \sin 0 \\ -p_1\sin 0 & p_1\cos 0 & \cdots & -p_m\sin 0 & p_m\cos 0 \\ -p_1^2\cos 0 & -p_1^2\sin 0 & \cdots & -p_m^2\cos 0 & -p_m^2\sin 0 \\ p_1^3\sin 0 & -p_1^3\cos 0 & \cdots & p_m^3\sin 0 & -p_m^3\cos 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1}p_1^{2m-2}\cos 0 & (-1)^{m-1}p_1^{2m-2}\sin 0 & \cdots & (-1)^{m-1}p_m^{2m-2}\cos 0 & (-1)^{m-1}p_m^{2m-2}\sin 0 \\ (-1)^{m}p_1^{2m-1}\sin 0 & (-1)^{m-1}p_1^{2m-1}\cos 0 & \cdots & (-1)^{m}p_m^{2m-1}\sin 0 & (-1)^{m-1}p_m^{2m-1}\cos 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 1 & 0 & \cdots & 1 & 0 \\ 0 & p_1 & 0 & p_2 & \cdots & 0 & p_m \\ -p_1^2 & 0 & -p_2^2 & 0 & \cdots & -p_m^2 & 0 \\ 0 & -p_1^3 & 0 & -p_2^3 & \cdots & 0 & -p_m^3 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ (-1)^{m-1}p_1^{2m-2} & 0 & (-1)^{m-1}p_2^{2m-2} & 0 & \cdots & (-1)^{m-1}p_m^{2m-2} & 0 \\ 0 & (-1)^{m-1}p_1^{2m-1} & 0 & (-1)^{m-1}p_2^{2m-1} & \cdots & 0 & (-1)^{m-1}p_m^{2m-1} \end{vmatrix}$$

The last determinant can be calculated by (2.5). It splits two factors:

$$W = \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -p_1^2 & -p_2^2 & \cdots & -p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1}p_1^{2m-2} & (-1)^{m-1}p_2^{2m-2} & \cdots & (-1)^{m-1}p_m^{2m-2} \end{vmatrix} \begin{vmatrix} p_1 & p_2 & \cdots & p_m \\ -p_1^3 & -p_2^3 & \cdots & -p_m^3 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1}p_1^{2m-1} & (-1)^{m-1}p_2^{2m-1} & \cdots & (-1)^{m-1}p_m^{2m-1} \end{vmatrix}$$

$$= \left(\prod_{k=1}^{m} p_k\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ -p_1^2 & -p_2^2 & \cdots & -p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ (-1)^{m-1}p_1^{2m-2} & (-1)^{m-1}p_2^{2m-2} & \cdots & (-1)^{m-1}p_m^{2m-2} \end{vmatrix}^2$$

$$= \left(\prod_{k=1}^{m} p_k\right) \begin{vmatrix} 1 & 1 & \cdots & 1 \\ p_1^2 & p_2^2 & \cdots & p_m^2 \\ \vdots & \vdots & \ddots & \vdots \\ p_1^{2m-2} & p_2^{2m-2} & \cdots & p_m^{2m-2} \end{vmatrix}^2$$

The last determinant is the Vandermonde determinant of the numbers p_1^2, p_2^2, $\cdots$, p_m^2 and it is calculated by multiplying the differences between them (for more details, see [16]). Then, the Wronskian of the trigonometric system takes the following form:

$$W = \left(\prod_{k=1}^{m} p_k\right)\left(\prod_{1\leq i<j\leq m}\left(p_j^2 - p_i^2\right)\right)^2$$

Thanks to the Theorem 2.3, the proof of the last Wronskian formula is much shorter and simpler than that in [17], provided by Kaya.

Corollary 2.6. The necessary and sufficient conditions for the linear independence of trigonometric system $\cos p_1x$, $\sin p_1x$, $\cos p_2x$, $\sin p_2x$, $\cdots$, $\cos p_mx$, and $\sin p_mx$ are the following:

i. $p_k \neq 0$, for all $k \in \overline{1,m}$

ii. $p_i \neq p_j$ and $p_i \neq -p_j$, for all $i \neq j$

Corollary 2.7. The Wronskian of the particular trigonometric system $\cos x$, $\sin x$, $\cos 2x$, $\sin 2x$, $\cdots$, $\cos mx$, and $\sin mx$ is

$$m!\left(\prod_{1\leq i<j\leq m}\left(j^2 - i^2\right)\right)^2$$

2.5. Some Properties of Block Matrices Whose Submatrices are Triangular

This section provides some properties of block matrices whose submatrices are triangular, such as sum, product, adjoint, inverse, and eigenvalues.

