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Coefficient Bounds for Subfamilies of Analytic Mappings of Complex Order Associated with the Ruscheweyh Derivative and Differential Equation

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Abstract. In this article, we introduced some new subfamilies of analytic mappings of complex order defined by using the Ruscheweyh derivative and a family of non-homogenous Cauchy-Euler differential equations. We estimated the nth coefficient bounds and Fekete-Szegö-type functional for functions in these subfamilies. The obtained results were then used in estimation of upper bounds on coefficients of logarithmic, bi-univalent, and second Hankel determinant for such functions. Various useful deductions relevant to recent results in the literature were made with some refinements of known results.

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1. Introduction and Preliminaries

Function theory of one complex variable is a very rich area of study and an elegant subject of classical mathematics. Analytic functions, univalent and multivalent, offer a very fascinating role in the interplay of analytic structure and geometric behavior. Early results that emerged at the beginning of the 20th century laid down the foundation of analytic (geometric) function theory. For example, the Koebe investigation (1907), Gronwall's proof of the area theorem (1914), and the famous Bieberbach's estimates of the second coefficients in 1916. Applications and extensions of univalent and multivalent functions theory have been used in fields like ordinary and partial differential equations, fractional calculus, operator theory, mathematical physics, and differential subordination [13].

Geometric function theory has also a close connection with geometry and analysis, thus attracted the attention of many function theorists from the beginning of the 20th century to recent times. Mostly the available results are reported over the open unit disc of the complex plane $\mathbb{C}$. The reason is that they are characterized by the fact that such a function provides one-to-one mapping onto its image domain. In the following, we briefly introduce some preliminary definitions and results.

Let A denote the family of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad (z \in E) \tag{1.1}$$

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defined in the open unit disk $E = \{z \in \mathbb{C} : |z| < 1\}$ and satisfying the normalization

$$f(0) = f'(0) - 1 = 0.$$

Further, let $S \subset A$ be the subclass of analytic functions that never takes the same value twice, that is

$$S = \{f \in A : f(z_1) = f(z_2) \Rightarrow z_1 = z_2,\ (z_1, z_2 \in E)\}.$$

Such functions are commonly known as univalent functions. Also for be a complex number $\gamma \neq 0$, we denote by $S^*(\gamma)$ and $C(\gamma)$ the familiar subfamilies of all univalent starlike and convex functions of complex order, characterized respectively as (see [14, 20, 24, 26, 35]):

$$f \in S^*(\gamma) \quad \Longleftrightarrow \quad \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf'(z)}{f(z)} - 1\right]\right) > 0, \quad \forall z \in E$$

and

$$f \in C(\gamma) \quad \Longleftrightarrow \quad \Re\left(1 + \frac{1}{\gamma}\left[\frac{zf''(z)}{f'(z)}\right]\right) > 0, \quad \forall z \in E.$$

It is easy to note that

$$S^*(1-\beta) \equiv S^*_\beta, \quad C(1-\beta) \equiv C_\beta, \quad (0 \leq \beta < 1),$$

where S^*_β and C_β are the classes of univalent starlike and convex functions of real order β. Furthermore, we have

$$S^*(1) \equiv S^*_0 \equiv S^*, \quad C(1) \equiv C_0 \equiv C$$

with the inclusion relation

$$C \subset S^* \subset S,$$

where C and S^* are the well-known classes of univalent starlike and convex functions, respectively. Coefficient bounds and related results for various subclasses of starlike and convex functions, as well as closely related families, have been extensively studied in the literature. Notable examples include starlike and convex functions of complex order [6, 7, 32, 36], q-starlike functions of Janowski type [9], and close-to-convex functions of complex order [12]. Further contributions encompass starlike and convex functions associated with Pascal distribution series [19], Janowski spiral-like functions of complex order [22], and strongly starlike functions of order α [25]. Additionally, research has explored Ruscheweyh-type starlike functions of complex order [27] and other specialized subclasses of starlike functions [31, 37].

The Hadamard product or convolution of two functions $f, g \in A$, denoted by $f * g$, is defined as

$$(f * g)(z) = z + \sum_{n=2}^{\infty} a_n b_n z^n = (g * f)(z), \quad (z \in E),$$

where $f(z)$ is given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$.

By using the concept of convolution, Ruscheweyh introduced the differential operator $D^\alpha : A \to A$ as [30]:

$$\begin{aligned} D^\alpha f(z) &= \frac{z}{(1-z)^{\alpha+1}} * f(z) \\ &= z + \sum_{n=2}^{\infty} \phi_n(\alpha)\, a_n z^n, \quad (\alpha \in \mathbb{R}, \alpha > -1), \end{aligned}$$

where

$$\phi_n(\alpha) = \frac{(\alpha+1)_{n-1}}{(n-1)!} \tag{1.2}$$

and $(v)_n$ is the Pochhamer symbol given as

$$(v)_n = \begin{cases} 1, & n = 0; \\ v(v+1)(v+2)\ldots(v+n-1), & n \in \mathbb{N}. \end{cases}$$

Now, let $D_\delta f(z) = (1-\delta)f(z) + \delta z f'(z),\ (\delta \geq 0)$, and consider the differential operator $D_\delta^\alpha : A \to A$ defined as:

$$\begin{aligned} D_\delta^\alpha f(z) &= \frac{z}{(1-z)^{\alpha+1}} * D_\delta f(z) \\ &= \frac{(1-\delta)z}{(1-z)^{\alpha+1}} * f(z) + \frac{\delta z}{(1-z)^{\alpha+1}} * zf'(z) \\ &= z + \sum_{n=2}^{\infty} [1 + (n-1)\delta]\, \phi_n(\alpha)\, a_n z^n, \quad (\alpha \in \mathbb{R}, \alpha > -1; \delta \geq 0). \end{aligned}$$

It can be seen that for $\delta = 0$, $D_0^\alpha = D^\alpha$ is the familiar Ruscheweyh differential operator [30]. Al-Shaqsi and Darus (see [4]) called $D_\delta^\alpha f$ as the generalized Ruscheweyh differential operator. For further generalization of operators $D^\alpha f$ and $D_\delta^\alpha f$ and their applications, the reader can consult e.g. [3, 5, 15, 18, 28].

Also, observe that, if $m \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, then

$$D_\delta^m f(z) = \frac{z\left(z^{m-1} D_\delta f(z)\right)^{(m)}}{m!}, \quad (z \in E).$$

In recent years, there has been growing interest in studying coefficient estimates for various subclasses of analytic functions of complex order; see, for example, [7,12,22,23,32]. These estimates are crucial for understanding geometric properties of analytic functions, such as their growth, distortion, and behavior near the boundary of the unit disk. Sharp bounds on coefficients also help determine whether functions belong to important geometric classes like univalent, starlike, or convex functions. Further interesting applications can be found in [17, 21].

