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A note on pointwise quasi hemi-slant submersions

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Abstract. As a generalization of hemi-slant and semi-slant submersions, we discuss pointwise quasi Hemi-slant (PQHS) submersions from almost Hermitian manifolds onto Riemannian manifolds. We obtain various results satisfied by these submersions from Kähler manifolds onto Riemannian manifolds. Moreover, we find necessary and sufficient conditions on integrability of the distributions, and explore the geometry of totally geodesic foliations of discussed submersions. At last, we construct some examples of a PQHS submersion from an almost Hermitian manifold onto a Riemannian manifold.

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1. Introduction

Let N_1 be a Riemannian manifold endowed with a Riemannian metric g_{N_1}. An almost Hermitian manifold is a subclass of almost complex manifold. Since there are many applications of Riemannian submersions in science and technology, especially in the theory of relativity, robotics and cosmology; therefore, it attracts many geometers to do the research in this area.

The theory of Riemannian submersion was initiated by O' Neill [17] and Gray [5], during 1966-1967. In 1976, Watson [30] proposed the study of almost complex type Riemannian submersions, and defined almost Hermitian submersions between almost Hermitian manifolds. A new class of Riemannian submersions (almost contact metric submersions) was studied by Chinea [3] which was an extension of almost Hermitian submersion.

In 2013, Sahin introduced the notion of semi-invariant submersions [23], which was a generalization of holomorphic submersions and anti-invariant submersions [22]. Additionaly, he also defined slant submersions from almost Hermitian manifolds onto arbitrary Riemannian manifolds [24]. Many geometers studied different types of Riemannian submersions between Riemannian manifolds and contributed many important results in [1, 4, 6–8, 11, 12, 14–16, 18–21, 25, 29].

The notion of pointwise slant submersions was introduced by Lee and Sahin [13] in 2014, and further studied by Kumar et al. in [9, 10]. Recently, Sepet et al. introduced pointwise slant submersions [28] and pointwise semi-slant submersions [27]. Pointwise semi-slant submersions whose total manifolds are locally product Riemannian manifolds was studied by Sayer et al. [26]. The above studies inspire us to introduce the notion of PQHS submersions from the almost Hermitian manifolds to the Riemannian manifolds and explore its geometrical properties. We exhibit our work as follows: After intoduction, in the second section we mention some defintions and properties related to the main topic. The third

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section deals with the study of PQHS submersions. The necessary and sufficient conditions for PQHS submersions to be integrable and totally geodesics are given in the fourth section. Finally, the last section is concerned with some non-trivial examples of PQHS submersion from Kaehler manifold.

2. Preliminaries

Let N_1 be an even-dimensional differentiable manifold and J be a $(1,1)$ tensor field on N_1 such that $J^2 = -I$, here I is the identity operator. Then J is named an almost complex structure on N_1. The manifold (N_1, J) is an almost complex manifold [31], and its Nijenhuis tensor N is defined as:

$$N(Z_1, Z_2) = [JZ_1, JZ_2] - [Z_1, Z_2] - J[JZ_1, Z_2] - J[Z_1, JZ_2], \quad \text{for all } Z_1, Z_2 \in \Gamma(TN_1).$$

If N on N_1 is zero, then N_1 is called a complex manifold.

Let g_{N_1} be a Riemannian metric on N_1 such that

$$g_{N_1}(JZ_1, JZ_2) = g_{N_1}(Z_1, Z_2), \quad g_{N_1}(Z_1, JZ_2) = -g_{N_1}(JZ_1, Z_2), \tag{1}$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$.

Then, g_{N_1} is called an almost Hermitian metric on N_1, and $(N_1,\ g_{N_1})$ is called almost Hermitian manifold. The Riemannian connection ∇ of the almost Hermitian manifold N_1 can be extended to the whole tensor algebra on N_1. Tensor fields $(\nabla_{Z_1} J)$ is defined as

$$(\nabla_{Z_1} J)Z_2 = \nabla_{Z_1} JZ_2 - J\nabla_{Z_1} Z_2,$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$.

The manifold (N_1, g_{N_1}, J) is called a Kähler manifold if

$$(\nabla_{Z_1} J)Z_2 = 0, \tag{2}$$

for all $Z_1, Z_2 \in \Gamma(TN_1)$.

Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be the Riemannian manifolds, where g_{N_1} and g_{N_2} are the Riemannian metrics on C^∞-manifolds N_1 and N_2, respectively. Let $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion.

Define O'Neill's tensors $\mathcal{T}$ and $\mathcal{A}$ by [17]

$$\mathcal{A}_E F = \mathcal{H}\nabla_{\mathcal{H}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{H}E}\mathcal{H}F, \tag{3}$$

$$\mathcal{T}_E F = \mathcal{H}\nabla_{\mathcal{V}E}\mathcal{V}F + \mathcal{V}\nabla_{\mathcal{V}E}\mathcal{H}F, \tag{4}$$

for any vector fields E and F on N_1, where ∇ is the Levi-Civita connection of g_{N_1}.

