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FUZZY NONLINEAR SECOND ORDER VOLTERRA INTEGRODIFFERENTIAL EQUATION

S.MELLIANI, A. MOUJAHID, A. KAJOUNI, AND L. S. CHADLI

Abstract. In this paper we generalized the definitions of family cosine and sine in the fuzzy case. secondly we proved the existence and uniqueness of the mild solution of nonlinear second order Volterra integrodifferential equation with fuzzy initial data. Finally we given an application example.

1. Introduction

The problems of existence, uniqueness and other properties of solutions for the second order systems have much attention in the recent years. It is advantageous to treat second order abstract differential equations directly rather than to convert into first order systems (refer, Fitzgibbon [8]). Fitzgibbon [8] used the second order abstract system for establishing the boundedness of solutions of the equation governing the transverse motion of an extensible beam. A useful technique for the study of abstract second order equations is the theory of strongly continuous cosine family of operators. We will make use of some of the basic ideas from cosine family theory [7, 9, 19, 21, 22]. Motivation for second order systems can be found in [7, 12, 16, 19, 20].
In this regard H. L. Tidke and M. B. Dhakene are study the existence and uniqueness of solutions of second order nonlinear Volterra integro-differential :

$$x''(t) = Ax(t) + f(t, x(t)) \int_{t_0}^{t} k(t, s, x(s))ds, \quad 0 \leq t_0 \leq T$$
$$x(t_0) = x_0, \quad x'(t_0) = y_0$$

where A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in Banach space X, $f : [t_0, T] \times X \times X \to X$, $k : [t_0, T] \times [t_0, T] \times X \to X$ are continuous functions and x_0, y_0 are given elements of X.
In this article we will generalize the definitions of family cosine and sine in the fuzzy case, in the second part we will prove the existence and uniqueness of the mild solution of the Previous problem with A is an infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ in space E^n, $f : [t_0, T] \times E^n \times E^n \to E^n$, $k : [t_0, T] \times [t_0, T] \times E^n \to E^n$ are continuous functions and x_0, y_0 are given elements of E^n.

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2. Preliminaries

We introduce notations, definitions and preliminary facts that will be used throughout the paper.

Definition 2.1. A one parameter family $\{C(t) : t \in \mathbb{R}\}$ of bounded linear operators in E^n is called a strongly continuous cosine family if and only if

- (a) $C(0) = I$ (I is the identity operator on E^n);
- (b) $C(t)x$ is strongly continuous in t on $\mathbb{R}$ for each fixed $x \in E^n$;
- (c) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$.

If $\{C(t) : t \in \mathbb{R}\}$ is a strongly continuous cosine family in E^n, then we define the associated sine family $\{S(t) : t \in \mathbb{R}\}$ by

$$S(t)x = \int_0^t C(s)xds\ , \quad x \in E^n,\ t \in \mathbb{R} \quad (3)$$

The infinitesimal generator $A : E^n \to E^n$ of a cosine family $C(t) : t \in \mathbb{R}$ is defined by

$$Ax = \frac{d^2}{dt^2} C(t)x|_{t=0},\ x \in D(A)$$

where $D(A) = \{x \in E^n : C(\cdot)x \in C^2(\mathbb{R}, E^n)\}$. Let $M \geq 1$ and N be two positive constants such that $\|C(t)\| \leq M$ and $\|S(t)\| \leq N$ for all $t \in [0, T]$.

Definition 2.2. We say that x is a mild solution of the problem (1)-(2) if:

(i) $x \in \mathcal{C}([t_0, T], E^n)$ and $x(t) \in D(A)$ for all $t \in [t_0, T]$

(ii) $x(t) = C(t-t_0)x_0 + S(t-t_0)y_0 + \int_{t_0}^t S(t-s)f(s, x(s), \int_{t_0}^s k(s, \tau, x(\tau))d\tau)dx, t \in [t_0, T]$ (4)

We list the following hypotheses for our convenience.

(H_1) For $t, s \in [t_0, T]$ and $x_i, y_i \in E^n, i = 1, 2$, there exist nonnegative constants L, K such that

$$D(f(t, x_1, y_1), f(t, x_2, y_2) \leq L[D(x_1, x_2) + D(y_1, y_2)]$$

and

$$D(k(t, s, x_1), k(t, s, x_2)) \leq KD(x_1, x_2)$$

(H_2) There exist two continuous functions$p, q : [t_0, T] \to \mathbb{R}^+$ such that

$$D(f(t, x_1, y), \hat{0}) \leq p(t)[D(x, \hat{0}) + D(y, \hat{0})]$$

and

$$D(k(t, s, x), \hat{0}) \leq q(t)D(x, \hat{0})$$

For all $x, y \in E^n$ and $t, s \in [t_0, T]$.

We require the following Lemmas in our further discussion.

