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On The Construction of the Laplace Transform via Gamma Function

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Abstract
The Laplace transform can be applied to integrable and exponential-type functions on the half-line $[0,\infty)$ by the formula $L\{f\}=\int_0^\infty f(x)e^{-sx}dx$. This transform reduces differential equations to algebraic equations and solves many non-homogeneous differential equations. However, the Laplace transform cannot be applied to some functions such as $x^{-\frac{9}{4}}$, because the given integral is divergent. So, the Laplace transform can not solve some differential equations with some terms such as $x^{-\frac{9}{4}}$. This transform requires revision to accommodate such functions and solve a wider class of differential equations. In this study, we defined the Ω-Laplace transform, which eliminates such insufficiency of the Laplace transform and is a generalization of it. We applied this new operator to previously unsolved differential equations and obtained solutions. Ω-Laplace transform given with the help of series:

$$f(x)=\sum_{n=0}^{\infty} c_n x^{r_n} \Rightarrow \Omega\{f\}=\sum_{n=0}^{\infty}\frac{c_n\Gamma(r_n+1)}{s^{r_n+1}}$$

Moreover, we compare the similarities and differences of this transform with the Laplace transform.

1. Introduction

The Laplace operator is a method for solving differential equations. It is an integral operator that transforms a function of a single variable. Laplace wrote extensive books on the use of functions in 1814, and the integral form of the Laplace transform was developed as a result [1]. Beginning in 1744, Leonhard Euler studied the integral transform of the form known as [2]:

$$z=\int X(x)e^{ax}\,dx\,,\;\; z=\int X(x)x^{A}\,dx.$$

Lagrange was an admirer of Euler and in his work on the integration of probability density functions he investigated expressions of the form [3]:

$$\int X(x)e^{-ax}a^{x}\,dx.$$

Integrals of this type attracted the attention of Laplace, and in 1782 he began to use integral operations to solve equations, following Euler. In 1785, instead of simply looking for a solution in integral form, he used an integral of the form [4]:

$$\int x^{s}\varphi(x)dx,$$

where the last expression is called Mellin transform. Laplace developed this transform and finally gave the transform as [5]

$$L\{f\}=\int_0^\infty f(x)e^{-sx}dx. \tag{1}$$

In sources M. Çağlayan et al. [6], A. Mısır [7], R. N. Bracewell [8], W. Feller [9], G. A. Korn [10], D. V. Widder [11] and J. William [12] Laplace's transformation is given in great detail.

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The form (1) is still in use today. In sources of differential equations or operator theory, there is usually a table of Laplace transform and inverse Laplace transform. This table is used when solving differential equations. In this table,

$$L\{f\} = \frac{\Gamma(r+1)}{s^{r+1}}$$

is the Laplace transform for $r > -1$of the function

$$f(x) = x^r.$$

When $r < -1$, this function does not have a Laplace transform, since the integral (1) for is divergent for the function $f(x) = x^r$. In this work, we generalize the Laplace transform to eliminate such problems. We will call this new operator Omega-Laplace transform and denote it by Ω-Laplace. The Ω-Laplace transform can only be applied for the functions of the form

$$f(x) = \sum_{n=1}^{\infty} c_n x^{r_n},$$

as

$$\Omega\{f\} = \sum_{n=1}^{\infty} \frac{c_n \Gamma(r_n+1)}{s^{r_n+1}} \tag{2}$$

We assume that these two series converge on at least one subinterval of ℝ.

In this study, we will show that the Ω transform given by the equality (2) is a generalization of the Laplace transform, we will give the similar and different properties of this operator with the Laplace transform, and we will solve some differential equations that have not been solved before.

Note that There are other operators that have been constructed to solve differential equations that the Laplace transform cannot solve, such as Mellin transform [5]:

$$\mathcal{M}\{f\}(s) = \int_0^{\infty} x^{s-1} f(x)dx,$$

fractional Laplace transform [13]:

$$L_\alpha\{f\}(s) = \int_0^{\infty} f(x)E_\alpha(-sx)dx,$$

where E_α is the Mittag-Leffler function, and Heaviside calculus [14]. The Mellin transform is used to solve some differential equations. However, it turns some terms like $x^k y$ into $\mathcal{M}\{f\}(s+k)$. This makes it difficult to solve the kind of differential equations we solve. Fractional Laplace transform is useful for fractional differential equation. It is also not useful for the kind of differential equations we solve. Heaviside calculus is a method to turn differential expression into algebraic expression like the Laplace transform. However, it is insufficient to solve the differential equations we solved in our examples.

There are recent studies on series representations. In [15-17], series representations of mathematical expressions, such as constants, functions, operators, etc., are obtained and their applications to differential equations or different fields are given. However, in these studies, the definition of the Laplace transform with the help of series and its application to unsolved differential equations are not available.