Theorem 2.8. The sum and product of two matrices of type (2.1) are also of type (2.1). Besides, the adjoint matrix of a matrix of type (2.1) is also of type (2.1).

Proof. The first part of the theorem can be easily proved. Therefore, we prove the second part of the theorem. It is sufficient that the cofactor of an entry a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$ is equal to 0. According to Corollary 2.2, the determinant of a matrix as in (2.1) is independent of the variable a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$. Then, the derivative of the determinant of a matrix as in (2.1) concerning a_{i_0,j_0} is 0. On the other hand, Jacobi's formula [18] for the matrix analysis says that the cofactor of an entry in a square matrix depending on the variables a_{i_0,j_0} is the derivative of the determinant of the matrix according to the considered entry. Then, the cofactors of the entries a_{i_0,j_0} with $\overline{i_0} < \overline{j_0}$ are equal to 0. □

Corollary 2.9. If a matrix as in (2.1) has an inverse, then the inverse is also of type (2.1).

The proof is obtained from the equality:

$$A^{-1} = \frac{1}{|A|}\mathrm{adj}(A)$$

Theorem 2.10. Let λ be a complex number. Then, λ is an eigenvalue of a matrix A as in (2.1) if and only if there exists a number $k \in \overline{1,n}$ such that λ is an eigenvalue of the matrix $\tilde{A}_k^*$ in (2.6).

Proof. λ is an eigenvalue of the matrix A if and only if the following relation holds:

$$|A-\lambda I|=\begin{vmatrix} a_{1,1}-\lambda & \cdots & a_{1,n} & a_{1,n+1} & \cdots & a_{1,2n} & \cdots & a_{1,n(m-1)+1} & \cdots & a_{1,nm} \\ 0 & \cdots & a_{2,n} & 0 & \cdots & a_{2,2n} & \cdots & 0 & \cdots & a_{2,nm} \\ 0 & \cdots & a_{3,n} & 0 & \cdots & a_{3,2n} & \cdots & 0 & \cdots & a_{3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{n,n}-\lambda & 0 & \cdots & a_{n,2n} & \cdots & 0 & \cdots & a_{n,nm} \\ a_{n+1,1} & \cdots & a_{n+1,n} & a_{n+1,n+1}-\lambda & \cdots & a_{n+1,2n} & \cdots & a_{n+1,n(m-1)+1} & \cdots & a_{n+1,nm} \\ 0 & \cdots & a_{n+2,n} & 0 & \cdots & a_{n+2,2n} & \cdots & 0 & \cdots & a_{n+2,nm} \\ 0 & \cdots & a_{n+3,n} & 0 & \cdots & a_{n+3,2n} & \cdots & 0 & \cdots & a_{n+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{2n,n} & 0 & \cdots & a_{2n,2n}-\lambda & \cdots & 0 & \cdots & a_{2n,nm} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n(m-1)+1,1} & \cdots & a_{n(m-1)+1,n} & a_{n(m-1)+1,n+1} & \cdots & a_{n(m-1)+1,2n} & \cdots & a_{n(m-1)+1,n(m-1)+1}-\lambda & \cdots & a_{n(m-1)+1,nm} \\ 0 & \cdots & a_{n(m-1)+2,n} & 0 & \cdots & a_{n(m-1)+2,2n} & \cdots & 0 & \cdots & a_{n(m-1)+2,nm} \\ 0 & \cdots & a_{n(m-1)+3,n} & 0 & \cdots & a_{n(m-1)+3,2n} & \cdots & 0 & \cdots & a_{n(m-1)+3,nm} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \cdots & \vdots & \ddots & \vdots \\ 0 & \cdots & a_{nm,n} & 0 & \cdots & a_{nm,2n} & \cdots & 0 & \cdots & a_{nm,nm}-\lambda \end{vmatrix}=0$$

By (2.5), the last relation can be rewritten as follows:

$$|A-\lambda I|=\prod_{k=1}^{n}\begin{vmatrix} a_{k,k}-\lambda & a_{k,n+k} & \cdots & a_{k,n(m-1)+k} \\ a_{n+k,k} & a_{n+k,n+k}-\lambda & \cdots & a_{n+k,n(m-1)+k} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n(m-1)+k,k} & a_{n(m-1)+k,n+k} & \cdots & a_{n(m-1)+k,n(m-1)+k}-\lambda \end{vmatrix}=0$$

□

3. Conclusion

This paper proves that the determinant of a large-scale block matrix whose submatrices are triangular does not need to be computed using classical and computational methods. The determinant of such matrices is equal to the product of the determinants of their special submatrices. This method greatly reduces the computational effort involved in calculating the determinant.

