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# Minimal Curves on Ruled Surfaces Generated by Legendre Curves

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**Abstract** — In this paper, we study the conditions under which Legendre curves on ruled surfaces are classified as minimal loci. By investigating the scenario where the directive curve is a binormal vector, we establish the criteria for these curves to have minimal loci on B-scroll-ruled surfaces. Furthermore, we present illustrative examples showcasing concrete instances of minimal curves within this context. Finally, we discuss the need for further research.

**Keywords** *Legendre curves, minimal curves, B-scroll surfaces*

**Mathematics Subject Classification (2020)** 53A10, 53A04

## 1. Introduction

A ruled surface is one of the special surfaces represented by moving a straight line continuously along a space curve called the base curve. More explicitly, a surface $\mathcal{M}$ in $\mathbb{R}^3$ is called a ruled surface if it admits a parameterization $\Phi_{(\sigma,\gamma)} : I \times J \to \mathcal{M}$ which consists of a collection of a one-parameter family of straight lines indexed by $u$ in the form of $\Phi_{(\sigma,\gamma)}(s,u) = \sigma(s) + u\gamma(s)$ where $s \in I$ and $u \in J$ such that $I$ and $J$ are open intervals in $\mathbb{R}$ [1]. Here, $\sigma$ and $\gamma$ are smooth mappings defined from the interval $I$ to $\mathbb{R}^3$. Moreover, $\sigma$ is the base curve or directrix, and the non-null curve $\gamma$ is the director curve. The straight lines $u \to \sigma(s) + u\gamma(s)$ are the rulings. Ruled surfaces have many important applications in various fields of science and technology, such as computer-aided geometric design (CAGD), architectural designing, manufacturing technology, and robotics [2–4]. In recent years, there has been intensive research on ruled surfaces in various spaces, including Euclidean, Lorentzian, and Minkowski spaces [5–7], where important properties and characterizations are presented. In [8], two developable ruled surfaces have been introduced using the principal normal indicatrix of a regular space as a base curve for both surfaces and the tangent indicatrix and the binormal indicatrix as the director curves. Afterward, [9] has extended the work of [8] by providing the condition for a minimal locus for the related surfaces.

In [10], a moving frame of a Legendre curve in the unit tangent bundle has been introduced, and a pair of smooth functions of a Legendre curve, analogous to the curvature of a regular plane curve, has been defined. The existence and uniqueness of Legendre curves have been proved. Legendre curves on the unit tangent bundle have been provided in [11] using rotation-minimizing (RM) vector fields.

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Besides, [12] has demonstrated that any Legendre curve in $TS^2$ corresponds to a developable ruled surface. For recent studies on Legendre curves, we refer to [13,14]. In [15], a ruled surface's mean and Gaussian curvatures are expressed in terms of its striction and director curves.

The remainder of the present study is organized as follows: Section 2 presents fundamental concepts and properties to be used in the following sections. Section 3 introduces two main results: Theorem 3.1 and Theorem 3.2. In Theorem 3.1, we calculate the mean and Gaussian curvature of a ruled surface, generating a Legendre curve together with the director curve, whose base curve differs from those in [15]. Hence, the results obtained in [15] are simplified. In Theorem 3.2, we investigate the conditions under which Legendre curves on ruled surfaces are classified as minimal loci. Section 4 obtains the criteria for these curves to be minimal loci on B-scroll ruled surfaces by taking the directive curve as a binormal vector, as provided in Theorem 4.1. Moreover, in Theorem 4.3, we obtain the minimality condition for the developable ruled surface with a special choice of the base curve. Section 5 provides some computational examples of minimal curves. The final section is dedicated to conclusions and final remarks.

## 2. Preliminaries

This section presents some basic concepts of the theory of surfaces in $\mathbb{R}^3$ to be needed to prove our main results. For a detailed discussion on the related subjects, see [1,16].