Schrödinger equation is very important for mathematics, physics and engineering. Particularly, it gives the mathematical formulation of quantum mechanics [18]:

$$i\hbar \frac{\partial}{\partial t}\Psi(x,t) = \left(-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x,t)\right)\Psi(x,t)$$

in nuclear physics. The function $V(x,t)$ is called potential and it determinates the behavior of particles. In some cases, the potential takes a value such as x^q $(q < -1)$. Then, the Laplace transform or other transform cannot give a solution for the Schrödinger equation. For such equations the Omega-Laplace transform can be used.

In recent years, the generalized Mellin transform, the generalized Fourier transform and the fractional Laplace transform have been investigated in [19-24]. However, the generalizations here are given in the integral form, as in the classical definitions. The series approach is present in our work.

2. Results and Discussion

Classical Laplace transform is defined by

$$L\{f\}(s) = \int_0^{\infty} f(x)e^{-sx}\,dx$$

and is frequently used in differential equations, physics and engineering problems. Although it is very effective in solving problems, we noticed some shortcomings in the definition given by Laplace above. For this reason, we have defined a new Laplace transform using infinite series. So as not to be confused with the L operator, we will use this new transform in the work of Ω. As we will show in the examples that it can solve some differential equations that the classical Laplace transform cannot solve with the Ω transform. Let's now examine this new definition, its properties and examples.

Definition 1. Let $x \in \mathbb{R}, r_n \in \mathbb{R} \setminus \mathbb{Z}^-$, $\{c_n\}, \{r_n\}$ be sequences of real numbers and the series

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n} \tag{3}$$

converges on at least one subinterval of $\mathbb{R}$. We define the transform of f as follows:

$$\Omega\{f\} = \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n + 1)}{s^{r_n+1}}. \tag{4}$$

In this study, we will work on functions of type (3) that are expanded into series. When we try to solve differential equations, we try to find solutions in the form of (3). The transform of (3) is given in (4). The transform in (4) forms the basis of the present study and is an alternative to the classical Laplace transform. There is no image of functions like $x^{-\frac{3}{2}}$ under the classical Laplace transform. However, this function has an Ω transform. The Ω transform of this function is $2\sqrt{\pi s}$. Now, let's give some theorems of this new transform and use them in our differential equations.

Theorem 1. We split the series (3) into positive and negative exponents as follows.

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n} + \sum_{n=0}^{\infty} d_n x^{-q_n}, \qquad r_n, q_n > 0,$$

Assume that the limits

$$R = \lim_{n\to\infty} |c_n|^{\frac{1}{r_n}}, \qquad Q = \lim_{n\to\infty} |d_n|^{\frac{1}{q_n}}$$

exist and $Q < R$. Then, the series (3) converges on the interval (Q, R). Also, the series converges on the set $(-R, -Q) \cup (Q, R)$ in the case that all the exponents of series f are integers.

Proof. Consider the series

$$\sum_{n=0}^{\infty} c_n x^{r_n} \tag{5}$$

and

$$\sum_{n=0}^{\infty} d_n x^{-q_n}. \tag{6}$$

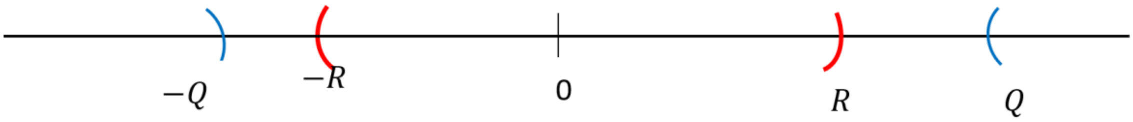
We investigate the convergence of the series (5) and (6) separately. We apply the Cauchy root test for the convergence of the series (5):

$$\begin{gathered}
\lim_{n\to\infty} \sqrt[n]{|c_n x^{r_n}|} = \lim_{n\to\infty} \sqrt{|c_n|}. \, \lim_{n\to\infty} |x|^{r_n/n} \\
= \lim_{n\to\infty} \sqrt[n]{|c_n|}. \, |x|^{\lim_{n\to\infty} \frac{r_n}{n}} < 1 \\
\Leftrightarrow |x|^{\lim_{n\to\infty} \frac{r_n}{n}} < \frac{1}{\lim_{n\to\infty} \sqrt[n]{|c_n|}} \\
\Leftrightarrow |x| < \frac{1}{\left(\lim_{n\to\infty} n_{\sqrt{|c_n|}}\right)^{1/\lim_{n\to\infty} \frac{r_n}{n}}} \\
\Leftrightarrow |x| < \frac{1}{\lim_{n\to\infty} \left(|c_n|^{\frac{1}{n}}\right)^{\frac{n}{r_n}}} \\
\Leftrightarrow |x| < \frac{1}{\lim_{n\to\infty} |c_n|^{\frac{1}{r_n}}} \\
\Leftrightarrow |x| < R = \lim_{n\to\infty} |c_n|^{-\frac{1}{r_n}} \\
-R < x < R, x \in (-R, R),
\end{gathered}$$