To explore these properties, researchers have developed different subclasses of analytic functions using tools such as differential operators and functional transformations. One such contribution is by Xu et al. [36], who studied coefficient bounds for the class $S_\psi(\lambda, \gamma)$ of functions of complex order $\gamma \neq 0$, defined as:

$$S_\psi(\lambda, \gamma) = \left\{ f \in A : 1 + \frac{1}{\gamma}\left(\frac{z\left((1-\lambda)f(z) + \lambda z f'(z)\right)'}{(1-\lambda)f(z) + \lambda z f'(z)} - 1 \right) \in \psi(E) \right\},$$

where $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, and $\psi : E \to \mathbb{C}$ is a convex function satisfying $\psi(0) = 1$ and $\Re\{\psi(z)\} > 0$ for $z \in E$. This function class, for $\psi(z) = (1+z)/(1-z)$, $z \in E$, gives

$$S_\psi(1, \gamma) \equiv S^*(\gamma), \quad S_\psi(0, \gamma) \equiv C(\gamma),$$

which correspond to the classes of starlike and convex functions of complex order, respectively.

Although much progress has been made in finding coefficient bounds for analytic functions [22, 23], most results focus on subclasses defined by linear or homogeneous operators. In this paper, we introduce two new subclasses of analytic functions of complex order $\gamma \neq 0$: $\mathcal{QS}\psi^\alpha(\delta, \lambda, \gamma)$ and $\mathcal{GS}\psi^\alpha(m, \lambda, \gamma; \mu)$. These are based on non-homogeneous Cauchy-Euler differential equations combined with the Ruscheweyh derivative, allowing for more flexible function structures. The main goal of this paper is to unify and extend earlier results and to provide sharper bounds for important quantities like logarithmic coefficients and second Hankel determinants, both of which play key roles in geometric function theory. Our findings also lead to improvements over some recent results in the literature, offering new insights into the structure and applications of such function classes. We now proceed to define these new subclasses.

Definition 1.1. Let $\psi : E \longrightarrow \mathbb{C}$ be a convex function with $\psi(0) = 1$ and $\Re\{\psi(z)\} > 0$. Then, by $\mathcal{QS}_\psi^\alpha(\delta, \lambda, \gamma)$ we mean the subfamily of analytic functions of complex order given as:

$$\mathcal{QS}_\psi^\alpha(\delta, \lambda, \gamma) = \left\{ f \in A : 1 + \frac{1}{\gamma}\left[\frac{z\left((1-\lambda)D_\delta^\alpha f(z) + \lambda D_\delta^{\alpha+1} f(z)\right)'}{(1-\lambda)D_\delta^\alpha f(z) + \lambda D_\delta^{\alpha+1} f(z)} - 1 \right] \in \psi(E) \right\},$$

where $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\alpha > -1$ and $\delta \geq 0$.

Definition 1.2. Let $f \in A$, then $f \in \mathcal{GS}_\psi^\alpha(m, \delta, \lambda, \gamma; \mu)$ if it satisfies the following m^{th} order non-homogenous Cauchy-Euler type differential equation

$$z^m \frac{d^m w}{dz^m} + \binom{m}{1}(\mu + m - 1) z^{m-1} \frac{d^{m-1} w}{dz^{m-1}} + \cdots + \binom{m}{m} \prod_{k=0}^{m-1} (\mu + k) w = \prod_{k=0}^{m-1} (\mu + k + 1) h(z), \tag{1.3}$$

where $w = f(z)$, $h \in \mathcal{QS}_\psi^\alpha(\delta, \lambda, \gamma)$, $\mu > -1$, and $m \in \mathbb{N}_2 = \{2, 3, 4, \cdots\}$.

Remark 1.3. As an immediate consequence of Definitions 1.1 and 1.2, we have the following special cases

$$\mathcal{QS}_{\psi}^{0}(0,\lambda,\gamma) \equiv S_{\psi}(\lambda,\gamma), \quad \mathcal{GS}_{\psi}^{0}(m,0,\lambda,\gamma;\mu) \equiv K_{\psi}(m,\lambda,\gamma;\mu),$$

which have been studied in [36].

Remark 1.4. For suitable choices of the involved parameters and function ψ, many earlier introduced subfamilies of analytic functions can be deduced easily. The following special cases are worthy to mention.
(i). For $\psi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, it is a simple exercise to verify the hypothesis of Definition 1.1. Consequently from Definitions 1.1 and 1.2, one gets

$$\mathcal{QS}_{\psi}^{0}(0,\lambda,\gamma) \equiv \mathcal{SC}(\lambda,\gamma,\beta), \quad \mathcal{GS}_{\psi}^{0}(2,0,\lambda,\gamma;\mu) \equiv \mathcal{B}(\lambda,\gamma,\beta;\mu),$$

which were studied in [7].
(ii) The function $\psi(z) = \frac{1+Lz}{1+Mz}$ $(-1 \leq M < L \leq 1, M \neq 0)$ also satisfies the hypothesis of Definition 1.1. Accordingly Definitions 1.1 and 1.2 gives the following

$$\mathcal{QS}_{\psi}^{0}(0,\lambda,\gamma) \equiv \mathcal{S}(\lambda,\gamma,L,M), \quad \mathcal{GS}_{\psi}^{0}(m,0,\lambda,\gamma;\mu) \equiv \mathcal{K}(\lambda,\gamma,L,M,m;\mu),$$

which were studied in [32].

In what follows the following key results will be required in deriving the main results.

Definition 1.5 ([20])**.** Let $f, g \in A$. Then, f is subordinate to g, denoted by $f \prec g$, if for some analytic function $q(z)$, there holds

$$f(z) = g(q(z)), \quad (z \in E),$$

where $q(z)$ is a Schwarz function satisfying $q(0) = 0$ and $|q(z)| < 1$ for $z \in E$.

In particular, if g is univalent in E, then $f \prec g$ can be written as

$$f(0) = g(0) \quad \text{and} \quad f(E) \subset g(E), \quad (z \in E).$$

The following result can be seen as an application of subordination that we will require in our later investigation.

Lemma 1.6 ([29])**.** *Let $g(z) = \sum_{k=1}^{\infty} b_k z^k$ be a convex analytic function in E and let $f(z) = \sum_{k=1}^{\infty} a_k z^k$ be analytic in E. If $f \prec g$ $(z \in E)$, then*

$$|a_k| \leq |b_1|, \quad \forall k \in \mathbb{N}.$$

Remark 1.7. The class $\mathcal{QS}_{\psi}^{\alpha}(\delta,\lambda,\gamma)$ is non-empty. Indeed, for analytic function $p(z)$ such that $p(0) = \psi(0)$ and $p(E) \subset \psi(E)$. Then $p(z) \prec \psi(z)$, for $z \in E$. Further, we define the analytic function $F(z)$ in terms of the operator $D_{\delta}^{\alpha} f$ as

$$F(z) = (1-\lambda)D_{\delta}^{\alpha} f(z) + \lambda D_{\delta}^{\alpha+1} f(z), \quad (z \in E). \tag{1.4}$$

Then, we can write

$$\begin{aligned} 1 + \frac{1}{\gamma}\left(\frac{zF'(z)}{F(z)} - 1\right) &= p(z), \\ \Leftrightarrow \frac{F'(z)}{F(z)} &= \left(\frac{1+\gamma[p(z)-1]}{z}\right), \\ \Leftrightarrow \log F(z) &= \int_0^z \left(\frac{1+\gamma[p(\zeta)-1]}{\zeta}\right) d\zeta, \\ \Leftrightarrow F(z) &= z\exp\left(\gamma \int_0^z \frac{p(\zeta)-1}{\zeta} d\zeta\right) = z + \cdots. \end{aligned}$$

This shows that functions in the subclass $\mathcal{QS}_{\psi}^{\alpha}(\delta,\lambda,\gamma)$ has the above integral representation.