Let $\Phi(s,u)$ be a regular surface with the first and second fundamental forms

$$Eds^2 + 2Fdsdu + Gdu^2 \quad \text{and} \quad Lds^2 + 2Mdsdu + Ndu^2$$

respectively. Traditionally, the mean and Gaussian curvatures of the surface are provided in the following form:

$$H = \frac{LG - 2MF + NE}{2\left(EG - F^2\right)} \quad \text{and} \quad K = \frac{LN - M^2}{EG - F^2}$$

where $E$, $F$, and $G$ and $L$, $M$, and $N$ are the coefficients of the first and second fundamental forms, respectively. In [15], for a ruled surface $\Phi\left(s,u\right) = \sigma(s) + u\gamma(s)$, the mean curvature is presented in the following form:

$$H = \frac{\langle \sigma'' + u\gamma'', \sigma' \times \gamma + u\left(\gamma' \times \gamma\right)\rangle - 2\left\langle \sigma', \gamma\right\rangle \left\langle \gamma', \sigma' \times \gamma\right\rangle}{2\left(EG - F^2\right)^{\frac{3}{2}}} \tag{2.1}$$

where $\sigma(s)$ is a striction curve on the surface. Moreover, the Gauss curvature is formulated as follows:

$$K = -\frac{\langle \gamma', \sigma' \times \gamma\rangle^2}{\left(\|\sigma' \times \gamma\|^2 + u^2\|\gamma'\|^2\right)\left(\|\sigma'\|^2 + u^2\|\gamma'\|^2 - \langle \sigma', \gamma\rangle^2\right)} \tag{2.2}$$

A minimal locus of a ruled surface is defined as follows:

**Definition 2.1.** Let $\Phi(s,u) = \sigma(s) + u\gamma(s)$ be a ruled surface. If the mean curvature of the surface $\Phi(s,u)$ along the curve $X(s) = \sigma(s) + u(s)\gamma(s)$ is zero, then the curve $X(s)$ is called the minimal curve on $\Phi(s,u)$.

Legendre curves can be provided by the following definition:

**Definition 2.2.** The smooth curve $\Gamma(s) = (\alpha(s), v(s)) : I \subset \mathbb{R} \to \mathcal{M}$ is called a Legendre curve in $\mathcal{M}$ if $\langle \alpha'(s), v(s)\rangle = 0$.

It can be observed that $\langle \alpha(s), v(s)\rangle = 0$ for a smooth curve $\Gamma(s) = (\alpha(s), v(s))$ in $\mathcal{M}$. Thus, we can define a new frame using the unit vector $\eta = \alpha(s) \times v(s)$ where the symbol $\times$ denotes the usual vector product in $\mathbb{R}^3$ [11]. It is obvious that

$$\langle \alpha(s), \eta(s) \rangle = \langle v(s), \eta(s) \rangle = 0$$

Hence, the following Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\alpha(s)$ is obtained as follows:

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & m(s) & n(s) \\ -m(s) & 0 & l(s) \\ -n(s) & -l(s) & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix} \tag{2.3}$$

where

$$l(s) = \langle \alpha'(s), v(s) \rangle, \quad m(s) = -\langle \alpha'(s), \eta(s) \rangle, \quad \text{and} \quad n(s) = -\langle v'(s), \eta(s) \rangle$$

Here, the elements of the set $\{l, m, n\}$ are called the curvature functions of $\Gamma$.

If $l(s) = 0$, then the curve $\Gamma(s) = (\alpha(s), v(s))$ is Legendre in $\mathcal{M}$ with the curvature functions $(m, n)$. Then, the Frenet frame provided in (2.3) for the Legendre condition can be provided by

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & m(s) & n(s) \\ -m(s) & 0 & 0 \\ -n(s) & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix}$$

For the Legendre curve $(T(s), B(s))$, the definition of B-scroll is as follows:

**Definition 2.3.** Let $\{T, N, B\}$ be the Frenet frame of the unit speed curve $\sigma\colon I \to E^3$. The ruled surface formed by the binormal vector $B$ along the curve $\sigma$ is called the binormal scroll (briefly B-scroll). Here, the curve $\sigma$ is called the base curve of the B-scroll, and the binormal vector $B$ is called its director curve. The parametric equation of B-scroll is written as follows: $\Phi(s, u) = \sigma(s) + uB(s)$.