We assume that R is positive. Otherwise, the series (2) cannot converge anywhere. If at least one of the exponents is not an integer, the interval takes the form $(0, R)$. Now, we investigate the series (6):

$$\begin{gathered}
\lim_{n\to\infty} \sqrt{|d_n \cdot x^{-q_n}|} = \lim_{n\to\infty} \sqrt[n]{|d_n|} \cdot \lim_{n\to\infty} |x|^{-\frac{q_n}{n}} \\
= \lim_{n\to\infty} \sqrt[n]{|d_n|} \cdot |x|^{-\lim_{n\to\infty} \frac{q_n}{n}} < 1 \\
\Leftrightarrow |x|^{-\lim_{n\to\infty} \frac{q_n}{n}} < \frac{1}{\lim_{n\to\infty} \sqrt[n]{|d_n|}} \\
\Leftrightarrow |x| > \frac{1}{\left(\lim_{n\to\infty} \sqrt[n]{|dn|}\right)^{-\frac{1}{\lim_{n\to\infty} \frac{q_n}{n}}}} \\
\Leftrightarrow |x| > \lim_{n\to\infty} \left(|d_n|^{\frac{1}{n}}\right)^{\frac{n}{q_n}} \\
\Leftrightarrow |x| > \lim_{n\to\infty} |d_n|^{\frac{1}{q_n}} = Q \\
x \epsilon (-\infty, Q) \cup (Q, +\infty).
\end{gathered}$$

If $R \leq Q$, the set of convergence of the series (3) is null. See the following



So, we assume that $Q < R$. When we take the intersection of the convergence sets, we obtain $(-R, -Q) \cup (Q, R)$. If at least one of the exponents is not an integer, the set of convergence is (Q, R).

Theorem 2. For a function f that has the series expansion (3), this expansion is unique.

Proof. We sketch the proof in the case where all the exponents r_n are ordered in the form $r_0 < r_1 < r_2 < \cdots$. The function f has 2 series expansion as

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n} = \sum_{n=0}^{\infty} d_n x^{q_n}, r_n, q_n \in \mathbb{R} \setminus \mathbb{Z}^-.$$

Assume that $r_0 = q_0, r_1 = q_1, r_2 = q_2, \ldots$. Even if the exponents are not equal, we can assume the equality as true by writing 0 to the coefficient of different exponents. The last equality can be written by

$$c_0 x^{r_0} + c_1 x^{r_1} + c_2 x^{r_2} + \cdots = d_0 x^{r_0} + d_1 x^{r_1} + d_2 x^{r_2} + \cdots.$$

We product both sides of the equality by x^{-r_0}:

$$c_0 + c_1 x^{r_1 - r_0} + c_2 x^{r_2 - r_0} + \cdots = d_0 + d_1 x^{r_1 - r_0} + d_2 x^{r_2 - r_0} + \cdots$$

The equality takes the form $c_0 = d_0$ for $x = 0$. Similarly, we have

$$\begin{aligned} c_1 &= d_1 \\ c_2 &= d_2 \\ &\vdots \\ c_n &= d_n \\ &\vdots \end{aligned}$$

Theorem 3. (Linearity Property). Let

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n}, \qquad g(x) = \sum_{n=0}^{\infty} d_n x^{q_n}$$

and $a, b \in \mathbb{R}$. Then, the equality

$$\Omega\{af(x) + bg(x)\} = a\Omega\{f(x)\} + b\Omega\{g(x)\}$$

holds.

Proof. Consider two functions f and g in the form

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n}, \qquad g(x) = \sum_{n=0}^{\infty} d_n x^{q_n}$$

and apply the operator Ω:

$$\begin{aligned} \Omega\{af(x) + bg(x)\} &= \Omega\left\{a \sum_{n=0}^{\infty} c_n x^{r_n} + b \sum_{n=0}^{\infty} d_n x^{q_n}\right\} \\ &= \Omega\left\{\sum_{n=0}^{\infty} a c_n x^{r_n} + \sum_{n=0}^{\infty} b d_n x^{q_n}\right\} \\ &= \sum_{n=0}^{\infty} \frac{a c_n \Gamma(r_n + 1)}{s^{r_n+1}} + \sum_{n=0}^{\infty} \frac{b d_n \Gamma(q_n + 1)}{s^{q_n+1}} \\ &= a \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n + 1)}{s^{r_n+1}} + b \sum_{n=0}^{\infty} \frac{d_n \Gamma(q_n + 1)}{s^{q_n+1}} \\ &= a\Omega\{f(x)\} + b\Omega\{g(x)\}. \end{aligned}$$

This completes the proof.