From now on, we assume the restrictions on the parameters, $0 \leq \lambda \leq 1$, $\gamma \in \mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mu > -1$, $m, n \in \mathbb{N}_2 = \{2, 3, 4, ...\}$, $\alpha > -1$, and $\delta \geq 0$, and $f \in A$ with series form (1.1), unless stated otherwise.

2. Coefficient Estimation for the Subfamily $\mathcal{QS}_{\psi}^{\alpha}(\delta, \lambda, \gamma)$

Before going towards the proof of the main result in this section, we introduce the following to simplify the matters. By reconsidering (1.4) with the series representation of $D_{\delta}^{\alpha} f$, we can write

$$\begin{aligned} F(z) &= (1-\lambda)\left(z + \sum_{n=2}^{\infty} [1+(n-1)\delta]\, \phi_n(\alpha)\, a_n z^n\right) + \lambda\left(z + \sum_{n=2}^{\infty} [1+(n-1)\delta]\, \phi_n(\alpha+1)\, a_n z^n\right) \\ &= z + \sum_{n=2}^{\infty} \left[(1-\lambda)\,\phi_n(\alpha) + \lambda\left(\frac{\alpha+n}{\alpha+1}\right)\phi_n(\alpha)\right][1+(n-1)\delta]\, a_n z^n. \end{aligned} \tag{2.1}$$

Now, we can write (2.1) as

$$F(z) = z + \sum_{n=2}^{\infty} A_n z^n, \tag{2.2}$$

where

$$A_n = C(n, \alpha, \delta, \lambda)\phi_n(\alpha)\, a_n, \tag{2.3}$$

in which $\phi_n(\alpha)$ is given by (1.2), and

$$C(n, \alpha, \delta, \lambda) = \frac{[\alpha + 1 + \lambda(n-1)]\,[1+(n-1)\delta]}{\alpha+1}. \tag{2.4}$$

Now, we are ready to state and prove the following result.

Theorem 2.1. *Let $f \in A$ with the series form (1.1). If $f \in \mathcal{QS}_{\psi}^{\alpha}(\delta, \lambda, \gamma)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} (k + |\gamma|\, |\psi'(0)|)}{(n-1)!\, C(n, \alpha, \delta, \lambda)\, \phi_n(\alpha)},$$

where $\phi_n(\alpha)$ is given by (1.2) and $C(n, \alpha, \delta, \lambda)$ is given by (2.4).

Proof. As it is clear from (2.2), the function $F(z)$ is analytic in E with $F(0) = F'(0) - 1 = 0$. Now, in view of Remark 1.7, we can write

$$1 + \frac{1}{\gamma}\left[\frac{zF'(z)}{F(z)} - 1\right] = p(z) \in \psi(E), \quad (z \in E), \tag{2.5}$$

where the function p satisfies the hypothesis in Remark 1.7. Consequently (2.5) can be expressed as

$$zF'(z) = (1 + \gamma(p(z) - 1))\, F(z).$$

Substituting (2.2) with $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ in above expression while considering $A_1 = 1$ gives us

$$\begin{aligned} z + \sum_{n=2}^{\infty} nA_n z^n &= \left(1 + \gamma \sum_{n=1}^{\infty} p_n z^n\right)\left(z + \sum_{n=2}^{\infty} A_n z^n\right), \\ &= z + \sum_{n=2}^{\infty} A_n z^n + \gamma \sum_{n=2}^{\infty} \left(\sum_{k=1}^{n-1} p_k A_{n-k}\right) z^n. \end{aligned}$$

Thus, comparison of coefficients of z^n on both sides with little calculus gives

$$(n-1)\,|A_n| \leq |\gamma| \sum_{k=1}^{n-1} |p_k|\, |A_{n-k}|\,. \tag{2.6}$$

Now, from Lemma 1.6 it follows that

$$\left|p_j\right| = \left|\frac{p^{(j)}(0)}{j!}\right| \leq \left|\psi'(0)\right|, \quad (j \in \mathbb{N}_2). \tag{2.7}$$

Using (2.7) into (2.6), one can observe

$$|A_n| \leq \frac{|\gamma||\psi'(0)|}{n-1} \sum_{k=1}^{n-1} |A_{n-k}|, \quad \forall n \geq 2,$$

where $A_1 = 1$. Next, we need to show that

$$\frac{|\gamma||\psi'(0)|}{n-1}\sum_{k=1}^{n-1}|A_{n-k}| \leq \frac{\prod_{k=0}^{n-2}(k+|\gamma||\psi'(0)|)}{(n-1)!}, \quad \forall n \geq 2.$$

It holds true for $n = 2$, since

$$|\gamma||\psi'(0)||A_1| = |\gamma||\psi'(0)|, \quad \Longrightarrow \quad |A_2| \leq |\gamma||\psi'(0)|.$$

Let it be true for $n = m$. That is,

$$|A_m| \leq \frac{|\gamma||\psi'(0)|}{m-1}\sum_{k=1}^{m-1}|A_{m-k}| \leq \frac{\prod_{k=0}^{m-2}(k+|\gamma||\psi'(0)|)}{(m-1)!}, \quad \forall m \geq 2.$$

To show that it is also true for $n = m+1$, we multiply both sides of the above inequality by $(m-1+|\gamma||\psi'(0)|)/m$ so that

$$\frac{\prod_{k=0}^{m-1}(k+|\gamma||\psi'(0)|)}{m!} \geq \frac{m-1+|\gamma||\psi'(0)|}{m}\frac{|\gamma||\psi'(0)|}{m-1}\sum_{k=1}^{m-1}|A_{m-k}|.$$

Then, a simple manipulation on the right hand side gives

$$\frac{\prod_{k=0}^{m-1}(k+|\gamma||\psi'(0)|)}{(m-1)!} \geq \frac{|\gamma||\psi'(0)|}{m}\left(\sum_{k=1}^{m-1}|A_{m-k}| + \frac{|\gamma||\psi'(0)|}{m-1}\sum_{k=1}^{m-1}|A_{m-k}|\right).$$

By assumption, we know that

$$\frac{|\gamma||\psi'(0)|}{n-1}\sum_{k=1}^{m-1}|A_{m-k}| \geq |A_m|, \quad \forall m \geq 2,$$

which subsequently gives

$$\begin{aligned}\frac{\prod_{k=0}^{m-1}(k+|\gamma||\psi'(0)|)}{(m-1)!} &\geq \frac{|\gamma||\psi'(0)|}{m}\left(\sum_{k=1}^{m-1}|A_{m-k}| + |A_m|\right)\\ &\geq \frac{|\gamma||\psi'(0)|}{m}\sum_{k=1}^{m}|A_{m+1-k}|.\end{aligned}$$

This verifies the statement

$$|A_n| \leq \frac{\prod_{k=0}^{n-2}(k+|\gamma|\,|\psi'(0)|)}{(n-1)!}, \quad (n \in \mathbb{N}_2).$$

Finally, by using the relation (2.3), one gets

$$|a_n| \leq \frac{\prod_{k=0}^{n-2}(k+|\gamma|\,|\psi'(0)|)}{(n-1)!C(n,\alpha,\delta,\lambda)\,\phi_n(\alpha)}, \quad (n \in \mathbb{N}_2).$$

This proves the statement of the theorem. □

Some special cases of Theorem 2.1 are noted as follow.