From equations (3) and (4), we have

$$\nabla_X Z = \mathcal{T}_X Z + \mathcal{V}\nabla_X Z, \tag{5}$$

$$\nabla_X Y = \mathcal{T}_X Y + \mathcal{H}\nabla_X Y, \tag{6}$$

$$\nabla_Y X = \mathcal{A}_Y X + \mathcal{V}\nabla_Y X, \tag{7}$$

$$\nabla_Y W = \mathcal{H}\nabla_Y W + \mathcal{A}_Y W, \tag{8}$$

for $X, Z \in \Gamma(\ker F_*)$ and $Y, W \in \Gamma(\ker F_*)^\perp$, where $\mathcal{H}\nabla_X W = \mathcal{A}_W X$, if W is basic [2].

Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be the Riemannian manifolds and $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a C^∞ map, then the second fundamental form of F is given by

$$(\nabla F_*)(Z, W) = \nabla^F_Z F_* W - F_*(\nabla_Z W), \tag{9}$$

for $Z, W \in \Gamma(TN_1)$, where ∇^F is the pullback connection and we denote for convenience by ∇ the Riemannian connections of the metrics g_{N_1} and g_{N_2}.

Now, we can easily prove the following lemma as in [2].

Lemma 1. *Let (N_1, g_{N_1}) and (N_2, g_{N_2}) be the Riemannian manifolds. If $F : (N_1, g_{N_1}) \to (N_2, g_{N_2})$ be a Riemannian submersion, then for any horizontal vector fields Z_1, Z_2 and vertical vector fields V_1, V_2, we have*

(i) $(\nabla F_*)(Z_1, Z_2) = 0$,
(ii) $(\nabla F_*)(V_1, V_2) = -F_*(\mathcal{T}_{V_1} V_2) = -F_*(\nabla_{V_1} V_2)$,
(iii) $(\nabla F_*)(V_1, Z_1) = -F_*(\nabla_{V_1} Z_1) = -F_*(\mathcal{A}_{V_1} Z_1)$.

3. PQHS Submersions

In this section, we discuss some results satisfying by PQHS submersion (F) from (N_1, g_{N_1}, J) onto manifold (N_2, g_{N_2}).

Definition 1. *A Riemannian submersion $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ is called a PQHS submersion if there exist three mutually orthogonal distributions D, D^θ and $D^\perp$ such that*

$$\ker F_* = D \oplus D^\theta \oplus D^\perp,\ J(D) = D,\ J(D^\perp) \subset (\Gamma \ker F_*)^\perp,$$

for any non-zero vector field $V_1 \in (D_1)_p$, $p \in N_1$, the angle θ_1 between JV_1 and $(D_1)_p$ is constant and is independent of the choice of point p and V_1 in $(D_1)_p$, where the vertical distribution $\Gamma(\ker F_*)$ admits three orthogonal complementary distributions D, D^θ and $D^\perp$ such that D is invariant, D^θ is slant with an angle funtion θ, and $D^\perp$ is anti-invariant.

Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ Then, we have

$$TN_1 = \ker F_* \oplus (\ker F_*)^\perp. \tag{10}$$

Now, for any vector field $V_1 \in \Gamma(\ker F_*)$, we put

$$V_1 = PV_1 + QV_1 + RV_1, \tag{11}$$

where P, Q and R are projection morphisms of $\ker F_*$ onto D, D^θ and $D^\perp$, respectively.

For $Y_1 \in (\Gamma \ker F_*)$, we set

$$JY_1 = \phi Y_1 + \omega Y_1, \tag{12}$$

where $\phi Y_1 \in (\Gamma \ker F_*)$ and $\omega Y_1 \in (\Gamma \ker F_*)^\perp$.

From (11) and (12), we have

$$\begin{aligned} JV_1 &= J(PV_1) + J(QV_1) + J(RV_1), \\ &= \phi(PV_1) + \omega(PV_1) + \phi(QV_1) + \omega(QV_1) + \phi(RV_1) + \omega(RV_1). \end{aligned}$$

Since $JD = D,\ J(D^\perp) \subset (\Gamma \ker F_*)^\perp$, we get $\omega PV_1 = 0, \phi(RV_1) = 0$.

Hence, the above equation reduces to

$$JV_1 = \phi(PV_1) + \phi(QV_1) + \omega(QV_1) + \omega(RV_1). \tag{13}$$

Thus, we have the following decomposition

$$J(\ker F_*) = D \oplus (\phi D_1 \oplus \phi D_2) \oplus (\omega D_1), \tag{14}$$

where $\oplus$ denotes orthogonal direct sum.

Also for any non-zero vector field $W_1 \in \Gamma(\ker F)^\perp$, we have

$$JW_1 = BW_1 + CW_1, \tag{15}$$

where $BW_1 \in \Gamma(\ker F_*)$ and $CW_1 \in \Gamma(\ker F_*)^\perp$.

Lemma 2. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, we have*

$$\phi^2 W_1 + B\omega W_1 = -W_1, \omega\phi W_1 + C\omega W_1 = 0,$$
$$\omega BU_1 + C^2 U_1 = -U_1, \phi BU_1 + BCU_1 = 0,$$

for all $W_1 \in \Gamma(\ker F_*)$ and $U_1 \in \Gamma(\ker F_*)^\perp$.

Proof. Using $(12), (15)$ and $J^2 = -I$, we have Lemma 2. □

The proof of the following result is the same as given in [27], therefore, we omit its proof.

Lemma 3. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, we have*

$$\phi^2 V_1 = -(\cos^2 \theta) V_1, \tag{16}$$

for $V_1 \in \Gamma(D^\theta)$, where θ is the slant function.