Theorem 4. Let f be a function of the series (3). Then the image of the function fe^{ax} under the transform of Ω is equal to the value of the image of f under the transform Ω at the point $s - a$. That is,

$$\Omega\{f(x)e^{ax}\}(s) = \Omega\{f\}(s - a).$$

Proof: We apply the transform Ω to the multiplication

$$f(x)e^{ax} = \left(\sum_{n=0}^{\infty} c_n x^{r_n}\right)\left(\sum_{n=0}^{\infty} \frac{a^n x^n}{n!}\right)$$
$$= (c_0 x^{r_0} + c_1 x^{r_1} + \cdots + c_n x^{r_n} + \cdots)\left(1 + ax + \frac{a^2x^2}{2!} + \frac{a^3x^3}{3!} + \cdots \frac{+a^n x^n}{n!} + \cdots\right)$$
$$= c_0 x^{r_0} + c_1 x^{r_1} + \cdots + c_n x^{r_n} + \cdots + a(c_0 x^{r_{0+1}} + c_1 x^{r_{1+1}} + \cdots + c_n x^{r_n+1} + \cdots)$$
$$+ \frac{a^2}{2!}(c_0 x^{r_0+2} + c_1 x^{r_1+2} + \cdots + c_n x^{r_n+2} + \cdots) + \cdots$$

we have

$$\frac{c_0\Gamma(r_0+1)}{s^{r_0+1}} + \frac{c_1\Gamma(r_1+1)}{s^{r_1+1}} + \cdots + \mathrm{a}\left(\frac{c_0\Gamma(r_0+2)}{s^{r_0+2}} + \frac{c_1\Gamma(r_1+2)}{s^{r_1+1}} + \cdots\right)$$
$$+ \frac{a^2}{2!}\left(\frac{c_0\Gamma(r_0+3)}{s^{r_0+3}} + \frac{c_1\Gamma(r_1+3)}{s^{r_1+3}} + \cdots\right) + \cdots.$$

For $t > -1$ and $\alpha \in \mathbb{R}$, we obtain

$$(1+t)^\alpha = \sum_{n=0}^{\infty} \binom{\alpha}{n} t^n$$

The last gives us the relation

$$\frac{1}{(s-a)^\alpha} = (s-a)^{-\alpha} = s^{-\alpha}\left(1 + \left(\frac{-a}{s}\right)\right)^{-\alpha}$$
$$= s^{-\alpha} \sum_{n=0}^{\infty} \binom{-\alpha}{n} (-1)^n \left(\frac{a^n}{s^n}\right),$$

where

$$\binom{-\alpha}{n} = \frac{-\alpha(-\alpha-1)\ldots(-\alpha-n+1)}{n!}$$

(see, [25]). Then, we have

$$\frac{1}{(s-a)^\alpha} = s^{-\alpha} \sum_{n=0}^{\infty} \frac{-\alpha(-\alpha-1)\ldots(-\alpha-n+1)}{n!}(-1)^n \frac{a^n}{s^n}$$
$$= \sum_{n=0}^{\infty} \frac{\alpha(\alpha+1)\ldots(\alpha+n-1)}{n!} . \frac{a^n}{s^{n+\alpha}}.$$

and,

$$\frac{1}{(s-\alpha)^{r_k+1}} = \sum_{n=0}^{\infty} \frac{(r_k+1)\ldots(r_k+n)}{n!} . \frac{a^n}{s^{n+r_k+1}}.$$

The image of the function f under the transform Ω at $s-a$ is

$$\Omega\{f\}(s-a) = \sum_{n=0}^{\infty} \frac{c_n\Gamma(r_n+1)}{(s-a)^{r_n+1}} = \frac{c_0\Gamma(r_0+1)}{(s-a)^{r_0+1}} + \frac{c_1\Gamma(r_1+1)}{(s-a)^{r_1+1}} + \cdots$$
$$c_0\Gamma(r_0+1)\left(\frac{1}{s^{r_0}+1} + a \cdot \frac{r_0+1}{s^{r_0+2}} + \frac{a^2}{2!}\frac{(r_0+1)(r_0+2)}{s^{r_0+3}} + \cdots\right)$$
$$+c_1\Gamma(r_1+1) + \left(\frac{1}{s^{r_1}+1} + a \cdot \frac{r_1+1}{s^{r_1+2}} + \frac{a^2}{2!}\frac{(r_1+1)(r_1+2)}{s^{r_1+3}} + \cdots\right).$$
$$+\cdots$$

This completes the proof.
For $f(x) = 1$, we have

$$\Omega\{e^{ax}\} = \frac{1}{s-a}.$$

Theorem 5. Let the function

$$f(x)=\sum_{n=0}^{\infty} c_n x^{r_n}, r_{k_0}=0, r_n \notin \mathbb{Z}^-$$

be a differentiable function and the series are uniform convergence on an interval of the reals. Then,

$$\Omega\{f'(x)\}=s\Omega\{f(x)\}-c_{k_0},$$

where, c_{k_0} is the coefficient of the term x^0 in the series. If, the function f is a differentiable function of order n, we have

$$\Omega\{f^{(n)}\}=s^n\Omega\{f\}-s^{n-1}c_{k_0}-\cdots-sc_{k_{n-2}}-c_{k_{n-1}}$$

where, c_{k_m} $(m=\overline{0,n-1})$ is the coefficient of the term in the series x^m.

