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Further Results for Hermite-Based Milne-Thomson Type Fubini Polynomials with Trigonometric Functions

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Abstract

This paper examines generating functions of r-parametric Hermite-based Milne-Thomson polynomials. Using generating function methods, the relationships among these polynomials, Fubini type polynomials, and trigonometric functions are given. Moreover, new formulas are derived by utilizing not only the generating functions of these polynomials but also associated functional equations. These formulas pertain to r-parametric Hermite-based sine-and cosine-Milne-Thomson Fubini polynomials, as well as Stirling type polynomials and numbers. Additionally, by analyzing special cases of newly obtained results, some known formulas are also derived. Furthermore, some identities involving secant and cosecant numbers are derived through the properties of trigonometric functions. Special polynomials and their generating functions are an important tool for solving some problems in many areas such as combinatorics and number theory. By introducing new formulas, this paper significantly enhances these problems-solving abilities in these areas. Consequently, these results have potential to shed light on important applications in mathematics, engineering, and mathematical physics.

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1. INTRODUCTION

Trigonometric functions have many significant applications. These functions are also related to special polynomials with their generating functions. These functions enable to define various families of special polynomials. Further, these functions play a crucial role in examining their fundamental properties of related polynomials. They used to construct mathematical modeling and solve differential equations in applied science. These frameworks of trigonometric functions and special polynomials in mathematical analysis, as well as in addressing physical problems and engineering applications. For example, many applications and formulas for these have been given by authors (Agyuz, 2024;-; Zayed et al., 2024). Inspired the above explanations, the motivation of this paper is to deal with the r-parametric Hermite-based Milne-Thomson polynomials with trigonometric functions and the Fubini type numbers in detail. The first motivation begins with the notations and definitions related to certain special polynomials and numbers. The second motivation use generating functions in order to obtain some novel results for certain families of polynomials.

Let $e^{v} = \exp(v)$ and

$$\frac{(\xi)_d}{d!} = \binom{\xi}{d} = \frac{\xi(\xi-1)(\xi-2)\dots(\xi-d+1)}{d!},$$

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where ξ is complex numbers: $\mathbb{C}$, $d \in \mathbb{N} = \{1,2,3, \dots\}$, and $(\xi)_0 = 1$ (Bayad & Simsek, 2014;-; Srivastava, 1976).

The generating function that pertains to secant numbers is expressed by

$$\sec(v) = \sum_{k=0}^{\infty} s_k \frac{v^{2k}}{(2k)!} \tag{1}$$

(Kim & Kim, 2018 (Equation (1.12))).

The first few terms of the numbers s_k are presented below:

$$s_0 = 1, \quad s_1 = 1, \quad s_2 = 5, \quad s_3 = 61, \quad s_4 = 1385, \quad s_5 = 50521, \quad s_6 = 2702765, \dots.$$

The secant numbers are also called “zig numbers” (Kim & Kim, 2018).

The generating function that pertains to hyperbolic cosecant numbers is expressed by

$$v\,\mathrm{csch}(v) = \sum_{k=0}^{\infty} d_k \frac{v^{k}}{k!} \tag{2}$$

(Kim & Kim, 2018 (Equation (2.22))).

By using (2), one has

$$v \csc(v) = (vi)\,\mathrm{csch}(vi) = \sum_{k=0}^{\infty} (-1)^k d_{2k} \frac{v^{2k}}{(2k)!}, \tag{3}$$

where $i^2 = -1$ (Kim & Kim, 2018 (Equation (2.24))).

The first few terms of the numbers d_k are presented below:

$$d_0 = 1, \quad d_1 = 0, \quad d_2 = -\frac{1}{3}, \quad d_3 = 0, \quad d_4 = \frac{7}{15}, \quad d_5 = 0, \quad d_6 = -\frac{31}{21}, \dots.$$

The numbers d_k are pertains to Bernoulli numbers and polynomials (Kim & Kim, 2018).