Lemma 4. *Let $F:(N_1,g_{N_1},J)\to(N_2,g_{N_2})$. Then, we have*

$$\mathcal{V}\nabla_{Z_1}\phi W_1+\mathcal{T}_{Z_1}\omega W_1=\phi\mathcal{V}\nabla_{Z_1}W_1+B\mathcal{T}_{Z_1}W_1, \tag{17}$$
$$\mathcal{T}_{Z_1}\phi W_1+\mathcal{H}\nabla_{Z_1}\omega W_1=\omega\mathcal{V}\nabla_{Z_1}W_1+C\mathcal{T}_{Z_1}W_1, \tag{18}$$
$$\mathcal{V}\nabla_{Y_1}BY_2+\mathcal{A}_{Y_1}CY_2=\phi\mathcal{A}_{Y_1}Y_2+B\mathcal{H}\nabla_{Y_1}Y_2, \tag{19}$$
$$\mathcal{A}_{Y_1}BY_2+\mathcal{H}\nabla_{Y_1}CY_2=\omega\mathcal{A}_{Y_1}Y_2+C\mathcal{H}\nabla_{Y_1}Y_2, \tag{20}$$
$$\mathcal{V}\nabla_{Z_1}BY_1+\mathcal{T}_{Z_1}CY_1=\phi\mathcal{T}_{Z_1}Y_1+B\mathcal{H}\nabla_{Z_1}Y_1, \tag{21}$$
$$\mathcal{T}_{Z_1}BY_1+\mathcal{H}\nabla_{Z_1}CY_1=\omega\mathcal{T}_{Z_1}Y_1+C\mathcal{H}\nabla_{Z_1}Y_1, \tag{22}$$
$$\mathcal{V}\nabla_{Y_1}\phi Z_1+\mathcal{A}_{Y_1}\omega Z_1=B\mathcal{A}_{Y_1}Z_1+\phi\mathcal{V}\nabla_{Y_1}Z_1, \tag{23}$$
$$\mathcal{A}_{Y_1}\phi Z_1+\mathcal{H}\nabla_{Y_1}\omega Z_1=\omega\mathcal{V}_{Y_1}Z_1+C\mathcal{A}_{Y_1}Z_1, \tag{24}$$

for any $Z_1,W_1\in\Gamma(\ker F_*)$ and $Y_1,Y_2\in\Gamma(\ker F_*)^\perp$.

Proof. Using (5)-(8), (12) and (15), we get the equations (17)-(24). □

Now, we define

$$(\nabla_{Z_1}\phi)W_1=\mathcal{V}\nabla_{Z_1}\phi W_1-\phi\mathcal{V}\nabla_{Z_1}W_1, \tag{25}$$
$$(\nabla_{Z_1}\omega)W_1=\mathcal{H}\nabla_{Z_1}\omega W_1-\omega\mathcal{V}\nabla_{Z_1}W_1, \tag{26}$$
$$(\nabla_{Y_1}C)Y_2=\mathcal{H}\nabla_{Y_1}CY_2-C\mathcal{H}\nabla_{Y_1}Y_2, \tag{27}$$
$$(\nabla_{Y_1}B)Y_2=\mathcal{V}\nabla_{Y_1}BY_2-B\mathcal{H}\nabla_{Y_1}Y_2, \tag{28}$$

for any $Z_1,W_1\in\Gamma(\ker F_*)$ and $Y_1,Y_2\in\Gamma(\ker F_*)^\perp$.

Lemma 5. *Let $F:(N_1,g_{N_1},J)\to(N_2,g_{N_2})$. Then, we have*

$$(\nabla_{Z_1}\phi)W_1=B\mathcal{T}_{Z_1}W_1-\mathcal{T}_{Z_1}\omega W_1,$$
$$(\nabla_{Z_1}\omega)W_1=C\mathcal{T}_{Z_1}W_1-\mathcal{T}_{Z_1}\phi W_1,$$
$$(\nabla_{Y_1}C)Y_2=\omega\mathcal{A}_{Y_1}Y_2-\mathcal{A}_{Y_1}BY_2,$$
$$(\nabla_{Y_1}B)Y_2=\phi\mathcal{A}_{Y_1}Y_2-\mathcal{A}_{Y_1}CY_2,$$

for any $Z_1,W_1\in\Gamma(\ker F_*)$ and $Y_1,Y_2\in\Gamma(\ker F_*)^\perp$.

Proof. Using (17)-(20) and (25)-(28), we get all the relations of Lemma 5. □

If the tensors ϕ and ω are parallel with respect to the linear connection ∇ on N_1, then

$$B\mathcal{T}_{W_1}W_2=\mathcal{T}_{W_1}\omega W_2, C\mathcal{T}_{W_1}W_2=\mathcal{T}_{W_1}\phi W_2,$$

for any $W_1,W_2\in\Gamma(TN_1)$.

4. Integrability of Distributions and Decomposition Theorems

In this section, we obtain necessary and sufficient conditions for the integrability of distributions related to PQHS submerions (F) and also for these distributions we define totally geodesic foliations. Here we denote a Kähler manifold by (N_1,g_{N_1},J).