Proof: Consider a function f in the form

$$f(x)=\sum_{n=0}^{\infty} c_n x^{r_n}, r_{k_0}=0, r_n \notin \mathbb{Z}^-,$$

then we have

$$f'(x)=\sum_{n=0}^{\infty} c_n r_n x^{r_n-1}.$$

Now, we apply the operator Ω:

$$\Omega\{f'\}=\sum_{n=0}^{\infty}\frac{c_n r_n \Gamma(r_n)}{s^{r_n}}=s\left(\sum_{n=0}^{\infty}\frac{c_n\Gamma(r_n+1)}{s^{r_n+1}}+\frac{c_{k_0}}{s}-\frac{c_{k_0}}{s}\right)=s\left(\Omega\{f\}-\frac{c_{k_0}}{s}\right)$$
$$=s\Omega\{f\}-c_{k_0}.$$

This completes the proof for first order derivative. If we apply the formula obtained for the first order derivative repeatedly for higher order derivatives, we have

$$\Omega\{f^{(n)}\}=s^n\Omega\{f\}-s^{n-1}c_{k_0}-\cdots-sc_{k_{n-2}}-c_{k_{n-1}}.$$

Theorem 6. If the series (3) and (4) are convergent, then

$$\frac{d}{ds}\Omega\{f(x)\}(s)=\Omega\{-xf(x)\}(s).$$

Proof: If we take the derivative of Ω of f in the form (4) with respect to s, we have

$$\frac{d}{ds}\Omega\{f(x)\}(s)=\sum_{n=0}^{\infty}-c_n(r_n+1)\cdot\Gamma(r_n+1)s^{-(r_n+2)}$$

$$=\sum_{n=0}^{\infty}-\frac{c_n\Gamma(r_n+2)}{s^{r_n+2}}$$
$$=\Omega\left\{\sum_{n=0}^{\infty}-c_n x^{r_n+1}\right\}$$
$$=\Omega\left\{(-x)\sum_{n=0}^{\infty}c_n x^{r_n}\right\}$$
$$=\Omega\{-xf(x)\}.$$

Similarly, we have the following for all the positive integers n

$$\frac{d^n s}{ds^n}\Omega\{f(x)\}=\Omega\{(-1)^n x^n f(x)\}.$$

Theorem 7. If the series (3) are uniformly convergent and the integral

$$\int_0^x f(u)\,du$$

is convergent on an interval $(0,R), R>0$, we have

$$\Omega\left\{\int_0^x f(u)\,du\right\}=\frac{1}{s}\Omega\{f(x)\}(s).$$

Proof. We write the series (3) as

$$f(u) = \sum_{n=0}^{\infty} c_n u^{r_n}.$$

$$\int_0^x \left(\sum_{n=0}^{\infty} c_n u^{r_n}\right) du = \sum_{n=0}^{\infty} \left(c_n \int_0^x u^{r_n}\, du\right) = \sum_{n=0}^{\infty} c_n \left(\frac{u^{r_n+1}}{r_n+1}\bigg|_0^x\right) = \sum_{n=0}^{\infty} c_n \frac{x^{r_n+1}}{r_n+1}.$$

We apply the Ω transform:

$$\Omega\left\{\int_0^x f(u)\, du\right\} = \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n+2)}{(r_n+1)s^{r_n+2}} = \sum_{n=0}^{\infty} \frac{c_n (r_n+1)\Gamma(r_n+1)}{(r_n+1)s^{r_n+1}s}$$

$$= \frac{1}{s}\sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n+1)}{s^{r_n+1}} = \frac{1}{s}\Omega\{f(x)\}(s).$$

This completes the proof.

Theorem 8. If the series (3) and (4) are uniformly convergent on intervals I and J respectively and the exponents r_n in (3) are not 0, then we have

$$\Omega\left\{\frac{f(x)}{x}\right\}(s) = \int_s^{\infty} \Omega\{f\}(u)\, du.$$

Proof: Since the following equality holds

$$\frac{f(x)}{x} = \sum_{n=0}^{\infty} c_n x^{r_n - 1}$$

Then we can write

$$\Omega\left\{\frac{f(x)}{x}\right\} = \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n)}{s^{r_n}}.$$

If we both multiply and divide the series by r_n, we have

$$\Omega\left\{\frac{f(x)}{x}\right\} = \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n) r_n}{s^{r_n} r_n} = \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n+1)}{s^{r_n} r_n} = \sum_{n=0}^{\infty} c_n \Gamma(r_n+1) \int_s^{\infty} \frac{du}{u^{r_n+1}} = \int_s^{\infty} \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n+1)}{u^{r_n+1}} du$$

$$= \int_s^{\infty} \Omega\{f\}(u) du.$$

Theorem 9 (Convolution). If f and g are functions given in the form of series (3), and those series both uniforms converge on an interval $(0,R), R > 0$, then the equality holds

$$\Omega\{(f * g)(x)\} = \Omega\{f(x)\} \cdot \Omega\{g(x)\}$$

where

$$f(x) = \sum_{n=0}^{\infty} c_n x^{r_n}, g(x) = \sum_{m=0}^{\infty} d_m x^{q_m}$$

and

$$(f * g)(x) = \int_0^x f(t) \cdot g(x-t)\, dt.$$

The last integral is known as the convolution of the functions f and g.