The generating function that pertains to generalized Hermite-Kampè de Fèriet polynomials is expressed by

$$\exp\left(\sum_{j=1}^{r} u_j v^j\right) = \sum_{k=0}^{\infty} H_k(\vec{u}, r) \frac{v^k}{k!}, \tag{4}$$

where $\vec{u} = (u_1, u_2, \dots, u_r)$ (Dattoli et al., 1994; 1996; Kilar, 2021; Kilar & Simsek, 2021). Many researchers have studied these types of polynomials using various techniques, such as degenerate versions and q-calculus (Cesarano et al., 2022; Fadel et al., 2024; Zayed et al., 2024).

From (4), we get

$$H_k(\vec{u},r) = k!\sum_{j=0}^{\left[\frac{k}{r}\right]} \frac{u_r^j H_{k-rj}(\vec{u},r-1)}{j!\,(k-jr)!},$$

where $[m]$ is the largest integer m (Dattoli et al., 1994; 1996; Kilar, 2021; Kilar & Simsek, 2021).

The generating function that pertains to λ-array polynomials is expressed by

$$(\lambda \exp(v) - 1)^p \exp(vx) = \sum_{k=0}^{\infty} \frac{p!\,S_p^k(x;\lambda)}{k!} v^k, \tag{5}$$

where $p \in \mathbb{N}_0 = \{0,1,2,3,\dots\}$ and $\lambda \in \mathbb{C}$ (Simsek, 2013).

For $x = 0$ in (5) yields λ-Stirling numbers of the second kind

$$(\lambda \exp(v) - 1)^p = \sum_{k=0}^{\infty} \frac{p!\,S_2(k,p;\lambda)}{k!} v^k \tag{6}$$

(Simsek, 2013).

For $\lambda = 1$ in (6) yields Stirling numbers of the second kind

$$S_2(k,p;1) \coloneqq S_2(k,p).$$

The generating function that pertains to numbers $S_{22}(k,p)$ is expressed by

$$(\exp(v) - v - 1)^p = \sum_{k=0}^{\infty} \frac{p!\,S_{22}(k,p)}{k!} v^k \tag{7}$$

(Charalambides, 2005 (pp. 123-127); Simsek, 2024).

The generating functions that pertain to polynomials are expressed by

$$\exp\left(\sum_{j=1}^{r} u_j v^j\right)\cos(yv) = \sum_{k=0}^{\infty} C_k(\vec{u},y;r)\frac{v^k}{k!} \tag{8}$$

and

$$\exp\left(\sum_{j=1}^{r} u_j v^j\right)\sin(yv) = \sum_{k=0}^{\infty} S_k(\vec{u},y;r)\frac{v^k}{k!} \tag{9}$$

(Kilar & Simsek, 2021; and also Kilar, 2021; 2023a).

When $r = 1$ in (8) and (9), one has

$$C_k(u_1,y;1) \coloneqq C_k(u_1,y) \quad \text{and} \quad S_k(u_1,y;1) \coloneqq S_k(u_1,y).$$

That is, the polynomials $C_k(x,y)$ and $S_k(x,y)$ are expressed by

$$\exp(vx)\cos(yv) = \sum_{k=0}^{\infty} C_k(x,y)\frac{v^k}{k!} \tag{10}$$

and

$$\exp(vx)\sin(yv) = \sum_{k=0}^{\infty} S_k(x,y)\frac{v^k}{k!}. \tag{11}$$

Kilar & Simsek (2021) defined the r-parametric Hermite-based Milne-Thomson type polynomials as follows:

$$2\big(b+f(v,a)\big)^p \exp(xv)\exp\left(\sum_{j=1}^{r} u_j v^j\right)\cos(yv) = \sum_{k=0}^{\infty} \mathfrak{h}_1(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!} \tag{12}$$

and

$$2\big(b+f(v,a)\big)^p \exp(xv)\exp\left(\sum_{j=1}^{r} u_j v^j\right)\sin(yv) = \sum_{k=0}^{\infty} \mathfrak{h}_2(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!}, \tag{13}$$

where $f(v,a)$ refers to a meromorphic function or an analytic function, $p \in \mathbb{N}_0$ and a,b are real numbers: $\mathbb{R}$ (see also Kilar, 2021).