Corollary 2.2. *If $f \in \mathcal{QS}_{\psi}^{0}(\delta,\lambda,\gamma)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2}(k+|\gamma|\,|\psi'(0)|)}{(1+(n-1)\,\delta)\,[1+\lambda\,(n-1)]\,(n-1)!}, \quad (n \in \mathbb{N}_2).$$

Corollary 2.3. *If $f \in \mathcal{QS}_{\psi}^{\alpha}(\delta,\lambda,\gamma)$ with $\psi(z) = \frac{1+Lz}{1+Mz}$ $(-1 \leq M < L \leq 1)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2}(k+|\gamma|\,|L-M|)}{(n-1)!C(n,\alpha,\delta,\lambda)\,\phi_n(\alpha)}, \quad (n \in \mathbb{N}_2),$$

where $\phi_n(\alpha)$ is given by (1.2) and $C(n,\alpha,\delta,\lambda)$ is given by (2.4).

Corollary 2.4. *If $f \in \mathcal{QS}_{\psi}^{\alpha}(\delta, \lambda, \gamma)$ with $\psi(z) = \frac{1+(1-2\beta)z}{1-z}$ $(0 \leq \beta < 1)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} (k + 2|\gamma|(1-\beta))}{(n-1)!C(n, \alpha, \delta, \lambda)\, \phi_n(\alpha)}, \quad (n \in \mathbb{N}_2),$$

where $\phi_n(\alpha)$ is given by (1.2) and $C(n, \alpha, \delta, \lambda)$ is given by (2.4).

Remark 2.5. We would like to mention that for $\delta = 0$, and for $\alpha = \delta = 0$, Corollaries 2.2 and 2.4 give the coefficient estimates of Xu et al. [36], and Altintaş et al. [7], respectively.

Remark 2.6. From Corollary 2.3, it can be seen that for $-1 \leq M < L \leq 1$

$$k + |\gamma|(L-M) \leq k + \frac{2|\gamma|(L-M)}{1-M}, \quad \forall k \geq 0.$$

Therefore, the present results (e.g. Corollary 2.3) for $\alpha = \delta = 0$ refine the corresponding coefficient estimate of Srivastava et al. [32, Theorem 1].

The following results provide coefficient estimates of generalized subclasses of starlike functions which appear to be new one.

Corollary 2.7. *If $f \in \mathcal{QS}_{\psi}^{\alpha}(\delta, 1, \gamma)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} (k + |\gamma|\,|\psi'(0)|)}{(n-1)!C(n, \alpha, \delta, 1)\, \phi_n(\alpha)}, \quad (n \in \mathbb{N}_2),$$

where $\phi_n(\alpha)$ is given by (1.2) and $C(n, \alpha, \delta, \lambda)$ is given by (2.4).

Corollary 2.8. *If $f \in \mathcal{QS}_{\psi}^{\alpha}(\delta, \lambda, 1)$, then*

$$|a_n| \leq \frac{\prod_{k=0}^{n-2} (k + |\psi'(0)|)}{(n-1)!C(n, \alpha, \delta, \lambda)\, \phi_n(\alpha)}, \quad (n \in \mathbb{N}_2),$$

where $\phi_n(\alpha)$ is given by (1.2) and $C(n, \alpha, \delta, \lambda)$ is given by (2.4).

3. Coefficient Estimation for the Subfamily $\mathcal{GS}_{\psi}^{\alpha}(m, \delta, \lambda, \gamma; \mu)$

Our next result is stated in the following theorem which is an immediate consequence of Theorem 2.1.

Theorem 3.1. *If $f \in \mathcal{GS}_{\psi}^{\alpha}(m, \delta, \lambda, \gamma; \mu)$, then for $m, n \in \mathbb{N}_2$ and $\mu > -1$*

$$|a_n| \leq \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + n}\right) \left(\frac{\prod_{k=0}^{n-2} (k + |\gamma|\,|\psi'(0)|)}{(n-1)!C(n, \alpha, \delta, \lambda)\, \phi_n(\alpha)}\right),$$

where $\phi_n(\alpha)$ and $C(n, \alpha, \delta, \lambda)$ are given by (1.2) and (2.4), respectively.

Proof. Let $h(z) = z + \sum_{n=2}^{\infty} c_n z^n \in \mathcal{QS}_{\psi}^{\alpha}(\delta, \lambda, \gamma)$. Then, from (1.3) we have (see the Appendix for detail)

$$a_n = \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + n}\right) c_n.$$

The desired result now follows from Theorem 2.1 by using the coefficient estimate of c_n. □

The following are some special cases of Theorem 3.1.

Corollary 3.2. *If $f \in \mathcal{GS}_{\psi}^{\alpha}(m, 0, \lambda, \gamma; \mu)$, then for $m, n \in \mathbb{N}_2$ and $\mu > -1$*

$$|a_n| \leq \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + n}\right) \left(\frac{\prod_{k=0}^{n-2} (k + |\gamma|\,|\psi'(0)|)}{(n-1)!C(n, \alpha, 0, \lambda)\, \phi_n(\alpha)}\right),$$

where $\phi_n(\alpha)$ and $C(n, \alpha, \delta, \lambda)$ are given by (1.2) and (2.4), respectively.

Corollary 3.3. *If $f \in \mathcal{GS}_{\psi}^{\alpha}(m,\delta,\lambda,\gamma;\mu)$ and $\psi(z)=\frac{1+Lz}{1+Mz}$ $(-1\le M<L\le 1)$, then for $m,n\in\mathbb{N}_2$ and $\mu>-1$*

$$|a_n| \le \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)\left(\frac{\prod_{k=0}^{n-2}(k+|\gamma|\,|L-M|)}{(n-1)!C(n,\alpha,\delta,\lambda)\,\phi_n(\alpha)}\right),$$

where $\phi_n(\alpha)$ and $C(n,\alpha,\delta,\lambda)$ are given by (1.2) and (2.4) respectively.