The notion of B-scroll surfaces has been introduced in [17]. The B-scroll's first and second fundamental forms with Cartan framed null directrix in the Minkowskian 3-space are investigated in [18]. B-scrolls in 3-dimensional Lorentzian space $L^3$ are studied in [19].

## 3. The Legendre Curves and Minimal Curves on Ruled Surface

In this study, we consider a ruled surface

$$\Phi(s, u) = \int \lambda(s)\alpha(s)ds + uv(s)$$

where $(\alpha(s), v(s))$ is a smooth Legendre curve. We present the mean curvature of this surface in the following theorem.

**Theorem 3.1.** Let $(\alpha(s), v(s))$ be a Legendre curve. For the Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\alpha(s)$, the mean and Gauss curvatures of the ruled surface $\Phi(s, u) = \int \lambda(s)\alpha(s)ds + uv(s)$ are as follows, respectively:

$$H = \frac{-n^2(s)m(s)u^2 + (\lambda'(s)n(s) - \lambda(s)n'(s))\,u - m(s)\lambda^2(s)}{2\left(\lambda^2(s) + u^2n^2\right)^{\frac{3}{2}}} \tag{3.1}$$

and

$$K = -\frac{\lambda^2 n^2}{\left(\lambda^2 + u^2 n^2\right)^2}$$

Proof. Consider

$$\sigma(s) = \int \lambda(s)\alpha(s)ds, \;\; \gamma(s) = v(s)$$

By the formula in (2.1), the following equalities hold.

$$\langle \sigma'(s), \gamma(s) \rangle = \langle \lambda(s)\alpha(s), v(s) \rangle = 0 \tag{3.2}$$

$$\langle\sigma'(s),\gamma'(s)\rangle = \langle\lambda(s)\alpha(s), v'(s)\rangle = \langle\lambda(s)\alpha(s), -n(s)\eta(s)\rangle = 0 \tag{3.3}$$

$$\sigma''(s) + u\gamma''(s) = -\left(m(s)\lambda(s) + un'(s)\right)\eta(s) + \left(\lambda'(s) - un(s)m(s)\right)\alpha(s) - un^2(s)v(s) \tag{3.4}$$

and

$$\sigma'(s)\times\gamma(s) + u\left(\gamma'(s)\times\gamma(s)\right) = \lambda(s)\eta(s) + un(s)\alpha(s) \tag{3.5}$$

Moreover, we need to compute the first fundamental forms of the surface $\Phi(s,u)$ to evaluate the mean curvature $H$. By the equality $\Phi(s,u) = \sigma(s) + u\gamma(s)$, $\Phi_s = \sigma'(s) + u\gamma'(s)$ and $\Phi_u = \gamma(s)$. Therefore,

$$\begin{aligned} E &= \|\Phi_s\|^2 = \|\sigma'(s) + u\gamma'(s)\|^2 \\ &= \langle\sigma'(s) + u\gamma'(s), \sigma'(s) + u\gamma'(s)\rangle \\ &= \|\sigma'\|^2 + u^2\|\gamma'\|^2 + 2u\langle\sigma',\gamma'\rangle \end{aligned}$$

By using (3.3), the equality $E = \|\sigma'\|^2 + u^2\|\gamma'\|^2$ is obtained. Besides, $G = \|\Phi_u\|^2 = \|\gamma\|^2 = 1$. Similarly,

$$F = \langle\Phi_s,\Phi_u\rangle = \langle\sigma'(s) + u\gamma'(s), \gamma(s)\rangle = \langle\sigma'(s),\gamma(s)\rangle + u\langle\gamma'(s),\gamma(s)\rangle$$

Since $\|\gamma\| = 1$, i.e., $\langle\gamma,\gamma\rangle = 1$, then $\langle\gamma',\gamma\rangle = 0$ which implies $F = \langle\sigma'(s),\gamma(s)\rangle$. Moreover,

$$EG - F^2 = \|\sigma'\|^2 + u^2\|\gamma'\|^2 - \left(\langle\sigma'(s),\gamma(s)\rangle\right)^2 = \lambda^2 + u^2n^2$$