While the results presented in this paper significantly simplify the calculation of determinants for block matrices with triangular submatrices, several promising directions remain for future research. One area of potential exploration is the extension of these methods to non-triangular block matrices or matrices with more complex structural patterns. Additionally, investigating the applications of these findings in other branches of linear algebra, such as in solving systems of linear equations or in eigenvalue analysis, could provide further insights. Researchers may also consider applying these techniques to real-world problems in physics, engineering, or data science, where large-scale matrix computations are essential. Finally, developing more advanced computational tools and algorithms that leverage the factorization methods discussed here could contribute to faster and more efficient determinant calculations in large matrices.

Author Contributions

All the authors equally contributed to this work. This paper is derived from the second author's master's thesis supervised by the first author. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

Ethical Review and Approval

No approval from the Board of Ethics is required.

References

[1] M. Saadetoglu, Ş. M. Dinsev, *Inverses and determinants of $n \times n$ block matrices*, Mathematics 11 (17) (2023) 3784 12 pages.

[2] M. El-Mikkawy, *A note on a three-term recurrence for a tridiagonal matrix*, Applied Mathematics and Computation 139 (2–3) (2003) 503–511.

[3] N. D. Buono, G. Pio, *Nonnegative matrix tri-factorization for co-clustering: An analysis of the block matrix*, Information Sciences 301 (2015) 13–26.

[4] Y. Zhang, M. Zhang, Y. Liu, S. Ma, S. Feng, *Localized matrix factorization for recommendation based on matrix block diagonal forms*, in: D. Schwabe, V. Almeida, H. Glaser (Eds.), Proceedings of the 22nd International Conference on World Wide Web, Rio de Janeiro, 2013, pp. 1511–1520.

[5] E. Kamgnia, L. B. Nguenang, *Some efficient methods for computing the determinant of large sparse matrices*, Revue Africaine de Recherche en Informatique et Mathématiques Appliquées 17 (2014) 73–92.

[6] R. Schachtner, G. Pöppel, E. W. Lang, *Towards unique solutions of nonnegative matrix factorization problems by a determinant criterion*, Digital Signal Processing 21 (4) (2011) 528–534.

[7] J. T. Jia, J. Wang, T. F. Yuan, K. K. Zhang, B. M. Zhong, *An incomplete block-diagonalization approach for evaluating the determinants of bordered k-tridiagonal matrices*, Journal of Mathematical Chemistry 60 (8) (2022) 1658–1673.

[8] M. S. Solary, *From matrix polynomial to the determinant of block Toeplitz-Hessenberg matrix*, Numerical Algorithms 94 (3) (2023) 1421–1434.

[9] P. Sakkaplangkul, N. Chuenjarern, *Computational efficiency for calculating determinants of block matrices*, RMUTP Research Journal Sciences and Technology 18 (1) (2024) 38–46.

[10] J. Liu, J. Bi, M. Li, *Secure outsourcing of large matrix determinant computation*, Frontiers of Computer Science 14 (6) (2020) Article Number 146807 12 pages.

[11] D. Grinberg, P. J. Olver, *The n body matrix and its determinant*, SIAM Journal on Applied Algebra and Geometry 3 (1) (2019) 67–86.

[12] J. Y. Shao, H. Y. Shan, L. Zhang, *On some properties of the determinants of tensors*, Linear Algebra and Its Applications 439 (10) (2013) 3057–3069.

[13] K. Neymeyr, M. Sawall, *On the set of solutions of the nonnegative matrix factorization problem*, SIAM Journal on Matrix Analysis and Applications 39 (2) (2018) 1049–1069.

[14] M. Cè, L. Giusti, S. Schaefer, *Local factorization of the fermion determinant in lattice QCD*, Physical Review D 95 (3) (2017) 034503 13 pages.

[15] W. E. Boyce, R.C. DiPrima, Elementary differential equations, 10th Edition, Wiley, New York, 2012.

[16] S. Lipschutz, Linear algebra, 4th Edition, McGraw-Hill, New York, 2009.

[17] U. Kaya, *Wronski determinant of trigonometric system*, Journal of Advanced Mathematics and Mathematics Education 5 (1) (2022) 1–8.

[18] R. Bellman, Introduction to matrix analysis, Society for Industrial and Applied Mathematics, Philadelphia, 1987.