Corollary 3.4. *If $f \in \mathcal{GS}_{\psi}^{\alpha}(m,\delta,\lambda,\gamma;\mu)$ and $\psi(z)=\frac{1+(1-2\beta)z}{1-z}$ $(0\le\beta<1)$, then for $m,n\in\mathbb{N}_2$ and $\mu>-1$*

$$|a_n| \le \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)\left(\frac{\prod_{k=0}^{n-2}(k+2|\gamma|(1-\beta))}{(n-1)!C(n,\alpha,\delta,\lambda)\,\phi_n(\alpha)}\right),$$

where $\phi_n(\alpha)$ and $C(n,\alpha,\delta,\lambda)$ are given by (1.2) and (2.4), respectively.

Remark 3.5. It is interesting to note that by setting $\alpha=0$ in Corollary 3.2 one gets the coefficient estimate of Xu et al. [36], whereas by setting $\delta=\alpha=0$ and $m=2$ in Corollary 3.3, one can have the coefficient estimate of Altintaş et al. [7].

Remark 3.6. In light of Remark 2.6, Corollary 3.4 can be seen as a refined version of the coefficient estimate of Srivastava et al. [32, Theorem 2] for the case $\delta=\alpha=0$.

4. Fekete-Szegö Type Inequalities for the Subfamilies $\mathcal{QS}_{\psi}^{\alpha}(\delta,\lambda,\gamma)$ and $\mathcal{GS}_{\psi}^{\alpha}(m,\delta,\lambda,\gamma;\mu)$

In this section, Fekete-Szegö type inequalities are obtained for the subfamilies

$$\mathcal{QS}_{\psi}^{\alpha}(\delta,\lambda,\gamma),\quad \text{and}\quad \mathcal{GS}_{\psi}^{\alpha}(m,\delta,\lambda,\gamma;\mu),$$

when $\psi(z)=\frac{1+Lz}{1+Mz}$ $(-1\le M<L\le 1, M\neq 0)$. Accordingly, we have the following classes

$$\mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M),\quad \text{and}\quad \mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu).$$

In order to derive the main results, we recall the following.

Lemma 4.1 ([1]). *Let $q(z)=q_1z+q_2z^2+q_3z^3+\cdots$, $(z\in E)$ be the Schwarz function, then for any real b,*

$$\left|q_2-bq_1^2\right| \le \begin{cases} -b, & b<-1,\\ 1, & -1\le b\le 1,\\ b, & b>1.\end{cases}$$

These sharp estimates are attained for $b>1$ or $b<-1$, iff $q(z)=z$ or one of its rotation. If $-1<b<1$, then equality occurs iff $q(z)=z^2$ or one of its rotation. Equality also occurs for $b=-1$, iff $q(z)=\frac{z(z+\lambda)}{1+\lambda z}$ $(0\le\lambda\le 1)$ or one of its rotation, while for $b=1$ iff $q(z)=-\frac{z(z+\lambda)}{1+\lambda z}$ $(0\le\lambda\le 1)$ or one of its rotation.

Lemma 4.2 ([1]). *Let $q(z)=q_1z+q_2z^2+q_3z^3+\cdots$, $(z\in E)$ be a Schwarz function, then for any complex number b*

$$\left|q_2-bq_1^2\right| \le \max\{1,|b|\}.$$

This estimate is sharp and attains for $q(z)=z$ or $q(z)=z^2$.

Theorem 4.3. *Let $f\in\mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$ and $-1\le M<L\le 1, M\neq 0$. If b is any complex number, then*

$$\left|a_3-ba_2^2\right| \le \frac{|\gamma||L-M|}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\max\{1,|\sigma|\}, \tag{4.1}$$

where σ is given as

$$\sigma = M+\gamma(L-M)\left[\frac{2bC(3,\alpha,\delta,\lambda)\phi_3(\alpha)}{\left(C(2,\alpha,\delta,\lambda)\phi_2(\alpha)\right)^2}-1\right]. \tag{4.2}$$

The parameters values $\phi_n(\alpha)$ and $C(n,\alpha,\delta,\lambda)$ are given by (1.2) and (2.4), respectively.

Proof. Let $f \in \mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$. Then, by using the subordination, we can write

$$1+\frac{1}{\gamma}\Big[\frac{zF'(z)}{F(z)}-1\Big] \prec \frac{1+Lz}{1+Mz},$$

or equivalently

$$\frac{zF'(z)}{F(z)}-1=\frac{\gamma(L-M)q(z)}{1+Mq(z)},$$

where q is the Schwarz function satisfying $q(0)=0$ and $|q(z)|<1$ for $z\in E$. By substituting the series expansions of $F(z), F'(z)$ and $q(z)$ in above equation, we get

$$A_2z+(2A_3-A_2^2)z^2+\cdots=\gamma(L-M)q_1z+\gamma(L-M)(q_2-Mq_1^2)z^2+\cdots. \tag{4.3}$$

By equating coefficients of like powers of z and using (2.3), one can write

$$a_2=\Big[\frac{\gamma(L-M)}{C(2,\alpha,\delta,\lambda)\phi_2(\alpha)}\Big]q_1,$$

$$a_3=\frac{\gamma(L-M)}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\Big[q_2-\Big(M-\gamma(L-M)\Big)q_1^2\Big],$$

where $\phi_n(\alpha)$ and $C(n,\alpha,\delta,\lambda)$ are given by (1.2) and (2.4), respectively.

Now, it is a simple exercise to see that

$$\left|a_3-ba_2^2\right|=\left(\frac{|\gamma||L-M|}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\right)\left|q_2-\sigma q_1^2\right|, \tag{4.4}$$

where σ is given by (4.2). Hence, by applying Lemma 4.2 to inequality (4.4), for the complex number b, gives the desired inequality (4.1). This completes the proof. □

Remark 4.4. A direct application of Lemma 4.1 to (4.3) gives immediately

$$|A_2|\le|\gamma||L-M| \quad\text{and}\quad \left|A_3-\frac{1}{2}A_2^2\right|\le\frac{1}{2}|\gamma||L-M|,$$

where we have used the fact that $|q_1|\le 1$ and $-1\le M<L\le 1, M\neq 0$.

Theorem 4.5. *Let $f\in\mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$ and $-1\le M<L\le 1, M\neq 0$. If b is any complex number, then*

$$\left|a_3-ba_2^2\right|\le\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+3}\right)\left(\frac{|\gamma||L-M|}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\right)\max\{1,|\tau|\}, \tag{4.5}$$

where τ is defined as

$$\tau=M+\gamma(L-M)\left[\prod_{j=0}^{m-1}\left(\frac{(\mu+j+1)(\mu+j+3)}{(\mu+j+2)^2}\right)\frac{2bC(3,\alpha,\delta,\lambda)\phi_3(\alpha)}{(C(2,\alpha,\delta,\lambda)\phi_2(\alpha))^2}-1\right]. \tag{4.6}$$

The parameters values $\phi_n(\alpha)$ and $C(n,\alpha,\delta,\lambda)$ are given by (1.2) and (2.4), respectively.