Lemma 2.3. *Let $u(t), p(t)$ and $q(t)$ be real valued nonnegative continuous functions defined on $\mathbb{R}^+$, for which the inequality*

$$u(t) \leq u_0 + \int_0^t p(s)[\int_0^s q(\tau)u(\tau)d\tau]ds,$$

holds for all $t \in \mathbb{R}^+$, where u_0 is a nonnegative constant, then

$$u(t) \leq u_0[1 + \int_0^t p(s)\exp(\int_0^s p(\tau)q(\tau)d\tau)ds],$$

holds for all $t \in \mathbb{R}^+$

3. Existence and Uniqueness of Mild Solution

Theorem 3.1. *Let $f : [t_0, T] \times E^n \times E^n \to E^n$ is continuous and satisfied the hypothesis (H_1). Then for each $x_0, y_0 \in E^n$, the initial value problem (1).(2) has a unique mild solution $x \in C_T$ on $[t_0, T]$. Moreover, the mapping $(x_0, y_0) \to E^n$ is Lipschitz continuous from $E^n \times E^n$ into C_T.*

Proof. Define a mapping $F : C_T \to C_T$ by

$$(Fx)(t) = C(t-t_0)x_0+S(t-t_0)y_0+\int_{t_0}^t S(t-s)f(s,x(s),\int_{t_0}^s k(s,\tau,x(\tau))d\tau)ds,\ t \in [t_0,T] \quad (5)$$

step 1

Let $x \in C_T$ and h a small reel, we have:

$$\begin{aligned}
D(Fx(t+h), Fx(t)) &= D(C(t+h-t_0)x_0 + S(t+h-t_0)y_0 + \int_{t_0}^{t+h} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau \\
&+ S(t-t_0)y_0 + \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau ds)) \\
&\leq D(C(t+h-t_0)x_0, C(t-t_0)x_0) + D(S(t+h-t_0)y_0, S(t-t_0)y_0) \\
&+ D(\int_{t_0}^{t+h} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x \\
&\leq D(C(t+h-t_0)x_0, C(t-t_0)x_0) + D(S(t+h-t_0)y_0, S(t-t_0)y_0) \\
&+ D(\int_{t_0}^{t} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau \\
&+ D(\int_{t}^{t+h} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \hat{0})
\end{aligned}$$

It is clear that

$$D(C(t+h-t_0)x_0, C(t-t_0)x_0) \to 0 \quad \text{as} \quad h \to 0$$

$$D(S(t+h-t_0)y_0, S(t-t_0)y_0) \to 0 \quad \text{as} \quad h \to 0$$

$$D(\int_{t_0}^{t} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds) \to 0 \quad \text{as} \quad h \to 0$$

$$D(\int_{t}^{t+h} S(t+h-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \hat{0}) \to 0 \quad \text{as} \quad h \to 0$$

Thus $F(x) \in C_T$ i.e F maps C_T into itself.
step 2

We observe that the mild solution of the equations (1)(2) is a fixed point of the operator equation $Fx = x$. Let $x, y \in C_T$ and using equation (5), and the hypothesis, we obtain:

$$\begin{aligned}
D((Fx)(t),(Fy)(t)) &= D(C(t-t_0)x_0 + S(t-t_0)y_0 + \int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, C(t-t_0)x_0 \\
&+ \int_{t_0}^{t} S(t-s)f(s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds) \\
&= D(\int_{t_0}^{t} S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau) \\
&\leq \int_{t_0}^{t} D(S(t-s)f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau) \\
&\leq N\int_{t_0}^{t} D(f(s,x(s),\int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds, f(s,y(s),\int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds) \\
&\leq N\int_{t_0}^{t} L[H(x,y) + KH(x,y)\int_{t_0}^{s} d\tau]ds \\
&\leq N\int_{t_0}^{t} L[H(x,y) + KH(x,y)(s-t_0)]ds \\
&\leq N(t-t_0)[L + LK\frac{(t-t_0)}{2}]H(x,y)
\end{aligned}$$

Similarly by using the equation (5),(6) and the hypothesis we get :