Using (3.2), (3.4), and (3.5) in (2.1),

$$H = \frac{-n^2(s)m(s)u^2 + \left(\lambda'(s)n(s) - \lambda(s)n'(s)\right)u - m(s)\lambda^2(s)}{2\left(\lambda^2(s) + u^2n^2\right)^{\frac{3}{2}}}$$

In addition, we have the following equalities to evaluate equality (2.2):

$$\langle\gamma',\sigma'\times\gamma\rangle = \langle -n\eta, \lambda\eta\rangle = -\lambda n \tag{3.6}$$

$$\|\sigma'\times\gamma\|^2 + u^2\|\gamma'\|^2 = \|\lambda\eta\|^2 + u^2\|-n\eta\|^2 = \lambda^2 + u^2n^2 \tag{3.7}$$

and

$$\|\sigma'\|^2 + u^2\|\gamma'\|^2 - \langle\sigma',\gamma\rangle^2 = \|\lambda\alpha\|^2 + u^2\|-n\eta\|^2 = \lambda^2 + u^2n^2 \tag{3.8}$$

If we substitute (3.6)-(3.8) into (2.2), then $K = -\frac{\lambda^2n^2}{(\lambda^2+u^2n^2)^2}$. □

In the following theorem, we provide the necessary condition for a curve to be minimal on the surface $\Phi(s,u)$.

**Theorem 3.2.** Let $(\alpha(s), v(s))$ be a smooth Legendre curve. Then, the curve $\beta(s) = \int\lambda(s)\alpha(s)ds + u_{1,2}v(s)$ is minimal on the ruled surface $\Phi(s,u) = \int\lambda(s)\alpha(s)ds + uv(s)$ where

$$u_{1,2} = \frac{\lambda'n - \lambda n' \pm \sqrt{\left(\lambda'n - \lambda n'\right)^2 - \left(4mn\lambda\right)^2}}{2mn^2}$$

Proof. From (3.1), the condition for the curve $\beta(s)$ to be a minimal curve on the surface $\Phi(s,u)$ is as follows:

$$n^2(s)m(s)u^2 + \left(\lambda(s)n'(s) - \lambda'(s)n(s)\right)u + m(s)\lambda^2(s) = 0 \tag{3.9}$$

Solution of this second-order equation is

$$u_{1,2} = \frac{\lambda'n - \lambda n' \pm \sqrt{\left(\lambda'n - \lambda n'\right)^2 - \left(4mn\lambda\right)^2}}{2mn^2}$$

which completes the proof. □

As a special case for $u = \lambda(s)$, we provide the following proposition:

**Proposition 3.3.** Let $(\alpha(s), v(s))$ be a smooth Legendre curve. Then, the curve $\beta(s)$ is minimal on the ruled surface $\Phi(s,u) = \int \lambda(s)\alpha(s)ds + uv(s)$ where

$$u = e^{\int \frac{\left(1+n^2(s)\right)m(s)+n'(s)}{n(s)}ds}$$

Proof. By (3.9), if $u = \lambda(s)$, then the following equation holds:

$$\lambda'(s)n(s) + \lambda(s)\left(-m(s) - n'(s) - n^2(s)m(s)\right) = 0$$

The solution of the equation is

$$\lambda(s) = e^{\int \frac{\left(1+n^2(s)\right)m(s)+n'(s)}{n(s)}ds}$$

Thus, for $u = \lambda(s)$, the curve $\beta(s) = \int \lambda(s)\alpha(s)ds + uv(s)$ is minimal on the ruled surface $\Phi(s,u)$. □

Specifically taking $\alpha(s) = T(s)$ and $v(s) = B(s)$, we get the following proposition.