Proof: First, we find the series expansion of the function $f(t) \cdot g(x-t)$:

$$f(t) \cdot g(x-t) = \left(\sum_{n=0}^{\infty} c_n t^{r_n}\right) \cdot \left(\sum_{m=0}^{\infty} d_m (x-t)^{q_m}\right)$$

$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m t^{r_n} (x-t)^{q_m}.$$

Second, we integrate this function on the interval $[0, x]$. By the uniform convergence, we have

$$(f * g)(x) = \int_0^x \left(\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m t^{r_n} (x - t)^{q_m} \right) dt$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m \int_0^x t^{r_n} (x - t)^{q_m} \, dt.$$

Third, by change of variable $(t = xu)$, we have

$$(f * g)(x) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m \int_0^1 x^{r_n} u^{r_n} (x - xu)^{q_m} \, du$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m x^{r_n + q_m + 1} \int_0^1 u^{r_n} (1 - u)^{q_m} \, du$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m x^{r_n + q_m + 1} \beta(r_n + 1, q_m + 1)$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m x^{r_n + q_m + 1} \frac{\Gamma(r_n + 1)\Gamma(q_m + 1)}{\Gamma(r_n + q_m + 2)}.$$

Finally, we apply the operator Ω:

$$\Omega\{(f * g)\} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_n d_m \frac{\Gamma(r_n + 1)\Gamma(q_m + 1)\Gamma(r_n + q_m + 2)}{\Gamma(r_n + q_m + 2) s^{r_n + q_m + 2}}$$
$$= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{c_n \Gamma(r_n + 1)}{s^{r_n + 1}} \frac{d_m \Gamma(q_m + 1)}{s^{q_m + 1}}$$
$$= \sum_{n=0}^{\infty} \frac{c_n \Gamma(r_n + 1)}{s^{r_n + 1}} \sum_{m=0}^{\infty} \frac{d_m \Gamma(q_m + 1)}{s^{q_m + 1}}$$
$$\Omega\{f(x)\} \cdot \Omega\{g(x)\},$$

where β is the beta function, see [8/10].

3. Application

Example 1. Let's calculate the Ω transform of the trigonometric function $\sin bx$ and $\cos bx, b > 0$ by using the Taylor expansion:

$$\sin bx = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1} b^{2n+1}}{(2n + 1)!}.$$

Apply the operator Ω to above series:

$$\Omega\{\sin bx\} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{2n+1} (2n + 1)!}{(2n + 1)! \, s^{2n+2}} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{2n+1}}{s^{2n+2}} = \frac{b}{s^2} \sum_{n=0}^{\infty} (-1)^n \left(\frac{b^2}{s^2}\right)^n$$
$$= \frac{b}{s^2} \frac{1}{1 + \frac{b^2}{s^2}} = \frac{b}{s^2 + b^2}.$$

Similarly, consider the following series and apply the same process:

$$\cos bx = \sum_{n=0}^{\infty} \frac{(-1)^n b^{2n} x^{2n}}{(2n)!}$$
$$\Omega\{\cos bx\} = \sum_{n=0}^{\infty} \frac{(-1)^n b^{2n} (2n)!}{(2n)! \, s^{2n+1}} = \frac{1}{s} \sum_{n=0}^{\infty} \frac{(-1)^n b^{2n}}{s^{2n}} = \frac{1}{s} \sum_{n=0}^{\infty} (-1)^n \left(\frac{b^2}{s^2}\right)^n$$
$$= \frac{1}{s} \frac{1}{1 + \frac{b^2}{s^2}} = \frac{s}{s^2 + b^2}.$$

We see that the images of cos and sin functions under Laplace transform and Ω transform are the same.

Example 2. Solve the first order equation $2xy' = -3y$ with the ; Ω transform.

Solution: Apply the Ω transform of both sides of the equation:

$$2\Omega\{(-x)y'\} = 3\Omega\{y\}.$$

By Theorem 6, we have

$$2\frac{d}{ds}[\Omega\{y'\}] = 3\Omega\{y\}.$$

By Theorem 5, we have

$$2\frac{d}{ds}\left[s\Omega\{y\} - c_{k_0}\right] = 3\Omega\{y\}.$$

By derivative of multiplication, we have

$$2\Omega\{y\} + 2s\frac{d\Omega\{y\}}{ds} = 3\Omega\{y\}$$
$$\Rightarrow 2s\frac{d\Omega\{y\}}{ds} = \Omega\{y\}.$$

Now, we separate the variables:

$$\frac{2\,d\Omega\{y\}}{\Omega\{y\}} = \frac{ds}{s},$$

then

$$2\ln\Omega\{y\} = \ln s + c_1.$$

Apply the exponential function to both sides of equation and reorganize the constant:

$$(\Omega\{y\})^2 = c_2^2 s \quad \left(c_2 = e^{\frac{c_1}{2}}\right)$$
$$\Rightarrow \Omega\{y\} = c_2\sqrt{s}$$
$$\Rightarrow \Omega\{y\} = \frac{c_2\sqrt{s}\Gamma(-\frac{1}{2})}{\Gamma(-\frac{1}{2})}.$$