When $b = 0$ and $f(v,1) = \frac{2}{(2-\exp(v))^2}$ in (12) and (13), we have the r-parametric Hermite-based cosine-and sine-Milne-Thomson type Fubini polynomials, respectively:

$$\frac{2^{p+1}}{\big(2-\exp(v)\big)^{2p}}\exp(xv)\exp\left(\sum_{j=1}^{r} u_j v^j\right)\cos(yv) = \sum_{k=0}^{\infty} {}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!} \tag{14}$$

and

$$\frac{2^{p+1}}{\big(2-\exp(v)\big)^{2p}}\exp(xv)\exp\left(\sum_{j=1}^{r} u_j v^j\right)\sin(yv) = \sum_{k=0}^{\infty} {}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)\frac{v^k}{k!} \tag{15}$$

(Kilar, 2021; Kilar & Simsek, 2021).

Using Eqs. (14) and (15), some special cases of these are given as follows.

When $y = 0$ and $\vec{u} = (0,0,\dots,0) = \vec{0}$ in (14), one has

$${}_{F}\mathfrak{h}_1\big(k,x,0,p;\vec{0},r\big) := 2a_k^{(p)}(x)$$

and putting $x = 0$ into the above equation,

$$ {}_{F}\mathfrak{h}_1\left(k,0,0,p;\vec{0},r\right):=2a_k^{(p)}, $$

where $a_k^{(p)}(x)$ and $a_k^{(p)}$ refer to the pth ordered Fubini type polynomials and numbers (Kilar & Simsek, 2017). We note that the Fubini type polynomials and numbers play a crucial role in combinatorics, probability, statistics, and number theory. Furthermore, these numbers and polynomials have comprehensive applications in fields such as mathematical physics, engineering, and optimization problems. Many researchers have extensively studied these numbers and polynomials using techniques including approximations, operators, special functions, and generating functions, and also they found many interesting results (see Agyuz, 2024; Ali & Paris, 2022; Diagana & Maïga, 2017; Kereskényi-Balogh & Nyul, 2021; Kilar, 2023b; Kilar & Simsek, 2017; Srivastava et al., 2021).

2. MAIN RESULTS

Here, using generating functions of the r-parametric Hermite-based Milne-Thomson type polynomials, novel formulas and identities for these type polynomials are obtained. These relations include certain special polynomials and numbers mentioned in the section above, such as the polynomials $a_k^{(p)}(x)$, the numbers $a_k^{(p)}$, the secant and cosecant numbers, the polynomials $S_p^k(x;\lambda)$, the numbers $S_2(k,p;\lambda)$, and the numbers $S_{22}(k,p)$.

Theorem 2.1. For $k\in\mathbb{N}_0$ and $\vec{u}=(u_1,u_2,\dots,u_r)$, we have

$$ C_k(\vec{u},y;r)=2^{p-1}(2\mathrm{p})!\sum_{j=0}^{k}\binom{k}{j}S_{2p}^{j}\left(-x;\frac{1}{2}\right){}_{F}\mathfrak{h}_1(k-j,x,y,p;\vec{u},r). \tag{16} $$

Proof. From (14), we have

$$ \exp\left(\sum_{j=1}^{r}u_jv^j\right)\cos(yv)=\frac{2^{2p}\left(\frac{1}{2}\exp(v)-1\right)^{2p}}{2^{p+1}}\exp(-xv)\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!}. $$

Merging the above equation with (5) and (8), we derive

$$ \sum_{k=0}^{\infty}C_k(\vec{u},y;r)\frac{v^k}{k!}=2^{p-1}(2\mathrm{p})!\sum_{k=0}^{\infty}S_{2p}^{k}\left(-x;\frac{1}{2}\right)\frac{v^k}{k!}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!}. $$

Thus

$$ \sum_{k=0}^{\infty}C_k(\vec{u},y;r)\frac{v^k}{k!}=2^{p-1}(2\mathrm{p})!\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}S_{2p}^{j}\left(-x;\frac{1}{2}\right){}_{F}\mathfrak{h}_1(k-j,x,y,p;\vec{u},r)\frac{v^k}{k!}. $$

Equating the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields Eq. (16).