**Proposition 3.4.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth Legendre curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. Then, the curve $\beta(s) = \int \lambda(s)\alpha(s)ds + uv(s)$ is minimal on a ruled surface $\Phi(s,u) = \int \lambda(s)T(s)ds + uB(s)$ where

$$u = \lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s)-\kappa(s)+\tau'(s)\right)ds}$$

Proof. After the necessary calculations, then the following equalities are obtained:

$$\sigma''(s) + u''(s) = \left(\lambda'(s) + u\kappa(s)\tau(s)\right)T(s) + \left(\lambda(s)\kappa(s) - u\tau'(s)\right)N(s) - u\tau^2(s)B(s)$$

and

$$\sigma'(s) \times \gamma(s) + u\left(\gamma'(s) \times \gamma(s)\right) = -u\tau(s)T(s) - \lambda(s)N(s)$$

Hence, the condition of the minimal curve on the ruled surface $\Phi(s,u)$ is equivalent to the following equation:

$$-u\tau(s)\lambda'(s) - u^2\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda(s)u\tau'(s) = 0$$

For $u = \lambda(s)$,

$$-u\tau(s)\lambda'(s) - u^2\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda(s)u\tau'(s) = 0$$

$$-\lambda(s)\lambda'(s)\tau(s) - \lambda^2(s)\kappa(s)\tau^2(s) - \lambda^2(s)\kappa(s) + \lambda^2(s)\tau'(s) = 0$$

$$-\lambda'(s)\tau(s) + \lambda(s)\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right) = 0 \tag{3.10}$$

After the integration of both sides of (3.10),

$$\int \frac{\lambda'(s)}{\lambda(s)}ds = \int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right)ds$$

which yields

$$\lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s)-\kappa(s)+\tau'(s)\right)ds}$$

Hence, for $u = \lambda(s)$, the curve $\beta(s)$ is minimal on the ruled surface $\Phi(s,u)$.

□

In particular, if $\lambda(s) = 1$, then we obtain a B-scroll ruled surface.

## 4. Minimal Curves on B-scroll Surfaces

In the following theorem, we investigate the condition that a curve is a minimal curve on the B-scroll surface.

**Theorem 4.1.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth Legendre curve with frame apparatus $\{T, N, B, \kappa, \tau\}$ and consider the B-scroll surface

$$\Phi(s,u) = \int T(s)ds + uB(s) = \alpha(s) + uB(s)$$

If $\sigma(s) = \alpha(s)$ and $\gamma(s) = B(s)$, then the curves $\beta(s) = \alpha(s) + u_{1,2}B(s)$ are minimal on the B-scroll ruled surface $\Phi(s,u)$ such that

$$u_{1,2} = \frac{\tau'(s) \pm \sqrt{\tau^2(s) - 4\kappa^2(s)\tau^2(s)}}{2\kappa(s)\tau^2(s)}$$

Proof. If the equalities

$$\sigma''(s) + u\gamma''(s) = u\kappa(s)\tau(s)T(s) + \left(\kappa(s) - u\tau'(s)\right)N(s) - u\tau^2(s)B(s)$$

and

$$\sigma'(s) \times \gamma(s) + u\left(\gamma'(s) \times \gamma(s)\right) = -u\tau(s)T(s) - N(s)$$

are used in (2.1), then the following relation is obtained as a minimality condition:

$$-\kappa(s)\tau^2(s)u^2 + \tau'(s)u - \kappa = 0 \tag{4.1}$$

The solution of (4.1) is obtained as follows:

$$u_{1,2} = \frac{\tau'(s) \pm \sqrt{\tau^2(s) - 4\kappa^2(s)\tau^2(s)}}{2\kappa(s)\tau^2(s)}$$

Therefore the curves $\beta(s) = \alpha(s) + u_{1,2}B(s)$ are minimals on B-scroll surface. Here, if $\tau$ is arbitrary constant and $\kappa = \dfrac{1}{2}$, then $u_1 = u_2 = \frac{\tau'(s)}{\tau^2(s)}$. □

Hence, we have the following proposition.

**Proposition 4.2.** Let $\alpha \subset E^3$ be a Salkowski curve with arbitrary $\tau$ and $\kappa = \dfrac{1}{2}$. Then, the curve $\beta(s) = \alpha(s) + \dfrac{\tau'(s)}{\tau^2(s)}B(s)$ are minimal on the B-scroll surface.

We present a theorem for a developable ruled surface whose base curves are minimal.

**Theorem 4.3.** Let $\gamma(s)$ be a unit speed curve. Then, the mean curvature of the developable ruled surface

$$\Phi(s,u) = \int f(s)\gamma(s)ds + u\gamma(s) \tag{4.2}$$

along its base curve, the striction curve for $u = 0$ is zero.