$$
\begin{aligned}
D(F^2x(E), F^2y(t)) &= D(F(Fx(t)), F(Fy)(t)) \\
&= D(\int_{t_0}^{t} S(t-s)f(s, Fx(s), \int_{t_0}^{s} k(s,\tau,Fx(\tau))d\tau)ds), \int_{t_0}^{t} S(t-s)f(s, Fy(s), \int_{t_0}^{s} k(s,\tau,Fy \\
&\leq \int_{t_0}^{t} D(S(t-s)f(s, Fx(s), \int_{t_0}^{s} k(s,\tau,Fx(\tau))d\tau)ds, \int_{t_0}^{t} S(t-s)f(s, Fy(s), \int_{t_0}^{s} k(s,\tau,Fy(\\
&\leq NL\int_{t_0}^{t} D(F(Fx(s)), F(Fy)(s))ds + NL\int_{t_0}^{t} K\int_{t_0}^{s} D(F(Fx(s)), F(Fy)(s))d\tau ds \\
&\leq NL[NL\frac{(t-t_0)^2}{2!} + NLK\frac{(t-t_0)^3}{3!}]H(x,y) + NL[NL\int_{t_0}^{t}\frac{(t-t_0)^2}{2!}ds + NLK\int_{t_0}^{t}\frac{(t-t}{3!} \\
&\leq N^2\frac{(t-t_0)^2}{2!}[L^2 + 2L^2K\frac{(t-t_0)}{3} + L^2K^2\frac{(t-t_0)^2}{4\times 3}]H(x,y) \\
&\leq N^2\frac{(t-t_0)^2}{2!}[L^2 + 2L^2K\frac{(t-t_0)}{2!} + L^2K^2\frac{(t-t_0)^2}{4}]H(x,y) \\
&\leq \frac{(t-t_0)^2}{2!}[L^2 + 2L^2K\frac{(t-t_0)}{2!} + LK\frac{(t-t_0)}{2}]^2H(x,y)
\end{aligned}
$$

By marking use of the equation (5),(7) and iteration it follows that:

$$D(F^nx(t), F^ny(t)) \leq \frac{(t-t_0)^n}{n!}[N(L + LK\frac{(t-t_0)}{2!})]^nH(x,y)$$

For n large enough, $\frac{1}{n!}[TN(L+\frac{LKT}{2})]^n < 1$ Thus, there exists a positive integer n such that F^n is a contraction in E^n. Then there exists a unique $x \in E^n$ such that $F^nx = x$. Furthermore, we have:

$$F^n(Fx) = F(F^nx) = Fx$$

Hence Fx is a unique fixed point of F^n, so we conclude that x is the unique mild solution of (1).(2).

Suppose that y is another mild solution of the initial value problem (1) with $y(t_0) = x_0^*, y'(t_0) = y_0^*$ on $[t_0, T]$. Using the equation (4) and the hypothesis (H_1), we have:

$$
\begin{aligned}
D(x(t), y(t)) &= D(C(t-t_0)x_0 + S(t-t_0)y_0 + \int_{t_0}^{t} S(t-s)f(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds), C(t-t_0)x_0^* \\
&+ S(t-t_0)y_0^* + \int_{t_0}^{t} S(t-s)f(s, y(s), \int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds) \\
&\leq D(C(t-t_0)x_0 + C(t-t_0)x_0^*) + D(C(t-t_0)y_0 + C(t-t_0)y_0^*) \\
&+ \int_{t_0}^{t} S(t-s)f(s, x(s), \int_{t_0}^{s} k(s,\tau,x(\tau))d\tau)ds), \int_{t_0}^{t} S(t-s)f(s, y(s), \int_{t_0}^{s} k(s,\tau,y(\tau))d\tau)ds) \\
&\leq MD(x_0, x_0^*) + ND(y_0, y_0^*) + \int_{t_0}^{t}[D(x(t), y(t)) + \int_{t_0}^{s} kD(x(\tau), y(\tau))d\tau]ds
\end{aligned}
$$

By applying lemma (2) know as the Pachpahe's inequality with $u(t) = D(x(t), y(t))$ and $u_0 = 0$ to the inequality (9), we get:

$$D(x(t),y(t)) \leq [MD(x_0,x_0^*)+ND(y_0,y_0^*)] \times [1+NL\int_{t_0}^{t}\exp(\int_{t_0}^{s}(NL+K)d\tau)ds]$$

This proves that the uniqueness of x, i. e. for $x_0, y_0 \in E^n$, the initial value problem (1).(2) has a unique mild solution $x \in C_T$ on $t_0 \leq t \leq T$ and also Lipschitz continuity of the mapping $(x_0, y_0) \to x$. This completes the proof of the Theorem 1.

□

Theorem 3.2. *Let the hypothesis (H2) be satisfied. Then all solutions of (1)(2) are bounded on* $[0,T]$.

Proof. Let

$$x(t) = C(t-t_0)x_0+S(t-t_0)y_0+\int_{t_0}^{t}S(t-s)f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau)ds, \quad t \in t_0,T]$$

be a solution of (1), (2). Using hypothesis (H_2), we have:

$$MD(x(t),\hat{0}) \leq MD(x_0,\hat{0})+ND(y_0,\hat{0})+\int_{t_0}^{t}ND(f(s,x(s),\int_{t_0}^{s}k(s,\tau,x(\tau))d\tau,\hat{0})ds$$

$$\leq MD(x(t),\hat{0}) \leq MD(x_0,\hat{0})+ND(y_0,\hat{0})+\int_{t_0}^{t}Np(s)[D(x(s),\hat{0})+\int_{t_0}^{t}q(\tau)D(x(\tau),\hat{0})d\tau]ds$$