Theorem 1. *Let $F:(N_1,g_{N_1},J)\to(N_2,g_{N_2})$ with the slant function θ. Then, the invariant distribution D is integrable if and only if*

$$\phi(\mathcal{V}\nabla_{Z_1}JZ_2-\mathcal{V}\nabla_{Z_2}JZ_1)=B(\mathcal{T}_{Z_2}\phi Z_1-\mathcal{T}_{Z_1}JZ_2),$$

for $Z_1,Z_2\in\Gamma(D)$ and $V_1\in\Gamma(D^\theta\oplus D^\perp)$.

Proof. For $Z_1,Z_2\in\Gamma(D)$, and $V_1\in\Gamma(D^\theta\oplus D^\perp)$, using the equations (1), (2), (5), (11), (12) and (15), we have

$$\begin{aligned}
&g_{N_1}([Z_1,Z_2],V_1)\\
=\;&g_{N_1}(\nabla_{Z_1}JZ_2,J(QV_1+RV_1))-g_{N_1}(\nabla_{Z_2}JZ_1,J(QV_1+RV_1)),\\
=\;&g_{N_1}(\mathcal{V}\nabla_{Z_1}JZ_2-\mathcal{V}\nabla_{Z_2}JZ_1,J(QV_1+RV_1))+\\
&g_{N_1}(\mathcal{T}_{Z_1}JZ_2-\mathcal{T}_{Z_2}\phi Z_1,J(QV_1+RV_1)),\\
=\;&g_{N_1}(\phi(\mathcal{V}\nabla_{Z_1}JZ_2-\mathcal{V}\nabla_{Z_2}JZ_1),V_1)+g_{N_1}(B(\mathcal{T}_{Z_1}JZ_2-\mathcal{T}_{Z_2}\phi Z_1),V_1),
\end{aligned}$$

which completes the proof. □

Theorem 2. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ with the slant function θ. Then, the slant distribution D^θ is integrable if and only if*

$$\begin{aligned} & g_{N_1}(\mathcal{T}_{X_1}\omega\phi X_2, V_1) - g_{N_1}(\mathcal{T}_{X_2}\omega\phi X_1, V_1) \\ = \; & g_{N_1}(\mathcal{T}_{X_1}\omega X_2 - \mathcal{T}_{X_2}\omega X_1, JPV_1) + g_{N_1}(\mathcal{H}\nabla_{X_1}\omega X_2 - \mathcal{H}\nabla_{X_2}\omega X_1, JRV_1), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(D^\theta)$ and $V_1 \in \Gamma(D \oplus D^\perp)$.

Proof. For $X_1, X_2 \in \Gamma(D^\theta)$ and $V_1 \in \Gamma(D \oplus D^\perp)$, using the equations (1), (2), (6), (11), (12) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}([X_1, X_2], V_1) \\ = \; & g_{N_1}(\nabla_{X_1} JX_2, JV_1) - g_{N_1}(\nabla_{X_2} JX_1, JV_1), \\ = \; & -g_{N_1}(\nabla_{X_1}\phi^2 X_2, V_1) + g_{N_1}(\nabla_{X_2}\phi^2 X_1, V_1) - g_{N_1}(\nabla_{X_1}\omega\phi X_2, V_1) + \\ & g_{N_1}(\nabla_{X_2}\omega\phi X_1, V_1) + g_{N_1}(\nabla_{X_1}\omega X_2, JV_1) - g_{N_1}(\nabla_{X_2}\omega X_1, JV_1), \\ = \; & \cos^2\theta g_{N_1}([X_1, X_2], V_1) - g_{N_1}(\mathcal{T}_{X_1}\omega\phi X_2, V_1) + g_{N_1}(\mathcal{T}_{X_2}\omega\phi X_1, V_1) + \\ & g_{N_1}(\mathcal{T}_{X_1}\omega X_2 - \mathcal{T}_{X_2}\omega X_1, JPV_1) + g_{N_1}(\mathcal{H}\nabla_{X_1}\omega X_2 - \mathcal{H}\nabla_{X_2}\omega X_1, JRV_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta g_{N_1}([X_1, X_2], V_1) \\ = \; & -g_{N_1}(\mathcal{T}_{X_1}\omega\phi X_2, V_1) + g_{N_1}(\mathcal{T}_{X_2}\omega\phi X_1, V_1) + \\ & g_{N_1}(\mathcal{T}_{X_1}\omega X_2 - \mathcal{T}_{X_2}\omega X_1, JPV_1) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}\omega X_2 - \mathcal{H}\nabla_{X_2}\omega X_1, JRV_1), \end{aligned}$$

which completes the proof. □

We also have the following thorem:

Theorem 3. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ with the slant function θ. Then, the anti-invariant distribution $D^\perp$ is always integrable.*