Proof. Let $f\in\mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$. Then, from (1.3) we have (see the Appendix for detail)

$$a_n=\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)c_n, \quad (n=2,3).$$

By using Theorem 4.3, a simple computation shows that

$$a_2=\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+2}\right)\left[\frac{\gamma(L-M)}{C(2,\alpha,\delta,\lambda)\phi_2(\alpha)}\right]q_1,$$

and

$$a_3=\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+3}\right)\left(\frac{\gamma(L-M)}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\right)\left[q_2-(M-\gamma(L-M))\,q_1^2\right].$$

A simple exercise then lead us to deduce that

$$\left|a_3 - ba_2^2\right| = \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+3}\right)\left(\frac{|\gamma||L-M|}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\right)\left|q_2 - \tau q_1^2\right|, \tag{4.7}$$

where τ is given by (4.6). Finally, for complex number b, by applying Lemma 4.2 to (4.7) gives the inequality (4.5). This completes the proof. □

Remark 4.6. The results reported in Theorems 4.3 and 4.5 are best possible. Equality holds for functions respectively belonging to the classes

$$\mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M), \qquad \mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$$

given as follows:

(1) For $f \in \mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$ equality occurs when f is defined by

$$\frac{z\left((1-\lambda)D_\delta^\alpha f(z) + \lambda D_\delta^{\alpha+1} f(z)\right)'}{(1-\lambda)D_\delta^\alpha f(z) + \lambda D_\delta^{\alpha+1} f(z)} = 1 + \frac{\gamma(L-M)q(z)}{1+Mq(z)},$$

or as a solution of the following differential equation (see Remark 1.7):

$$(1-\lambda)D_\delta^\alpha f(z) + \lambda D_\delta^{\alpha+1} f(z) = \begin{cases} z\left(1+Mz^{j-1}\right)^{\frac{\gamma(L-M)}{(j-1)M}}, & \text{if } M \neq 0,\ j \geq 2, \\ z\exp\left(\frac{\gamma L}{j-1}z^{j-1}\right), & \text{if } M = 0,\ j \geq 2, \end{cases} \tag{4.8}$$

where $q(z) = z^{j-1}$ ($j \in \mathbb{N} \setminus \{1\}$) is a Schwarz function.

(2) For $f \in \mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$ equality occurs when $w = f(z)$ is defined by

$$z^m\frac{d^m w}{dz^m} + \binom{m}{1}(\mu+m-1)z^{m-1}\frac{d^{m-1}w}{dz^{m-1}} + \cdots + \binom{m}{m}\prod_{k=0}^{m-1}(\mu+k)w = \prod_{k=0}^{m-1}(\mu+k+1)h(z), \quad h \in \mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M),$$

where h is the solution of the differential equation (4.8).

Remark 4.7. To the author's knowledge, the Fekete-Szegö problem is completely solved for the first time for the classes

$$\mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M) \quad \text{and} \quad \mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu).$$

Therefore, any deduction made from the obtained results, especially from that of class $\mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$, appears to be completely new one.

5. Applications

In this section, some applications of the main results are demonstrated. In particular, solution of Fekete-Szegö problems lead the way to an estimates on upper bounds of logarithmic coefficients, bi-univalent coefficients, and second-order Hankel determinant for functions in the subclasses $\mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$ and $\mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$, where $-1 \leq M < L \leq 1, M \neq 0$.

Corollary 5.1. *Let* $\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=2}^{\infty}\xi_n z^n$ *such that* $2\xi_2 = a_2$ *and* $2\xi_3 = a_3 - a_2^2/2$. *If* $f \in \mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$ *and* $-1 \leq M < L \leq 1, M \neq 0$, *then*

$$|\xi_2| \leq \frac{|\gamma||L-M|}{2C(2,\alpha,\delta,\lambda)\phi_2(\alpha)},$$

$$|\xi_3| \leq \frac{|\gamma||L-M|}{4C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\max\{1,|\sigma_0|\},$$

where $\phi_n(\alpha)$ *and* $C(n,\alpha,\delta,\lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* σ_0 *is given as*

$$\sigma_0 = M + \gamma(L-M)\left[\frac{C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}{\left(C(2,\alpha,\delta,\lambda)\phi_2(\alpha)\right)^2} - 1\right].$$

Proof. The first inequality follows from

$$|\xi_2| = \frac{1}{2}|a_2| \le \frac{|\gamma||L-M|}{2C(2,\alpha,\delta,\lambda)\phi_2(\alpha)}|q_1|,$$

together with the fact that $|q_1| \le 1$. For the second inequality, we have

$$|\xi_3| = \frac{1}{2}\left|a_3 - \frac{a_2^2}{2}\right|,$$

which follows at once by using $b = 1/2$ in inequality (4.1) of Theorem 4.3. □

Corollary 5.2. *Let* $\log\left(\frac{f(z)}{z}\right) = 2\sum_{n=2}^{\infty}\xi_n z^n$ *such that* $2\xi_2 = a_2$ *and* $2\xi_3 = a_3 - a_2^2/2$. *If* $f \in \mathcal{GS}^{\alpha}(m,\delta,\lambda,\gamma,L,M;\mu)$ *and* $-1 \le M < L \le 1, M \ne 0$, *then*

$$|\xi_2| \le \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)\frac{|\gamma||L-M|}{2C(2,\alpha,\delta,\lambda)\phi_2(\alpha)},$$

$$|\xi_3| \le \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+3}\right)\left(\frac{|\gamma||L-M|}{4C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\right)\max\{1,|\tau_0|\},$$

where $\phi_n(\alpha)$ *and* $C(n,\alpha,\delta,\lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* τ_0 *is given as*

$$\tau_0 = M + \gamma(L-M)\left[\prod_{j=0}^{m-1}\left(\frac{(\mu+j+1)(\mu+j+3)}{(\mu+j+2)^2}\right)\frac{C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}{(C(2,\alpha,\delta,\lambda)\phi_2(\alpha))^2} - 1\right].$$

Proof. The first inequality simply follows from

$$|\xi_2| = \frac{1}{2}|a_2| = \frac{1}{2}\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)|c_2|,$$
$$= \frac{1}{2}\prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)\frac{|\gamma||L-M|}{2C(2,\alpha,\delta,\lambda)\phi_2(\alpha)}|q_1|,$$

which gives the desired result by using $|q_1| \le 1$. The second inequality follows by using $b = 1/2$ in inequality (4.5) of Theorem 4.5. □

Remark 5.3. The coefficients ξ_n are known as logarithmic coefficients which played a central role in the proof of Beiberbach's conjecture by de-Branges [16]. Further, the Brannan, Milin, and Robertson conjecture as well as Lebedev-Milin inequalities are also implied by logarithmic coefficients [2].