Substituting $x=0$ into (16) yields the below formula:

Corollary 2.2. For $k\in\mathbb{N}_0$ and $\vec{u}=(u_1,u_2,\dots,u_r)$, we have

$$C_k(\vec{u},y;r)=2^{p-1}(2\mathrm{p})!\sum_{j=0}^{k}\binom{k}{j}S_2\left(j,2p;\frac{1}{2}\right){}_{F}\mathfrak{h}_1(k-j,0,y,p;\vec{u},r).$$

Remark 2.3. Substituting $y=0$ and $\vec{u}=(0,0,\ldots,0)$ into (16), after some calculations, for $k\in\mathbb{N}$, we get

$$0=\sum_{j=0}^{k}\binom{k}{j}S_{2p}^{j}\left(-x;\frac{1}{2}\right)a_{k-j}^{(p)}(x)$$

(Kilar, 2023b).

Theorem 2.4. For $k\in\mathbb{N}_0$ and $\vec{u}=(u_1,u_2,\ldots,u_r)$, we have

$$S_k(\vec{u},y;r)=2^{p-1}(2\mathrm{p})!\sum_{j=0}^{k}\binom{k}{j}S_{2p}^{j}\left(-x;\frac{1}{2}\right){}_{F}\mathfrak{h}_2(k-j,x,y,p;\vec{u},r). \tag{17}$$

Proof. By aid of (15), we can write

$$\exp\left(\sum_{j=1}^{r}u_jv^j\right)\sin(yv)=2^{p-1}\left(\frac{1}{2}\exp(v)-1\right)^{2p}\exp(-xv)\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Merging the above equation with (5) and (9), we derive

$$\sum_{k=0}^{\infty}S_k(\vec{u},y;r)\frac{v^k}{k!}=2^{p-1}(2\mathrm{p})!\sum_{k=0}^{\infty}S_{2p}^{k}\left(-x;\frac{1}{2}\right)\frac{v^k}{k!}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Thus

$$\sum_{k=0}^{\infty}S_k(\vec{u},y;r)\frac{v^k}{k!}=2^{p-1}(2\mathrm{p})!\sum_{k=0}^{\infty}\sum_{j=0}^{k}\binom{k}{j}S_{2p}^{j}\left(-x;\frac{1}{2}\right){}_{F}\mathfrak{h}_2(k-j,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Equating the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields Eq. (17).

When $x=0$ in (17) yields the below formula:

Corollary 2.5. For $k\in\mathbb{N}_0$ and $\vec{u}=(u_1,u_2,\ldots,u_r)$, we have

$$S_k(\vec{u},y;r)=2^{p-1}(2\mathrm{p})!\sum_{j=0}^{k}\binom{k}{j}{}_{F}\mathfrak{h}_2(k-j,0,y,p;\vec{u},r)S_2\left(j,2p;\frac{1}{2}\right).$$

Theorem 2.6. For $\vec{t}=(x+u_1,u_2,u_3,\ldots,u_r)$, $\vec{u}=(u_1,u_2,u_3,\ldots,u_r)$ and $k\in\mathbb{N}$, we have

$$C_k(\vec{t},y;r)=2^{-p-1}\sum_{j=0}^{2p}(2p)_j\sum_{q=0}^{k}\sum_{m=0}^{q}\binom{q}{m}\binom{k}{q}(-1)^{q-m+j}(2p-j)_{q-m}S_{22}(m,j)\,{}_{F}\mathfrak{h}_1(k-q,x,y,p;\vec{u},r)\,.$$

Proof. Using (14), we get

$$\exp(xv)\exp\left(\sum_{j=1}^{r}u_jv^j\right)\cos(yv)=\frac{(\exp(v)-2)^{2p}}{2^{p+1}}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Merging the above equation with (8), we have

$$\sum_{k=0}^{\infty}C_k(\vec{t},y;r)\frac{v^k}{k!}=\frac{1}{2^{p+1}}\sum_{j=0}^{2p}\binom{2p}{j}(\exp(v)-v-1)^j\,(v-1)^{2p-j}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!},$$

where $\vec{t}=(x+u_1,u_2,u_3,\dots,u_r)$. Combining the above equation with (7), we obtain