Theorem 4. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$ with the slant function θ. Then the horizontal distribution $(\ker F_*)$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & \sin^2\theta g_{N_1}([X_1, W_1], X_2) \\ = \; & -g_{N_1}(\mathcal{V}\nabla_{W_1} JPX_1, \phi X_2) - g_{N_1}(\mathcal{A}_{W_1} JPX_1, \omega X_2) + \\ & \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{W_1} PX_1, X_2) + \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{W_1} RX_1, X_2) + \\ & \sin 2\theta W_1[\theta] g_{N_1}(QX_1, X_2) + g_{N_1}(\mathcal{A}_{W_1}\omega\phi QX_1, X_2) - \\ & g_{N_1}(\mathcal{A}_{W_1}\omega QX_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{W_1}\omega QX_1, \omega X_2) - \\ & g_{N_1}(\mathcal{A}_{W_1} JRX_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{W_1} JRX_1, \omega X_2), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker F_*)$ and $W_1 \in \Gamma(\ker F_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(\ker F_*)$ and $W_1 \in \Gamma(\ker F_*)^\perp$, using the equations (1), (2), (7), (8), (11), (12) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1} X_2, W_1) \\ = \; & -g_{N_1}([X_1, W_1], X_2) - g_{N_1}(\nabla_{W_1} X_1, X_2), \\ = \; & -g_{N_1}([X_1, W_1], X_2) - g_{N_1}(\nabla_{W_1} JPX_1, JX_2) + \cos^2\theta g_1([X_1, W_1], X_2) + \\ & -\cos^2\theta g_{N_1}(\nabla_{X_1} W_1, X_2) + \cos^2\theta g_{N_1}(\nabla_{W_1} RX_1, X_2) + \\ & \cos^2\theta g_{N_1}(\nabla_{W_1} PX_1, X_2) + \sin 2\theta W_1[\theta] g_{N_1}(QX_1, X_2) + \\ & g_{N_1}(\nabla_{W_1}\omega\phi QX_1, X_2) - g_{N_1}(\nabla_{W_1}\omega QX_1, JX_2) - g_{N_1}(\nabla_{W_1} JRX_1, JX_2). \end{aligned}$$

Now, we obtain

$$\begin{aligned} & \sin^2\theta g_{N_1}(\nabla_{X_1} X_2, W_1) \\ = \; & -\sin^2\theta g_{N_1}([X_1, W_1], X_2) - g_{N_1}(\mathcal{V}\nabla_{W_1} JPX_1, \phi X_2) - g_{N_1}(\mathcal{A}_{W_1} JPX_1, \omega X_2) + \\ & \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{W_1} PX_1, X_2) + \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{W_1} RX_1, X_2) + \end{aligned}$$

$$\sin 2\theta W_1[\theta] g_{N_1}(QX_1, X_2) + g_{N_1}(\mathcal{A}_{W_1}\omega\phi QX_1, X_2) - g_{N_1}(\mathcal{A}_{W_1}\omega QX_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{W_1}\omega QX_1, \omega X_2) - g_{N_1}(\mathcal{A}_{W_1}JRX_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{W_1}JRX_1, \omega X_2).$$

□

Theorem 5. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then the vertical distribution $(\ker F_*)^\perp$ defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & -\cos^2\theta g_{N_1}(\mathcal{A}_{X_1}X_2, QZ_1) \\ = \ & g_{N_1}(\mathcal{V}\nabla_{X_1}BX_2, JPZ_1) + g_{N_1}(\mathcal{A}_{X_1}CX_2, JPZ_1) - g_{N_1}(\mathcal{H}\nabla_{X_1}X_2, \omega QZ_1) + \\ & g_{N_1}(\mathcal{A}_{X_1}BX_2, Z_1) + g_{N_1}(\mathcal{H}\nabla_{X_1}CZ_1, \omega QZ_1) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}CX_2, JRZ_1) + g_{N_1}(\mathcal{A}_{X_1}CX_2, JRZ_1), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(\ker F_*)^\perp$ and $Z_1 \in \Gamma(\ker F_*)$.

Proof. For $X_1, X_2 \in \Gamma(\ker F_*)^\perp$ and $Z_1 \in \Gamma(\ker F_*)$, using equations (1) and (2), we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1}X_2, Z_1) \\ = \ & g_{N_1}(J\nabla_{X_1}X_2, JPZ_1 + JQZ_1 + JRZ_1). \end{aligned}$$

Now, using the equations (7), (8), (11), (12), (15) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1}X_2, Z_1) \\ = \ & g_{N_1}(\mathcal{V}\nabla_{X_1}BX_2, JPZ_1) + g_{N_1}(\mathcal{A}_{X_1}CX_2, JPZ_1) + \cos^2\theta g_{N_1}(\mathcal{A}_{X_1}X_2, QZ_1) - \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}X_2, \omega QZ_1) + g_{N_1}(\mathcal{A}_{X_1}BX_2, Z_1) + g_{N_1}(\mathcal{H}\nabla_{X_1}CZ_1, \omega QZ_1) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1}CX_2, JRZ_1) + g_{N_1}(\mathcal{A}_{X_1}CX_2, JRZ_1). \end{aligned}$$

□

Theorem 6. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, the invariant distribution D defines a totally geodesic foliation on N_1 if and only if*

$$\begin{aligned} & \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{Z_1}Z_2, QU_1) \\ = \ & g_{N_1}(\mathcal{T}_{Z_1}Z_2, \omega\phi QU_1) - g_{N_1}(\mathcal{T}_{Z_1}JZ_2, \omega QU_1 + JRU_1), \\ & g_{N_1}(\mathcal{V}\nabla_{Z_1}JZ_2, BW_1) = -g_{N_1}(\mathcal{T}_{Z_1}JZ_2, CW_1), \end{aligned}$$

for $Z_1, Z_2 \in \Gamma(D), U_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $W_1 \in \Gamma(\ker F_*)^\perp$.