applying lemma (2), with $u(t) = D(x(t))$, we get :

$$D(x(t)) \leq [MD(x_0+ND(y_0][1+\int_{t_0}^{t}Np(s)\exp(\int_{t_0}^{t}[Np(\tau)+q(\tau)]d\tau)ds]$$

$$\leq [MD(x_0)+N(y_0)][1+\int_{0}^{t}NP\exp(T[NP+Q])ds]$$

$$\leq [MD(x_0)+N(y_0)][1+TNP\exp(T[NP+Q])], (13)$$

Where

$$P = \max_{t\in[0,T]} p(t) \quad \text{and} \quad Q = \max_{t\in[0,T]} q(t).$$

Thus, the boundedness of $x(t)$ follows from inequality (13). This completes the proof of the theorem 2 □

Example

In order to illustrate the applications of some of our result established in previous section, we consider the following partial nonlinear differential equation of the from:

$$\frac{\partial^2 w(t,u)}{\partial t^2} = \frac{\partial^2 w(t,u)}{\partial u^2} + \frac{w(t,u)\sin(w(t,u))}{(1+t)(1+t^2)}$$

$$+\int_0^t \frac{sw(s,u)}{(1+t)}ds, \quad t \in [0,1], \quad u \in I = [0,\pi], \quad (14)$$

$$w(t,0) = w(t,\pi) = \hat{0}, \quad t \in [0,1], \quad (15)$$

$$w(0,u) = x_0(u) \in E^n, \quad u \in I \quad (16)$$

$$\frac{\partial w(t,u)}{\partial t}|_{t=0} = y_0(u) \in E^n, \quad u \in I, \quad (17)$$

Let us take $w(t,u) = x(t)(u)$. Since

$$f(t, x(t), \int_{t_0}^{t} k(t,s,x(s))ds) = \frac{x(t)\sin(x(t))}{(1+t)(1+t^2)} + \int_0^t \frac{sx(s)}{(1+t)}ds$$

Second order integrodifferential equations
and

$$k(t,s,x(s)) = \frac{sx(s)}{(1+t)},$$

we have

$$D(f(t,x_1,Kx_1), f(t,x_2,Kx_2) \leq \frac{2}{(1+t)(1+t^2)}D(x_1,x_2) + \int_0^t \frac{s}{(1+t)}D(x_1(s),x_2(s))ds$$

$$\leq \frac{2}{(1+t)(1+t^2)}H(x_1,x_2) + \frac{t^2}{2(1+t)}H(x_1,x_2)$$

$$\leq LH(x_1,x_2)$$

Were

$$L = \max_{t\in[0,1]}\{\frac{2}{(1+t)(1+t^2)}, \frac{t^2}{2(1+t)} \text{and} kx := \int_{t_0}^{t} k(t,s,x(s))ds\}$$

Also, we obtain

$$D(f(t,x,Kx),\hat{0}) \leq \frac{1}{(1+t)(1+t^2)}H(x,\hat{0}) + \int_0^t \frac{s}{(1+t)}H(x,\hat{0})ds$$

$$\leq [\frac{1}{(1+t)(1+t^2)} + \frac{t^2}{2(1+t)}]H(x)$$

$$\leq p(t)H(x,\hat{0})$$

where

$$p(t) = [\frac{1}{(1+t)(1+t^2)} + \frac{t^2}{2(1+t)}]$$

Similarly, we can estimate for the function k:

$$D(f(t,x_1,Kx_1), f(t,x_2,Kx_2) \leq \frac{s}{(1+t)}D(x_1,x_2)$$

$$\leq KH(x_1,x_2)$$

Were

$$K = \sup_{0\leq s\leq t\leq 1}\{\frac{s}{(1+t)}\},$$

and for $0 \leq s \leq t \leq 1$

$$D(k(t,s,x)) \leq \frac{s}{(1+t)}H(x,\hat{0}) \leq q(t)H(x,\hat{0})$$

where

$$q(t) = \frac{t}{(1+t)}$$

We define the operator $A : D(A) \subset E^n \to E^n$ by $Aw = w_{uu}$, where $D(A) = \{w(\cdot) \in E^n : w(0) = w(\pi) = 0\}$. It is well known that A is the generator of strongly continuous cosine function $\{C(t) : t \in \mathbb{R}\}$ on E^n.

Where the hypothesis H_1 and H_2 are satisfied by using theorem (1) and (2). Then the problem (14),(15),(16) and (17) has a unique mild solution which is bounded on $[0, 1]$.

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Laboratoire de Mathématiques Appliquées & Calcul Scientifique Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

Current address: Sultan Moulay Slimane University, BP 523, 23000 Beni Mellal, Morocco

E-mail address: saidmelliani@gmail.com