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Properties of a Subclass of Harmonic Univalent Functions Using the Al-Oboudi $q-$Differential Operator

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Abstract

In this paper, we introduce the Al-Oboudi $q-$differential operator, a generalized Sălăgean operator, for harmonic functions and define a new subclass of harmonic univalent functions using this operator. We investigate several fundamental properties of this subclass, including coefficient conditions, extreme points, distortion bounds, convex combination, and radii of convexity.

1. Introduction

Let $\mathbb{C}$ denote the complex plane and consider the open unit disk $\mathbb{E} = \{z \in \mathbb{C} : |z| < 1\}$. For a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ to be sense-preserving and locally univalent in the open unit disk $\mathbb{E}$, it is necessary and sufficient that the inequality $|\mathfrak{v}'(z)| < |\mathfrak{u}'(z)|$ holds in $\mathbb{E}$ (see [1]).

The class of functions that are harmonic, sense-preserving, univalent, and normalized by $\mathfrak{f}(0) = \mathfrak{f}_z(0) - 1 = 0$ in the open unit disk $\mathbb{E}$ is denoted by $\mathcal{SH}$. Within this class, the subclass of functions $\mathfrak{f} \in \mathcal{SH}$ that additionally satisfy $\mathfrak{v}'(0) = b_1 = 0$ is denoted by $\mathcal{SH}^0$. The functions $\mathfrak{u}$ and $\mathfrak{v}$ are analytic in the open unit disk $\mathbb{E}$ and have series expansions:

$$\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} a_s z^s, \quad \mathfrak{v}(z) = \sum_{s=2}^{\infty} b_s z^s. \tag{1.1}$$

A function $\mathfrak{f} \in \mathcal{SH}^0$ can be expressed as $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$. If we choose $\mathfrak{v}(z) = 0$, we obtain the class $\mathcal{S}$, which consists of analytic, univalent and normalized functions in $\mathbb{E}$. The relationships $\mathcal{S} \subset \mathcal{SH}^0 \subset \mathcal{SH}$ hold for the function classes $\mathcal{S}$, $\mathcal{SH}$, and $\mathcal{SH}^0$.

The subclasses $\mathcal{K}$ and $\mathcal{S}^*$ of $\mathcal{S}$ are characterized by their mappings of the unit disk $\mathbb{E}$ onto convex and starlike domains, respectively. Similarly, the subclasses of $\mathcal{SH}^0$ that map the unit disk $\mathbb{E}$ onto corresponding domains are denoted by $\mathcal{SH}^{0,*}$ and $\mathcal{KH}^0$. For a more detailed discussion, see [1, 2].

Jackson's q-derivative for a function $\psi \in \mathcal{S}$, where $0 < q < 1$, is defined as follows [3]:

$$D_q \psi(z) = \begin{cases} \dfrac{\psi(z) - \psi(qz)}{(1-q)z}, & \text{if } z \neq 0, \\ \\ \psi'(0), & \text{if } z = 0. \end{cases} \tag{1.2}$$

Note that if ψ is differentiable at z, then as $q \to 1^-$, we have $D_q \psi(z) \to \psi'(z)$.

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Jackson also defined the q-integral as follows [4]:

$$\int_0^z \psi(\zeta)\, d_q\zeta = z(1-q)\sum_{k=0}^{\infty} q^k \psi(zq^k), \tag{1.3}$$

provided that the series on the right-hand side converges.

Jahangiri et al. [5] introduced the *modified Sălăgean q-differential operator* for harmonic functions of the form $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where $\mathfrak{m} \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. This operator is defined as

$$D_q^{\mathfrak{m}}\mathfrak{f}(z) = D_q^{\mathfrak{m}}\mathfrak{u}(z) + (-1)^{\mathfrak{m}}\overline{D_q^{\mathfrak{m}}\mathfrak{v}(z)}, \tag{1.4}$$

where

$$D_q^{\mathfrak{m}}\mathfrak{u}(z) = z + \sum_{s=2}^{\infty} [s]_q^{\mathfrak{m}} a_s z^s \quad \text{and} \quad D_q^{\mathfrak{m}}\mathfrak{v}(z) = \sum_{s=2}^{\infty} [s]_q^{\mathfrak{m}} b_s z^s. \tag{1.5}$$