Proof. For all $Z_1, Z_2 \in \Gamma(D), U_1 \in \Gamma(D^\theta \oplus D^\perp)$ and $W_1 \in \Gamma(\ker F_*)^\perp$, using the equations (1), (2), (5), (11), (12) and (13), we have

$$\begin{aligned} g_{N_1}(\nabla_{Z_1}Z_2, U_1) &= g_{N_1}(\nabla_{Z_1}JZ_2, JU_1), \\ &= -g_{N_1}(\nabla_{Z_1}Z_2, \phi^2 QU_1) - g_{N_1}(\nabla_{Z_1}Z_2, \omega\phi QU_1) + \\ & \quad g_{N_1}(\nabla_{Z_1}JZ_2, \omega QU_1 + JRU_1), \\ &= \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{Z_1}Z_2, QU_1) - g_{N_1}(\mathcal{T}_{Z_1}Z_2, \omega\phi QU_1) + \\ & \quad g_{N_1}(\mathcal{T}_{Z_1}JZ_2, \omega QU_1 + JRU_1). \end{aligned}$$

Now, again using the equations (1), (2), (5) and (15), we have

$$\begin{aligned} g_{N_1}(\nabla_{Z_1}Z_2, W_1) &= g_{N_1}(\nabla_{Z_1}JZ_2, JW_1), \\ &= g_{N_1}(\mathcal{V}\nabla_{Z_1}JZ_2, BW_1) + g_{N_1}(\mathcal{T}_{Z_1}JZ_2, CW_1), \end{aligned}$$

which completes the proof. □

Theorem 7. *$F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, the slant distribution D^θ defines a totally geodesic foliation on N_1 if and only if*

$$g_{N_1}(\mathcal{T}_{X_1}\omega\phi X_2, Z_1) = g_{N_1}(\mathcal{T}_{X_1}\omega X_2, JPZ_1) + g_{N_1}(\mathcal{H}\nabla_{X_1}\omega X_2, JRZ_1),$$

$$\begin{aligned} & \sin^2\theta_1 g_{N_1}([X_1, Z_2], X_2) = \sin 2\theta Z_2[\theta] g_{N_1}(X_1, X_2) + \\ & g_{N_1}(\mathcal{A}_{Z_2}\omega\phi X_1, X_2) - g_{N_1}(\mathcal{A}_{Z_2}\omega X_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{Z_2}\omega X_1, \omega X_2), \end{aligned}$$

for all $X_1, X_2 \in \Gamma(D^\theta), Z_1 \in \Gamma(D \oplus D^\perp)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $X_1, X_2 \in \Gamma(D^\theta), Z_1 \in \Gamma(D \oplus D^\perp)$ and $Z_2 \in \Gamma(\ker F_*)^\perp$, using the equations (1), (2), (6), (12) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{X_1} X_2, Z_1) \\ = & -g_{N_1}(\nabla_{X_1} \phi^2 X_2, Z_1) - g_{N_1}(\nabla_{X_1} \omega\phi X_2, Z_1) + g_1(\nabla_{X_1} \omega X_2, JZ_1), \\ = & \cos^2\theta_1 g_{N_1}(\nabla_{X_1} X_2, Z_1) - g_{N_1}(\mathcal{T}_{X_1} \omega\phi X_2, Z_1) + \\ & g_{N_1}(\mathcal{T}_{X_1} \omega X_2, JPZ_1) + g_{N_1}(\mathcal{H}\nabla_{X_1} \omega X_2, JRZ_1). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_{N_1}(\nabla_{X_1} X_2, Z_1) \\ = & -g_{N_1}(\mathcal{T}_{X_1} \omega\phi X_2, Z_1) + g_{N_1}(\mathcal{T}_{X_1} \omega X_2, JPZ_1) + \\ & g_{N_1}(\mathcal{H}\nabla_{X_1} \omega X_2, JRZ_1). \end{aligned}$$

Next, from the equations (1), (2), (8), (12) and Lemma 3, we have

$$\begin{aligned} g_{N_1}(\nabla_{X_1} X_2, Z_2) &= -g_{N_1}([X_1, Z_2], X_2) - g_{N_1}(\nabla_{Z_2} X_1, X_2), \\ &= -g_{N_1}([X_1, Z_2], X_2) - g_{N_1}(\nabla_{Z_2} \cos^2\theta X_1, X_2) + \\ &\quad + g_{N_1}(\nabla_{Z_2} \omega\phi X_1, X_2) + g_{N_1}(\nabla_{Z_2} \omega X_1, JX_2). \end{aligned}$$

Now, we have

$$\begin{aligned} & \sin^2\theta_1 g_{N_1}(\nabla_{X_1} X_2, Z_2) \\ = & -\sin^2\theta_1 g_{N_1}([X_1, Z_2], X_2) + \sin 2\theta Z_2[\theta] g_{N_1}(X_1, X_2) + \\ & g_{N_1}(\mathcal{A}_{Z_2} \omega\phi X_1, X_2) - g_{N_1}(\mathcal{A}_{Z_2} \omega X_1, \phi X_2) - g_{N_1}(\mathcal{H}\nabla_{Z_2} \omega X_1, \omega X_2). \end{aligned}$$

□

Theorem 8. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, the slant distribution $D^\perp$ defines a totally geodesic foliation on N_1 if and only if*

$$g_{N_1}(\mathcal{T}_{V_1} JV_2, BW_2) = -g_{N_1}(\mathcal{H}\nabla_{V_1} JV_2, CW_2),$$

$$\begin{aligned} & \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{V_1} V_2, QW_1) \\ = & g_{N_1}(\mathcal{T}_{V_1} V_2, \omega\phi QW_1) - g_{N_1}(\mathcal{H}\nabla_{V_1} JV_2, \omega QW_1) - g_{N_1}(\mathcal{T}_{V_1} JV_2, JPW_1), \end{aligned}$$

for all $V_1, V_2 \in \Gamma(D^\perp), W_1 \in \Gamma(D \oplus D^\theta)$ and $W_2 \in \Gamma(\ker F_*)^\perp$.