$$\sum_{k=0}^{\infty}C_k(\vec{t},y;r)\frac{v^k}{k!}=\frac{1}{2^{p+1}}\sum_{j=0}^{2p}\binom{2p}{j}j!\sum_{k=0}^{\infty}S_{22}(k,j)\frac{v^k}{k!}\sum_{k=0}^{\infty}(-1)^{k+j}(2p-j)_k\frac{v^k}{k!}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Thus

$$\sum_{k=0}^{\infty}C_k(\vec{t},y;r)\frac{v^k}{k!}$$

$$=\frac{1}{2^{p+1}}\sum_{j=0}^{2p}(2p)_j\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{m=0}^{q}\binom{q}{m}\binom{k}{q}(-1)^{q-m+j}(2p-j)_{q-m}S_{22}(m,j)\,{}_{F}\mathfrak{h}_1(k-q,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Equating the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields the result.

Theorem 2.7. For $k\in\mathbb{N}$, $\vec{t}=(x+u_1,u_2,u_3,\dots,u_r)$ and $\vec{u}=(u_1,u_2,u_3,\dots,u_r)$, we have

$$S_k(\vec{t},y;r)=2^{-p-1}\sum_{j=0}^{2p}(2p)_j\sum_{q=0}^{k}\sum_{m=0}^{q}\binom{q}{m}\binom{k}{q}(-1)^{q-m+j}(2p-j)_{q-m}\,{}_{F}\mathfrak{h}_2(k-q,x,y,p;\vec{u},r)S_{22}(m,j)\,.$$

Proof. From (15), we get

$$\exp(xv)\exp\left(\sum_{j=1}^{r}u_jv^j\right)\sin(yv)=\frac{(\exp(v)-2)^{2p}}{2^{p+1}}\sum_{k=0}^{\infty}{}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)\frac{v^k}{k!}.$$

Merging the previous equation with (9) and (7) yields

$$\sum_{k=0}^{\infty} S_k(\vec{t}, y; r)\frac{v}{k!} = \frac{1}{2^{p+1}}\sum_{j=0}^{2p}\binom{2p}{j} j! \sum_{k=0}^{\infty} S_{22}(k,j)\frac{v^k}{k!}\sum_{k=0}^{\infty}(-1)^{k+j}(2p-j)_k\frac{v^k}{k!}\sum_{k=0}^{\infty} {}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)\frac{v^k}{k!},$$

where $\vec{t} = (x+u_1, u_2, u_3, \dots, u_r)$. Hence

$$\sum_{k=0}^{\infty} S_k(\vec{t}, y; r)\frac{v^k}{k!}$$

$$= \frac{1}{2^{p+1}}\sum_{j=0}^{2p}(2p)_j\sum_{k=0}^{\infty}\sum_{q=0}^{k}\sum_{m=0}^{q}\binom{q}{m}\binom{k}{q}(-1)^{q-m+j}(2p-j)_{q-m}\, {}_{F}\mathfrak{h}_2(k-q,x,y,p;\vec{u},r)S_{22}(m,j)\frac{v^k}{k!}.$$

Comparing the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields the result.

Theorem 2.8. For $k \in \mathbb{N}_0$ and $\vec{u} = (u_1, u_2, \dots, u_r)$, we have

$$\mathfrak{h}_1(k,x,y,p;\vec{u},r,a,b) = \sum_{j=0}^{\left[\frac{k+1}{2}\right]}\binom{k+1}{2j}\frac{y^{2j-1}}{k+1}\mathfrak{h}_2(1-2j+k,x,y,p;\vec{u},r,a,b)\frac{(-1)^j}{2}\, d_{2j}. \tag{18}$$

Proof. From (13), we derive

$$2\big(b+f(v,a)\big)^p \exp(xv)\exp\left(\sum_{j=1}^{r}u_j v^j\right)\sin(2yv) = \sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,2y,p;\vec{u},r,a,b)\frac{v^k}{k!}.$$

Using the above equation and (12), we have

$$\sum_{k=0}^{\infty}\mathfrak{h}_1(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!} = \frac{1}{2\sin(yv)}\sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,2y,p;\vec{u},r,a,b)\frac{v^k}{k!}.$$