Proof. Consider the following Frenet frame $\{\gamma, T, S\}$ along $\gamma(s)$:

$$\begin{pmatrix} \gamma' \\ T' \\ S' \end{pmatrix} = \begin{pmatrix} 0 & m(s) & 0 \\ -m(s) & 0 & n(s) \\ 0 & -m(s) & 0 \end{pmatrix} \begin{pmatrix} \gamma \\ T \\ S \end{pmatrix}$$

Then,

$$\sigma'(s) = f(s)\gamma(s), \quad \sigma''(s) = f'(s)\gamma(s) + f(s)m(s)T(s)$$

$$\gamma'(s) = m(s)T(s), \quad \gamma''(s) = -m^2\gamma(s) + m'(s)T(s) + m(s)n(s)S(s)$$

and

$$\sigma'(s) \times \gamma(s) = 0, \quad \gamma'(s) \times \gamma(s) = -m(s)S(s)$$

If we substitute these equalities into formula (2.1), then the following minimality condition is obtained:

$$\begin{aligned}\langle f'(s)\gamma + f(s)m(s)T + u(-m^2\gamma(s) + m'(s)T + m(s)n(s)S), -um(s)S\rangle &= \langle m(s)n(s)S(s), -um(s)S(s)\rangle \\ &= -u^2m^2(s)n(s) \\ &= 0\end{aligned}$$

Hence, the condition is a minimal curve on the surface $\Phi(s,u)$ in the following form: $u = 0$, $m \neq 0$, and $n \neq 0$. $\square$

By using Theorem 4.3, we provide two examples for the surfaces in [8], minimal along the striction curve:

**Example 4.4.** Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. By (4.2), if $\gamma(s) = N(s)$ and $f(s) = k(s)$, then the following ruled surface is obtained:

$$\Phi\,(s,u) = T(s) + uN(s) \tag{4.3}$$

The striction curve $T(s)$ is minimal on the ruled surface (4.3) for $u = 0$.

**Example 4.5.** Let $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ be a unit speed curve with frame apparatus $\{T, N, B, \kappa, \tau\}$. By (4.2), if $\gamma(s) = N(s)$ and $f(s) = -\tau(s)$, the ruled surface $\Phi\,(s,u) = B(s) + uN(s)$ is obtained. Thus, the base curve $B(s)$ is minimal on this surface.

## 5. Some Computational Examples

This section provides two computational examples to illustrate the obtained results.

**Example 5.1.** Let $\alpha : I \subset \mathbb{R} \to \mathbb{R}^3$ be a smooth curve defined by $\alpha(s) = (\cos s, \sin s, s)$. Then, the tangent and binormal vector fields of $\alpha$ are as follows, respectively:

$$T(s) = (-\sin s, \cos s, 1) \quad \text{and} \quad B(s) = \frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$$

with the curvature $\kappa = \dfrac{1}{2}$ and the torsion $\tau = \dfrac{1}{2}$. The curve $\Gamma(s) = (T(s), B(s))$ is Legendre in the ruled surface $\Phi\,(s,u) = \int \lambda T(s)ds + uB(s)$. If $\lambda(s) = e^{\int \frac{1}{\tau(s)}\left(-\kappa(s)\tau^2(s) - \kappa(s) + \tau'(s)\right)ds} = e^{-\frac{5}{4}s}$, then the ruled surface $\Phi\,(s,u) = \int e^{-\frac{5}{4}s}(-\sin s, \cos s, 1) + u\frac{1}{\sqrt{2}}(\sin s, -\cos s, 1)$ is obtained (see Figure 1).