Corollary 5.4. *Let* $g(z) = f^{-1}(z) = z + \sum_{n=2}^{\infty}\rho_n z^n$ *such that* $\rho_2 = -a_2$ *and* $\rho_3 = 2a_2^2 - a_3$. *If* $f \in \mathcal{QS}^{\alpha}(\delta,\lambda,\gamma,L,M)$ *and* $-1 \le M < L \le 1, M \ne 0$, *then*

$$|\rho_2| \le \frac{|\gamma||L-M|}{C(2,\alpha,\delta,\lambda)\phi_2(\alpha)},$$
$$|\rho_3| \le \frac{|\gamma||L-M|}{2C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}\max\{1,|\sigma_1|\},$$

where $\phi_n(\alpha)$ *and* $C(n,\alpha,\delta,\lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* σ_1 *is given as*

$$\sigma_1 = M + \gamma(L-M)\left[\frac{4C(3,\alpha,\delta,\lambda)\phi_3(\alpha)}{\left(C(2,\alpha,\delta,\lambda)\phi_2(\alpha)\right)^2} - 1\right].$$

Proof. The first inequality is straightforward whereas the second inequality is achieved by using $b = 2$ in inequality (4.1) of Theorem 4.3. □

Corollary 5.5. *Let* $g(z) = f^{-1}(z) = z + \sum_{n=2}^{\infty} \rho_n z^n$ *such that* $\rho_2 = -a_2$ *and* $\rho_3 = 2a_2^2 - a_3$. *If* $f \in \mathcal{GS}^{\alpha}(m, \delta, \lambda, \gamma, L, M; \mu)$ *and* $-1 \leq M < L \leq 1, M \neq 0$, *then*

$$|\rho_2| \leq \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + n} \right) \frac{|\gamma||L - M|}{C(2, \alpha, \delta, \lambda)\phi_2(\alpha)},$$

$$|\rho_3| \leq \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + 3} \right) \left(\frac{|\gamma||L - M|}{2C(3, \alpha, \delta, \lambda)\phi_3(\alpha)} \right) \max\{1, |\tau_1|\},$$

where $\phi_n(\alpha)$ *and* $C(n, \alpha, \delta, \lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* τ_1 *is given as*

$$\tau_1 = M + \gamma(L - M) \left[\prod_{j=0}^{m-1} \left(\frac{(\mu + j + 1)(\mu + j + 3)}{(\mu + j + 2)^2} \right) \frac{4C(3, \alpha, \delta, \lambda)\phi_3(\alpha)}{(C(2, \alpha, \delta, \lambda)\phi_2(\alpha))^2} - 1 \right].$$

Proof. The first inequality is straightforward whereas the second inequality is achieved by using $b = 2$ in inequality (4.5) of Theorem 4.5. □

Remark 5.6. The coefficients ρ_n are known as inverse coefficients for univalent functions. By Koebe's one-quarter theorem, it is well known that every univalent function $f \in S$ has an inverse f^{-1}, defined as

$$f^{-1}(f(z)) = z, \ (\forall z \in E); \quad f(f^{-1}(\omega)) = \omega, \ |\omega| < r_*(f), \ r_*(f) \geq 1/4,$$

with

$$g(\omega) = f^{-1}(\omega) = \omega - a_2\omega^2 + (2a_2^2 - a_3)\omega^3 + \cdots .$$

A univalent function $f \in A$ whose inverse f^{-1} is also univalent in E is called bi-univalent function, while assuming f^{-1} has a univalent analytic continuation to E. Theory of bi-univalent functions revived back with the work of Srivastav et al. [33] following Brannan and Taha [11].

Corollary 5.7. *If* $f \in \mathcal{QS}^{\alpha}(\delta, \lambda, \gamma, L, M)$ *and* $-1 \leq M < L \leq 1, M \neq 0$, *then*

$$|H_2(1)| = \left|a_3 - a_2^2\right| \leq \frac{|\gamma||L - M|}{2C(3, \alpha, \delta, \lambda)\phi_3(\alpha)} \max\{1, |\sigma_2|\},$$

where $\phi_n(\alpha)$ *and* $C(n, \alpha, \delta, \lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* σ_2 *is given as*

$$\sigma_2 = M + \gamma(L - M) \left[\frac{2C(3, \alpha, \delta, \lambda)\phi_3(\alpha)}{\left(C(2, \alpha, \delta, \lambda)\phi_2(\alpha)\right)^2} - 1 \right].$$

Proof. By using $b = 1$ in inequality (4.1) of Theorem 4.3 gives at once the desired result. □

Corollary 5.8. *If* $f \in \mathcal{GS}^{\alpha}(m, \delta, \lambda, \gamma, L, M; \mu)$ *and* $-1 \leq M < L \leq 1, M \neq 0$, *then*

$$|H_2(1)| = \left|a_3 - a_2^2\right| \leq \prod_{j=0}^{m-1} \left(\frac{\mu + j + 1}{\mu + j + 3} \right) \left(\frac{|\gamma||L - M| \max\{1, |\tau|\}}{2C(3, \alpha, \delta, \lambda)\phi_3(\alpha)} \right),$$

where $\phi_n(\alpha)$ *and* $C(n, \alpha, \delta, \lambda)$ *are given by* (1.2) *and* (2.4), *respectively, and* τ_2 *is given as*

$$\tau_2 = M + \gamma(L - M) \left[\prod_{j=0}^{m-1} \left(\frac{(\mu + j + 1)(\mu + j + 3)}{(\mu + j + 2)^2} \right) \frac{2C(3, \alpha, \delta, \lambda)\phi_3(\alpha)}{(C(2, \alpha, \delta, \lambda)\phi_2(\alpha))^2} - 1 \right].$$

Proof. By using $b = 1$ in inequality (4.5) of Theorem 4.5 gives at once the desired result. □

Remark 5.9. The functional $a_3 - ba_2^2$ is also known as Hankel determinant with Fekete-Szegö parameter b and read as $H_2^b(1)$ [10]. As shown, by using $b = 1$ in Theorem 4.3 and 4.5 gives us (no-sharp) bounds of Hankel determinant $H_2(1) = a_3 - a_2^2$. For further results in this direction, the reader may consult e.g., [8, 22, 23, 34] and cited works therein.

Remark 5.10. By specializing the involved parameters in Theorems 2.1, 3.1, 4.3, 4.5, as well as in Corollaries 5.1, 5.2, 5.4, 5.5, 5.7, 5.8, one can deduce coefficient inequalities, Fekete-Szegö, logarithmic, bi-univalent, and second Hankel determinant inequalities for the classes discussed in Remarks 1.3 and 1.4 as well as for the classes $C_\gamma[L,M]$, $S^*_\gamma[L,M]$, $C_\beta[L,M]$, $S^*_\beta[L,M]$, $C[L,M]$, $S^*[L,M]$, $C(\gamma)$, $S^*(\gamma)$ and C_β, S^*_β where $0 \neq \gamma \in \mathbb{C}$ and $0 \leq \beta < 1$. For details see [22, 23] and the cited works therein.

6. Conclusion

In this paper, we introduced and studied new subclasses of analytic functions of complex order, defined using the Ruscheweyh derivative and non-homogeneous Cauchy–Euler differential equations. For these subclasses, sharp coefficient bounds were obtained, and the Fekete–Szegö-type problems were fully solved. The results not only generalize but also improve many earlier known bounds in the literature. In addition, upper bounds for logarithmic coefficients and the second Hankel determinant were also reported, which further support the usefulness of the proposed subclasses. The results of this paper may be helpful in extending similar problems to related classes of functions, such as Ruscheweyh-type subclasses of close-to-convex and quasi-convex functions—see, for example, [12].