Proof. For $V_1, V_2 \in \Gamma(D^\perp), W_1 \in \Gamma(D \oplus D^\theta)$ and $W_2 \in \Gamma(\ker F_*)^\perp$, using the equations (1), (2), (6), (15) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{V_1} V_2, W_2) \\ = & g_{N_1}(\nabla_{V_1} JV_2, JW_2), \\ = & g_{N_1}(\mathcal{T}_{V_1} JV_2, BW_2) + g_{N_1}(\mathcal{H}\nabla_{V_1} JV_2, CW_2). \end{aligned}$$

Next, from the equations (1), (2), (6), (11), (12), (15) and Lemma 3, we have

$$\begin{aligned} & g_{N_1}(\nabla_{V_1} V_2, W_1) \\ = & g_{N_1}(J\nabla_{V_1} V_2, JPW_1) + g_{N_1}(J\nabla_{V_2} V_2, \phi QW_1) + g_{N_1}(J\nabla_{V_2} JV_2, \omega QW_1), \\ = & g_{N_1}(\mathcal{T}_{V_1} JV_2, JPW_1) + \cos^2\theta g_{N_1}(\mathcal{V}\nabla_{V_1} V_2, QW_1) - \\ & -g_{N_1}(\mathcal{T}_{V_1} V_2, \omega\phi QW_1) + g_{N_1}(\mathcal{H}\nabla_{V_1} JV_2, \omega QW_1). \end{aligned}$$

□

Theorem 9. *Let $F : (N_1, g_{N_1}, J) \to (N_2, g_{N_2})$. Then, F is a totally geodesic map if and only if*

$$\begin{aligned} & C\mathcal{T}_{V_1} JPV_2 + \omega\mathcal{V}\nabla_{V_1} JPV_2 - \cos^2\theta \mathcal{T}_{V_1} QV_2 + \mathcal{H}\nabla_{V_1} \omega\phi QV_2 + \\ & C\mathcal{H}\nabla_{V_1} \omega QV_2 + \omega\mathcal{T}_{V_1} \omega QV_2 + C\mathcal{H}\nabla_{V_1} JRV_2 + \omega\mathcal{T}_{V_1} JRV_2 \\ = & 0, \\ & C\mathcal{T}_{Y_1} BV_1 + \omega\mathcal{V}\nabla_{Y_1} BV_1 + C\mathcal{H}\nabla_{Y_1} CV_1 + \omega\mathcal{T}_{Y_1} CV_1 = 0, \end{aligned}$$

for all $V_1, V_2 \in \Gamma(\ker F_*)$ and $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

Proof. Since F is a PQHS submersion, then we have

$$(\nabla F_*)(Y_1, Y_2) = 0,$$

for $Y_1, Y_2 \in \Gamma(\ker F_*)^\perp$.

For $V_1, V_2 \in \Gamma(\ker F_*)$, using the equations $(1), (2), (5), (6), (11), (12), (15)$ and Lemma 3, we have

$$\begin{aligned} & (\nabla F_*)(V_1, V_2) \\ = \; & F_*(J\nabla_{V_1} JV_2), \\ = \; & F_*(J\nabla_{V_1} JPV_2 + \nabla_{V_1}\phi^2 QV_2 + \nabla_{V_1}\omega\phi QV_2 + J\nabla_{V_1}\omega QV_2 + J\nabla_{V_1} JRV_2), \\ = \; & F_*(C\mathcal{T}_{V_1} JPV_2 + \omega\mathcal{V}\nabla_{V_1} JPV_2 - \cos^2\theta\mathcal{T}_{V_1} QV_2 + \mathcal{H}\nabla_{V_1}\omega\phi QV_2 + \\ & B\mathcal{T}_{V_1} JPV_2 + \phi\mathcal{V}\nabla_{V_1} JPV_2 - \cos^2\theta\mathcal{V}\nabla_{V_1} QV_2 + \mathcal{T}_{V_1}\omega\phi QV_2 + \\ & C\mathcal{H}\nabla_{V_1}\omega QV_2 + \omega\mathcal{T}_{V_1}\omega QV_2 + C\mathcal{H}\nabla_{V_1} JRV_2 + \omega\mathcal{T}_{V_1} JRV_2 + \\ & B\mathcal{H}\nabla_{V_1}\omega QV_2 + \phi\mathcal{T}_{V_1}\omega QV_2 + B\mathcal{H}\nabla_{V_1} JRV_2 + \phi\mathcal{T}_{V_1} JRV_2). \end{aligned}$$