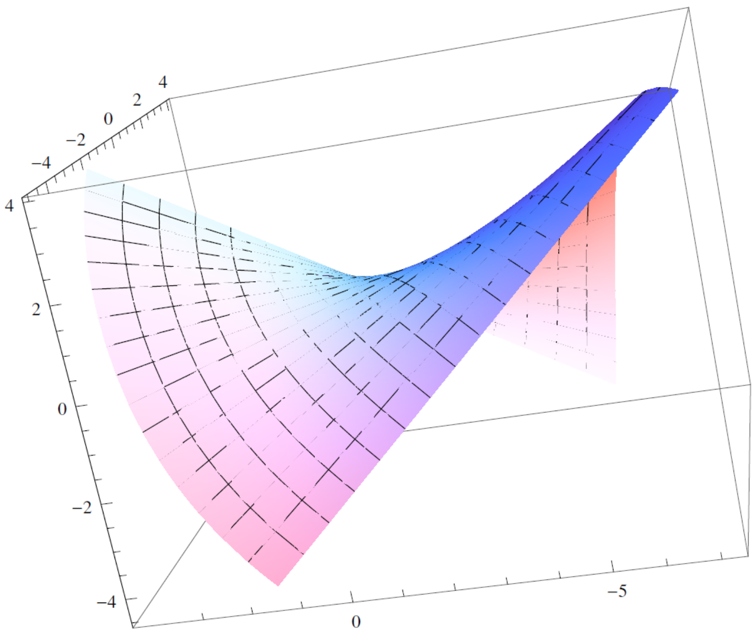


**Figure 1.** Ruled surface $\Phi_{(T,B)}$

Moreover, the curve

$$\beta(s) = e^{-\frac{5}{4}s}\left(\frac{16}{41}\cos s + \left(\frac{20}{41}+\frac{1}{\sqrt{2}}\right)\sin s, -\left(\frac{20}{41}+\frac{1}{\sqrt{2}}\right)\cos s + \frac{16}{41}\sin s, \left(-\frac{4}{5}+\frac{1}{\sqrt{2}}\right)\right)$$

is the minimal in the ruled surface $\Phi(s,u)$ (see Figure 2).

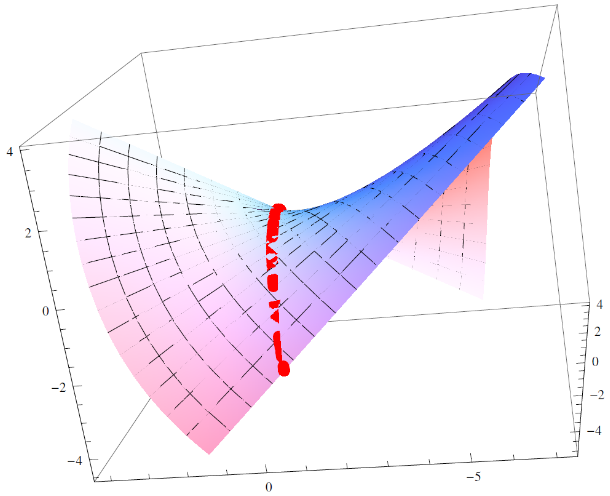

**Figure 2.** Minimal curve $\beta(s)$ in the red color on the ruled surface $\Phi(s,u)$

**Example 5.2.** Consider the smooth curve $\gamma : I \subset \mathbb{R} \to \mathbb{R}^3$ defined by $\gamma(s) = \frac{1}{\sqrt{2}}(-\cos s, -\sin s, 1)$ and the unit vector $v(s) = \dfrac{1}{\sqrt{2}}(\cos s, \sin s, 0)$. Then, $\Gamma(s) = (\gamma(s), v(s))$ is a Legendre curve and the following Frenet frame $\{\alpha(s), v(s), \eta(s)\}$ along $\gamma(s)$ is obtained:

$$\begin{pmatrix} \eta'(s) \\ \alpha'(s) \\ v'(s) \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta(s) \\ \alpha(s) \\ v(s) \end{pmatrix}$$

Hence, the ruled surface

$$\Phi(s,u) = \gamma(s) + uv(s) = -\frac{1}{\sqrt{2}}e^{-\frac{3}{2}s}\left(-\frac{6}{13}\cos s + \frac{4}{13}\sin s, -\frac{6}{13}\sin s - \frac{4}{13}\cos s, \frac{2}{3}\right) + \frac{u}{\sqrt{2}}(\cos s, \sin s, 0)$$

is obtained (see Figure 3).