Appendix

Herein, the following equality is elaborated

$$a_n = \prod_{j=0}^{m-1}\left(\frac{\mu+j+1}{\mu+j+n}\right)c_n.$$

On considering $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ and $h(z) = z + \sum_{n=2}^{\infty} c_n z^n$ together with the Cauchy-Euler differential equation (1.3), we can get

$$\begin{aligned}
&\sum_{n=2}^{\infty}\prod_{k=0}^{m-1}(n-k)a_n z^n + m(\mu+m-1)\sum_{n=2}^{\infty}\prod_{k=0}^{m-2}(n-k)a_n z^n +\\
&+\frac{m(m-1)}{2}(\mu+m-1)(\mu+m-2)\sum_{n=2}^{\infty}\prod_{k=0}^{m-3}(n-k)a_n z^n + \cdots\\
&+\frac{m(m-1)}{2}\prod_{k=2}^{m-1}(\mu+k)\sum_{n=2}^{\infty}n(n-1)a_n z^n + m\prod_{k=1}^{m-1}(\mu+k)\left(z+\sum_{n=2}^{\infty}na_n z^n\right)\\
&+\prod_{k=0}^{m-1}(\mu+k)\left(z+\sum_{n=2}^{\infty}a_n z^n\right) = \prod_{k=0}^{m-1}(\mu+k+1)\left(z+\sum_{n=2}^{\infty}c_n z^n\right).
\end{aligned}$$

A simplification then gives

$$\begin{aligned}
&\sum_{n=2}^{\infty}\Bigg[\prod_{k=0}^{m-1}(n-k) + m\prod_{k=m-1}^{m-1}(\mu+k)\prod_{k=0}^{m-2}(n-k)\\
&+\frac{m(m-1)}{2}\prod_{k=m-2}^{m-1}(\mu+k)\prod_{k=0}^{m-3}(n-k) + \cdots + \frac{m(m-1)}{2}\prod_{k=2}^{m-1}(\mu+k)\prod_{k=0}^{1}(n-k)\\
&+m\prod_{k=1}^{m-1}(\mu+k)\prod_{k=0}^{0}(n-k) + \prod_{k=0}^{m-1}(\mu+k)\Bigg]a_n z^n\\
&+\left[m\prod_{k=1}^{m-1}(\mu+k) + \prod_{k=0}^{m-1}(\mu+k)\right]z = \prod_{k=0}^{m-1}(\mu+k+1)z + \prod_{k=0}^{m-1}(\mu+k+1)\sum_{n=2}^{\infty}c_n z^n.
\end{aligned}$$

Now, observe that

$$m\prod_{k=1}^{m-1}(\mu+k) + \prod_{k=0}^{m-1}(\mu+k) = (m+\mu)\prod_{k=1}^{m-1}(\mu+k) = \prod_{k=1}^{m}(\mu+k) = \prod_{k=0}^{m-1}(\mu+k+1).$$

Hence, the coefficient of z^n on both sides simplify to the following:

$$\Big[\prod_{k=0}^{m-1}(n-k)+m\prod_{k=m-1}^{m-1}(\mu+k)\prod_{k=0}^{m-2}(n-k)+\cdots+m\prod_{k=1}^{m-1}(\mu+k)\prod_{k=0}^{0}(n-k)+\prod_{k=0}^{m-1}(\mu+k)\Big]a_n=\prod_{k=0}^{m-1}(\mu+k+1)c_n.$$

It remains to show the identity

$$\prod_{k=0}^{m-1}(n-k)+m\prod_{k=m-1}^{m-1}(\mu+k)\prod_{k=0}^{m-2}(n-k)+\cdots+m\prod_{k=1}^{m-1}(\mu+k)\prod_{k=0}^{0}(n-k)+\prod_{k=0}^{m-1}(\mu+k)=\prod_{k=0}^{m-1}(\mu+k+n),\quad \forall n\geq 2,$$

holds for $m \geq 2$. We show this by using the mathematical induction. For $m = 2$:

$$\begin{aligned}\prod_{k=0}^{1}(n-k)+2\prod_{k=1}^{1}(\mu+k)\prod_{k=0}^{0}(n-k)+\prod_{k=0}^{1}(\mu+k)&=n(n-1)+2(\mu+1)n+\mu(\mu+1)\\&=n(n-1)+(\mu+1)n+(\mu+1)n+\mu(\mu+1)\\&=n(n+\mu)+(\mu+1)(n+\mu)\\&=(\mu+n)(\mu+1+n).\end{aligned}$$

Thus, it holds for $m = 2$. Assume the identity holds for some $m = j \geq 2$. Then, we need to show the identity holds for $m = j + 1$, that is

$$\prod_{k=0}^{j}(n-k)+(j+1)\prod_{k=j}^{j}(\mu+k)\prod_{k=0}^{j-1}(n-k)+\cdots+(j+1)\prod_{k=1}^{j}(\mu+k)\prod_{k=0}^{0}(n-k)+\prod_{k=0}^{j}(\mu+k)=\prod_{k=0}^{j}(\mu+k+n).$$

Using the induction hypothesis, we can write the left-hand side of the above equation as:

$$\prod_{k=0}^{j}(n-k)+(j+1)\sum_{i=0}^{j-1}\prod_{k=i}^{j}(\mu+k)\prod_{k=0}^{i-1}(n-k)+\prod_{k=0}^{j}(\mu+k),$$

which by recognizing the pattern from the hypothesis gives

$$\prod_{k=0}^{j}(n-k)+(j+1)\prod_{k=0}^{j}(\mu+k)+\prod_{k=0}^{j}(\mu+k),$$

where we have used the fact that

$$\sum_{i=0}^{j-1}\prod_{k=i}^{j}(\mu+k)\prod_{k=0}^{i-1}(n-k)=\prod_{k=0}^{j}(\mu+k).$$

Since for each i, $\prod_{k=i}^{j}(\mu+k)$ is just the partial product of $\prod_{k=0}^{j}(\mu+k)$, and summing this over i from 0 to $j-1$ gives the full product. Now after rearrangement we can write

$$\prod_{k=0}^{j}(n-k)+(j+1)\prod_{k=0}^{j}(\mu+k)+\prod_{k=0}^{j}(\mu+k)=(j+1)\prod_{k=0}^{j}(n-k)+\prod_{k=0}^{j}(\mu+k)=\prod_{k=0}^{j}(\mu+k+n).$$

This completes the induction step, and hence the identity holds for all $m \geq 2$. Now, the equality

$$\prod_{k=0}^{m-1}(\mu+k+n)a_n=\prod_{k=0}^{m-1}(\mu+k+1)c_n$$

gives the relation

$$a_n=\frac{\prod_{k=0}^{m-1}(\mu+k+1)}{\prod_{k=0}^{m-1}(\mu+k+n)}c_n=\prod_{k=0}^{m-1}\left(\frac{\mu+k+1}{\mu+k+n}\right)c_n,\quad \forall m,n\geq 2.$$

Note that, this result first appeared in [32], which was subsequently followed in various articles [12,23,36].

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has made sole contribution to this work. He read and approved the final draft of this manuscript.

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