Joining the above equation with (3) yields

$$\sum_{k=0}^{\infty}\mathfrak{h}_1(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!} = \frac{1}{2yv}\sum_{k=0}^{\infty}(-1)^k y^{2k} d_{2k}\frac{v^{2k}}{(2k)!}\sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,2y,p;\vec{u},r,a,b)\frac{v^k}{k!}.$$

After some performing calculations, we have

$$\sum_{k=0}^{\infty}\mathfrak{h}_1(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!} = \sum_{k=0}^{\infty}\sum_{j=0}^{\left[\frac{k+1}{2}\right]}\binom{k+1}{2j}\frac{y^{2j-1}(-1)^j}{2(k+1)}\mathfrak{h}_2(1-2j+k,x,2y,p;\vec{u},r,a,b)\, d_{2j}\frac{v^k}{k!}.$$

Comparing the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields Eq. (18).

Theorem 2.9. For $k \in \mathbb{N}_0$ and $\vec{u} = (u_1, u_2, \dots, u_r)$, we have

$$\mathfrak{h}_2(k,x,y,p;\vec{u},r,a,b)=\frac{1}{2}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{k}{2j}\mathfrak{h}_2(k-2j,x,2y,p;\vec{u},r,a,b)y^{2j}\,s_j. \tag{19}$$

Proof. From (13), we derive

$$\sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!}=\frac{1}{2\cos(yv)}\sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,2y,p;\vec{u},r,a,b)\frac{v^k}{k!}. \tag{20}$$

Joining (20) with (1), some performing calculations, we get

$$2\sum_{k=0}^{\infty}\mathfrak{h}_2(k,x,y,p;\vec{u},r,a,b)\frac{v^k}{k!}=\sum_{k=0}^{\infty}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{k}{2j}\mathfrak{h}_2(k-2j,x,2y,p;\vec{u},r,a,b)y^{2j}\frac{s_j}{k!}v^k\,.$$

Comparing the terms of $\frac{v^k}{k!}$ on each side of the equation mentioned above yields Eq. (19).

When $b=0$ and $f(v,1)=\frac{2}{\left(2-\exp(v)\right)^2}$ in (18) and (19) yields the below formula:

Corollary 2.10. For $k\in\mathbb{N}_0$ and $\vec{u}=(u_1,u_2,\dots,u_r)$, we have

$${}_{F}\mathfrak{h}_1(k,x,y,p;\vec{u},r)=\sum_{j=0}^{\left[\frac{k+1}{2}\right]}\binom{k+1}{2j}\frac{y^{2j-1}}{k+1}\,{}_{F}\mathfrak{h}_2(1-2j+k,x,2y,p;\vec{u},r)\frac{(-1)^j}{2}d_{2j}$$

and

$${}_{F}\mathfrak{h}_2(k,x,y,p;\vec{u},r)=\frac{1}{2}\sum_{j=0}^{\left[\frac{k}{2}\right]}\binom{k}{2j}\,{}_{F}\mathfrak{h}_2(k-2j,x,2y,p;\vec{u},r)y^{2j}\,s_j.$$

3. CONCLUSION

In this paper, by aid of generating function methods, the r-parametric Hermite-based Milne-Thomson polynomials were investigated. By these functions and trigonometric functions, many formulas and identities pertaining to the Stirling type numbers and polynomials, the r-parametric Hermite-based sine-and cosine-Milne-Thomson Fubini polynomials, and the secant and cosecant numbers were obtained. These types of polynomials, Fubini and Hermite polynomials, have significant potential for applications in various fields such as combinatorics, probability theory, statistics, and number theory. Furthermore, the solutions offered by these polynomials are not limited to pure mathematics but can also be applied to solve real-word problems encountered in engineering, physics, and other areas. Therefore, the results presented in this paper provide valuable contributions to both theoretical mathematics and applied sciences, offering a broad range of benefits across many areas.

In the future, we plan to investigate the relations among r-parametric Hermite-based sine-and cosine-Milne-Thomson Fubini polynomials, determinantal expressions, Faa di Bruno formula, and other special numbers and polynomials, as well as their applications in differential equations and mathematical models.

CONFLICT OF INTEREST

The author declares no conflict of interest.

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