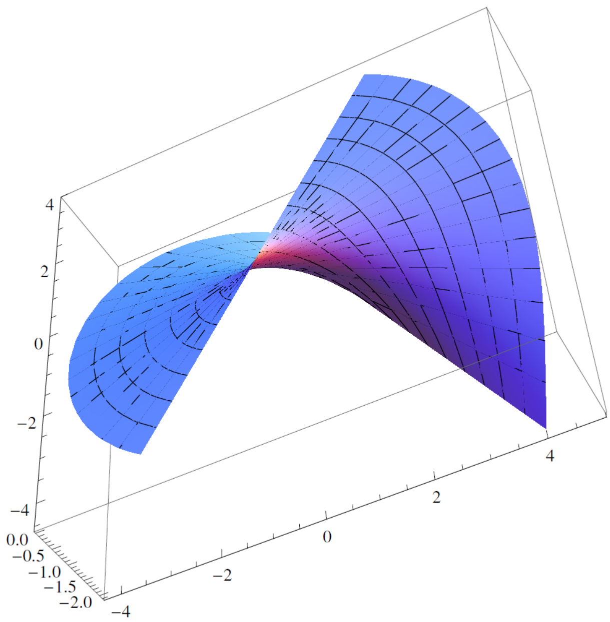

**Figure 3.** Ruled surface $\Phi(s,u)$

Then, the curve

$$\beta(s) = -\frac{1}{\sqrt{2}} e^{-\frac{3}{2}s} \left( -\frac{19}{13}\cos s + \frac{4}{13}\sin s, -\frac{19}{13}\sin s - \frac{4}{13}\cos s, \frac{2}{3} \right)$$

is minimal in $\Phi(s,u)$ (see Figure 4).

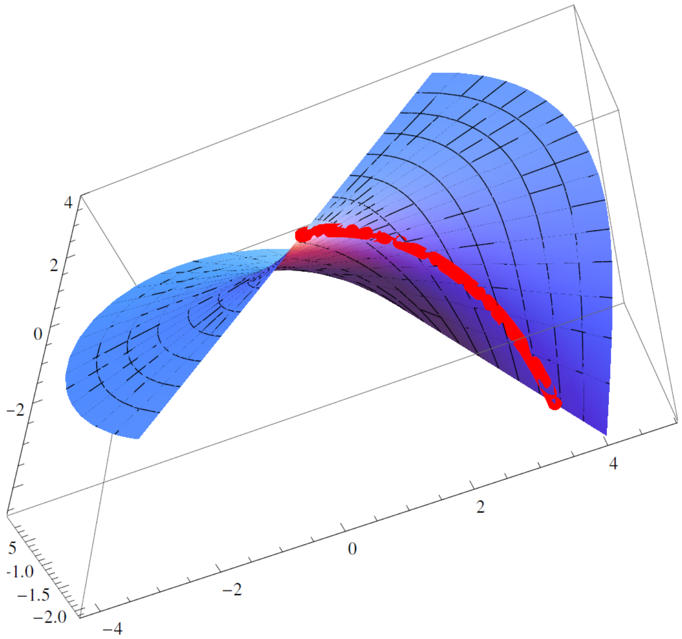


**Figure 4.** $\beta(s)$ in the red color on the ruled surface $\Phi(s,u)$

## 6. Conclusion

In this paper, we consider the ruled surfaces generated by Legendre curves and obtain the condition that these curves are minimal on them. We also study Legendre curves on B-scroll surfaces and provide the condition that a curve is minimal on these surfaces. Finally, we offer some computational examples and graphs of the related minimal curves on the ruled surfaces.

In [20], the relation between two Darboux frames of the standard curve $c(s)$ relatively to $\varphi$ and $\Psi$ was presented. It was proved that the ruled surface is minimal along its base curve if and only if the base curve is a geodesic curve on the regular surface. In this context, new results can be obtained for the ruled surfaces generated by Legendre curves considered in this paper. Furthermore, similar characterizations can be explored in Minkowskian and Lorentzian spaces.

## Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

## Conflicts of Interest

All the authors declare no conflict of interest.

## Ethical Review and Approval

No approval from the Board of Ethics is required.

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