By the definition of Ω, we find the solution:

$$y = \frac{c}{x^{\frac{3}{2}}} \quad \left(c = \frac{c_2}{\Gamma(-\frac{1}{2})}\right).$$

The general solution of the equation is $y = \frac{c}{x^{\frac{3}{2}}}$. Since the equation is separable, it could be solved even without Ω or Laplace transform. Now, we consider a second-order nonhomogeneous linear differential equation that cannot be solved by classical methods. Note that the analytical solution of the following equation cannot be obtained by classical methods.

Example 4. Solve the equation $4y'' + 2xy' + 5y = 35x^{-9/2}$ with the transform Ω. Let's transform both sides of the equation by Ω.

$$\Omega\{4y'' + 2xy' + 5y\} = \Omega\{35x^{-9/2}\}.$$

By linearity and definition of Ω, we have

$$4\Omega\{y''\} + 2\Omega\{xy'\} + 5\Omega\{y\} = 35\Gamma\left(-\frac{7}{2}\right)s^{\frac{7}{2}}.$$

By Theorem 5 and 6, we have

$$4\left(s^2\Omega\{y\} - sc_{k_0} - c_{k_1}\right) - 2\frac{d}{ds}\Omega\{y'\} + 5\Omega\{y\} = \frac{35.16\sqrt{\pi}}{105}s^{\frac{7}{2}}$$
$$\Rightarrow 4s^2\Omega\{y\} - 4sc_{k_0} - 4c_{k_1} - 2\frac{d}{ds}\left(s\Omega\{y\} - c_{k_0}\right) + 5\Omega\{y\} = \frac{16\sqrt{\pi}s^{\frac{7}{2}}}{3}$$
$$\Rightarrow 4s^2\Omega\{y\} - 4sc_{k_0} - 4c_{k_1} - 2\Omega\{y\} - 2s\frac{d}{ds}\Omega\{y\} + 5\Omega\{y\} = \frac{16\sqrt{\pi}s^{\frac{7}{2}}}{3}$$
$$\Rightarrow -2s\frac{d}{ds}\Omega\{y\} + (4s^2 + 3)\Omega\{y\} = \frac{16\sqrt{\pi}s^{\frac{7}{2}}}{3} + 4sc_{k_0} + 4c_{k_1}$$
$$\Rightarrow \frac{d}{ds}\Omega\{y\} - \left(2s + \frac{3}{2s}\right)\Omega\{y\} = -\frac{8\sqrt{\pi}s^{\frac{5}{2}}}{3} - 2c_{k_0} - \frac{2c_{k_1}}{s}.$$

To solve this differential equation, we multiply each side by $s^{-\frac{3}{2}}e^{-s^2}$ the integrating factor used for linear equations of order 1. To find a particular solution of the differential equation, we choose the coefficients $c_{k_0} = c_{k_1} = 0$ of the terms x^0 and x^1.

$$s^{\frac{-3}{2}}e^{-s^2}\frac{d}{ds}\Omega\{y\}-s^{\frac{-3}{2}}\left(2s+\frac{3}{2s}\right)e^{-s^2}\Omega\{y\}=-\frac{8\sqrt{\pi}se^{-s^2}}{3}$$

$$\Rightarrow \Omega\{y\}=s^{\frac{3}{2}}e^{s^2}\int\left(-\frac{8\sqrt{\pi}se^{-s^2}}{3}ds\right)=s^{\frac{3}{2}}e^{s^2}\left(\frac{4\sqrt{\pi}}{3}e^{-s^2}+c\right)=\frac{4\sqrt{\pi}s^{\frac{3}{2}}}{3}+cs^{\frac{3}{2}}e^{s^2}$$

We choose again $c=0$. Since $\Omega\{y\}=\frac{4\sqrt{\pi}}{3s^{-3/2}}$, by the definition of the transform Ω, we have

$$y=x^{-\frac{5}{2}}$$

It is an analytical, particular, solution of the differential equation. The general solution of the equation can be obtained by classical methods of differential equations theory using that particular solution.

Ω operator is an operator like Laplace. It has similar and different properties with the Laplace operator. Below is an Ω transform table for the operator like Laplace transform table.