Next, using the equations $(1), (2), (7), (8), (12), (15)$ and Lemma 3, we have

$$\begin{aligned} & (\nabla F_*)(Y_1, V_1), \\ = \; & -F_*(\nabla_{Y_1} V_1) \\ = \; & F_*(J\nabla_{Y_1} BV_1 + J\nabla_{Y_1} CV_1) \\ = \; & F_*(C\mathcal{T}_{Y_1} BV_1 + \omega\mathcal{V}\nabla_{Y_1} BV_1 + C\mathcal{H}\nabla_{Y_1} CV_1 + \omega\mathcal{T}_{Y_1} CV_1 \\ & B\mathcal{T}_{Y_1} BV_1 + \phi\mathcal{V}\nabla_{Y_1} BV_1 + B\mathcal{H}\nabla_{Y_1} CV_1 + \phi\mathcal{T}_{Y_1} CV_1). \end{aligned}$$

□

5. Examples

Let R^{2s} be the $2s$-dimensional Euclidean space endowed with an almost complex structure J defined by

$$\begin{aligned} & J(b_1\frac{\partial}{\partial y_1} + b_2\frac{\partial}{\partial y_2} + ... + b_{2s-1}\frac{\partial}{\partial y_{2s-1}} + b_{2n}\frac{\partial}{\partial y_{2s}}) \\ = \; & -b_2\frac{\partial}{\partial y_1} + b_1\frac{\partial}{\partial y_2} + ... - b_{2n}\frac{\partial}{\partial y_{2s-1}} + b_{2s-1}\frac{\partial}{\partial y_{2s}}, \end{aligned}$$

where $(y_1, y_2, ..., y_{2s-1}, y_{2s})$ are cartesian coordinates and $b_1, b_2, ..., b_{2s}$ are C^∞ functions defined on R^{2s}. We will use this notation in this section.

Example 1. *Let $(R^{10}, g_{R^{10}}, J)$ be a Kähler manifold endowed with usual metric $g_{R^{10}}$, and (R^4, g_{R^4}) be a Riemannian manifold endowed with Riemannian metric* $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sin^2 y_3 + \cos^2 y_5} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$, *where* $\sin^2 y_3 + \cos^2 y_5 \neq 0$.

Define a map $F : R^{10} \to R^4$ by

$$F(y_1, y_1, ..., y_{10}) = (\cos y_3 + \sin y_5, y_6, y_7, y_{10}),$$

which is a PQHS submersion such that

$$\begin{aligned} X_1 &= \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial y_2}, X_3 = \cos y_5\frac{\partial}{\partial y_3} + \sin y_3\frac{\partial}{\partial y_5}, \\ X_4 &= \frac{\partial}{\partial y_4}, X_5 = \frac{\partial}{\partial y_8}, X_6 = \frac{\partial}{\partial y_9}, \end{aligned}$$

$$\ker F_* = D \oplus D^\theta \oplus D^\perp,$$

where

$$D = < X_1 = \frac{\partial}{\partial y_1}, X_2 = \frac{\partial}{\partial y_2} >,$$

$$D^\theta = < X_3 = \cos y_5 \frac{\partial}{\partial y_3} + \sin y_3 \frac{\partial}{\partial y_5}, X_4 = \frac{\partial}{\partial y_4} >,$$
$$D^\perp = < X_5 = \frac{\partial}{\partial y_8}, X_6 = \frac{\partial}{\partial y_9} >,$$

$$(\ker F_*)^\perp = < H_1 = \frac{\partial}{\partial y_6}, H_2 = \sin y_3 \frac{\partial}{\partial y_3} - \cos y_5 \frac{\partial}{\partial y_5}, H_3 = \frac{\partial}{\partial y_7}, H_4 = \frac{\partial}{\partial y_{10}} >,$$

with slant function y_5.

Example 2. *Let* (R^8, g_{R^8}, J) *be a Kähler manifold endowed with usual metric* g_{R^8}*, and* (R^3, g_{R^3}) *be a Riemannian manifold endowed with Riemannian metric* $\begin{bmatrix} \frac{1}{\sin^2 y_1 + \cos^2 y_4} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, *where* $\sin^2 y_1 + \cos^2 y_4 \neq 0$.

Define a map $F : R^8 \to R^3$ *by*

$$F(y_1, y_1, ..., y_8) = (\cos y_1 + \sin y_4, y_3, y_7),$$

which is a PQHS submersion such that

$$X_1 = (\cos y_4 \frac{\partial}{\partial y_1} + \sin y_1 \frac{\partial}{\partial y_4}), X_2 = \frac{\partial}{\partial y_2},$$
$$X_3 = \frac{\partial}{\partial y_5}, X_4 = \frac{\partial}{\partial y_6}, X_5 = \frac{\partial}{\partial y_8}$$
$$\ker F_* = D \oplus D^\theta \oplus D^\perp,$$
$$D = < X_3 = \frac{\partial}{\partial y_5}, X_4 = \frac{\partial}{\partial y_6} >,$$
$$D^\theta = < X_1 = (\cos y_4 \frac{\partial}{\partial y_1} + \sin y_1 \frac{\partial}{\partial y_4}), X_2 = \frac{\partial}{\partial y_2} >,$$
$$D^\perp = < X_5 = \frac{\partial}{\partial y_8} >,$$

$$(\ker F_*)^\perp = < H_1 = (\sin y_1 \frac{\partial}{\partial y_1} - \cos y_4 \frac{\partial}{\partial y_4}), H_2 = \frac{\partial}{\partial y_3}, H_3 = \frac{\partial}{\partial y_7} >$$

with slant function $\theta = y_4$.

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