Table 1. *Ω* Transform table

$f(x)$	$\Omega\{f\}$
1	$\frac{1}{s}$
$x^r e^{ax}$	$\frac{\Gamma(r+1)}{(s-a)^{r+1}}$
$e^{ax}\sin bx$	$\frac{b}{(s-a)^2+b^2}$
$e^{ax}\cos bx$	$\frac{s-a}{(s-a)^2+b^2}$
$(-1)^n x^n f(x)$	$\frac{d^n s}{ds^n}\Omega\{f(x)\}$
$\sin bx$	$\frac{b}{s^2+b^2}$
$\cos bx$	$\frac{s}{s^2+b^2}$
e^{ax}	$\frac{1}{s-a}$
$f(x)e^{ax}$	$\Omega\{f\}(s-a)$
$\int_0^x f(u)\,du$	$\frac{1}{s}\Omega\{f(x)\}(s)$
$f^{(n)}$	$s^n\Omega\{f\}-s^{n-1}c_{k_0}-\cdots-sc_{k_{n-2}}-c_{k_{n-1}}$
$\frac{f(x)}{x}$	$\int_s^\infty \Omega\{f\}(u)\,du$
$\Omega\{(f*g)(x)\}$	$\Omega\{f(x)\}\cdot\Omega\{g(x)\}$

Now, we explain the similarities and differences between Laplace and Ω-Laplace operators. First, we give the similarities

1. Both operators turn a differential equation into an algebraic equation.
2. Both operators turn the functions like $x^r \sin bx$, $\cos bx$ into $\frac{\Gamma(r+1)}{s^{r+1}}, \frac{b}{s^2+b^2}, \frac{s}{s^2+b^2}$ respectively.
3. Both operators have shifting property (see Theorem 4)
4. Both operators have derivative property (see Theorem 6).
5. Both operators have two integration properties (see Theorem 7 and 8).
6. Both operators have convolution property (see Theorem 9).

Second, we give a differences table

Table 2. Differences between Laplace and Ω-Laplace Transform

Laplace transform	Ω-Laplace transform
can be applied exponential order functions integrable on $[0,\infty)$	can be applied functions of the form $f(x)=\sum_{n=0}^{\infty} c_n x^{r_n}$
can sometimes lead to difficult integrals	to take the transformation of a function, it is enough to know that it transforms x^r into $\frac{\Gamma(r+1)}{s^{r+1}}$
can not be applied to functions of the form x^r, $(r<-1)$	can be applied to functions of the form x^r, $(r<-1)$
to be applied to the derivative of a function, the function must be defined at 0	to be applied to the derivative of a function, the function need not be defined at 0
the derivative formula is $L\{f'\}=sL\{f\}-f(0)$	the derivative formula is $L\{f'\}=sL\{f\}-c_{k_0}$
to be applied to the n^{th} derivative of a function, the function and derivatives must be defined at 0	to be applied to the n^{th} derivative of a function, the function and derivatives need not be defined at 0
the higher derivative formula is $L\{f^{(n)}\}=s^nL\{f\}-s^{n-1}f(0)-s^{n-2}f'(0)-\cdots-f^{(n-1)}(0)$	the higher derivative formula is $L\{f^{(n)}\}=s^nL\{f\}-s^{n-1}c_{k_0}-s^{n-2}c_{k_1}-\cdots-c_{k_{n-1}}$
can not solve all differential equations that Ω-Laplace transform can solve	can solve all differential equations that Laplace transform can solve, and it can also solve differential equations other than this

4. Conclusion

In this study, we introduced the Ω-Laplace transform, a significant generalization of the classical Laplace transform. The classical Laplace transform, while powerful in reducing complex differential equations to algebraic forms, faces limitations when applied to functions with terms like x^r where $r<-1$, as the integral involved becomes divergent. The Ω-Laplace transform overcomes this limitation by incorporating a series-based approach, extending its applicability to a broader class of functions. Through several examples, we demonstrated that this new transform is capable of solving differential equations that were previously unsolvable with traditional methods. In addition, the similarities and differences between the Ω-Laplace and classical Laplace transforms were thoroughly analyzed, leading to the development of a comprehensive transformation table for Ω-Laplace.

The practical utility of the Ω-Laplace transform lies in its ability to handle functions that arise in fields like quantum mechanics, where potential functions can take values outside the scope of the classical Laplace transform. By providing an alternative approach to solving complex differential equations, the Ω-Laplace transform opens new avenues for mathematical analysis in applied contexts.

The introduction of the Ω-Laplace transform suggests several promising directions for future research. First, the exploration of further generalizations of this operator may yield even more versatile tools for solving a wider variety of differential equations. Additionally, applying the Ω-Laplace transform to higher-dimensional problems, such as partial differential equations in physics and engineering, could significantly enhance its practical applications. Moreover, studying the connections between Ω-Laplace and other integral transforms, such as the Mellin or Fourier transforms, might provide new insights and powerful hybrid techniques for advanced mathematical and physical models.

Furthermore, implementing the Ω-Laplace transform in computational software could make it accessible to a broader audience of researchers in applied sciences. This work sets the foundation for a deeper exploration of series-based transformations and their potential to revolutionize the solution of intricate mathematical problems.

Contributions of the authors

This study is the master's thesis of Ş.E. U.K. is the thesis supervisor. The authors' contribution to the study is equal.

Conflict of Interest Statement

There is no conflict of interest between the authors.

Statement of Research and Publication Ethics

The study is complied with research and publication ethics.

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