For a harmonic function $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, where $\mathfrak{m} \in \mathbb{N}_0$ and $\delta \geq 0$, we define the *modified Al-Oboudi q-differential operator* $D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)$ as follows:

$$\begin{aligned}
D_{\delta,q}^{0}\mathfrak{f}(z) &= D_q^0\mathfrak{f}(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)}, &(1.6)\\
D_{\delta,q}^{1}\mathfrak{f}(z) &= (1-\delta)D_q^0\mathfrak{f}(z) + \delta D_q^1\mathfrak{f}(z), &(1.7)\\
&\vdots\\
D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z) &= D_{\delta,q}^{1}\left(D_{\delta,q}^{\mathfrak{m}-1}\mathfrak{f}(z)\right). &(1.8)
\end{aligned}$$

Using the expression of $\mathfrak{f}$ given in (1.1), it follows from (1.7) and (1.8) that

$$D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z) = z + \sum_{s=2}^{\infty}\left[\delta([s]_q - 1) + 1\right]^{\mathfrak{m}} a_s z^s + (-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}\left[\delta([s]_q + 1) - 1\right]^{\mathfrak{m}}\overline{b_s z^s}. \tag{1.9}$$

We note that the operator $D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)$ reduces to several known differential operators for specific choices of the parameters δ, q. More precisely:

- For $\delta = 1$, the operator coincides with the $q-$analogue of the modified Sălăgean operator studied by Jahangiri et al. [5].
- As $q \to 1^-$, the operator becomes the generalization of the modified Sălăgean operator investigated by Yaşar and Yalçın [6].
- For $\delta = 1$ and $q \to 1^-$, we recover the modified Sălăgean differential operator defined by Jahangiri et al. [7].
- If $\mathfrak{v}(z) \equiv 0$, the operator reduces to the generalized $q-$Sălăgean operator introduced by Aouf et al. [8].
- If $\mathfrak{v}(z) \equiv 0$ and $q \to 1^-$, the operator reduces to the Al-Oboudi differential operator [9].
- For $\mathfrak{v}(z) \equiv 0$, $q \to 1^-$, and $\delta = 1$, we obtain the classical Sălăgean differential operator [10].

In 2019, Ahuja and Çetinkaya [11] introduced the class of $q-$harmonic, sense-preserving, and univalent functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$, denoted by $\mathcal{SH}_q$. For a function $\mathfrak{f}$ to be included in class $\mathcal{SH}_q$, it must meet the following requirements:

$$\omega(z) = \left|\frac{D_q\mathfrak{v}(z)}{D_q\mathfrak{u}(z)}\right| < 1.$$

Additionally, as $q \to 1^-$, the class $\mathcal{SH}$ is recovered.

For $0 \leq \alpha < 1$, the class of harmonic functions $f = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q$ that satisfy the inequality

$$\operatorname{Re}\left\{\frac{zD_q\mathfrak{u}(z) - \overline{zD_q\mathfrak{v}(z)}}{\mathfrak{u}(z) + \overline{\mathfrak{v}}(z)}\right\} > \alpha$$

is denoted by $\mathcal{SH}_q^*(\alpha)$. Functions in this class are referred to as $q-$starlike harmonic functions of order α. Similarly, for $0 \leq \alpha < 1$, the class of harmonic functions $f = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q$ satisfying the inequality

$$\operatorname{Re}\left\{\frac{zD_q(zD_q\mathfrak{u}(z)) - \overline{zD_q(zD_q\mathfrak{v}(z))}}{zD_q\mathfrak{u}(z) - \overline{zD_q\mathfrak{v}(z)}}\right\} > \alpha$$

is denoted by $\mathcal{KH}_q(\alpha)$. The functions in this class are called $q-$convex harmonic functions of order α. For a more detailed discussion, see [12, 13, 14].

Let

$$\mathfrak{f}_k(z)=z+\sum_{s=2}^{\infty}a_{k,s}z^s+\sum_{s=2}^{\infty}\overline{b_{k,s}z^s} \quad (z\in\mathbb{E},\, k=1,2),$$

then the functions $\mathfrak{f}_1$ and $\mathfrak{f}_2$ have the following Hadamard product (or convolution):

$$(\mathfrak{f}_1*\mathfrak{f}_2)(z)=z+\sum_{s=2}^{\infty}a_{1,s}a_{2,s}z^s+\sum_{s=2}^{\infty}\overline{b_{1,s}b_{2,s}z^s} \quad (z\in\mathbb{E}).$$

Furthermore, if $\mathfrak{f}\in\mathcal{SH}_q$, we obtain

$$\begin{aligned} D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z) &= \mathfrak{f}(z)*\underbrace{\left(\chi_1(z)+\overline{\chi_2(z)}\right)*\cdots*\left(\chi_1(z)+\overline{\chi_2(z)}\right)}_{\mathfrak{m}\text{ times}}, \\ &= \mathfrak{u}(z)*\underbrace{\chi_1(z)*\cdots*\chi_1(z)}_{\mathfrak{m}\text{ times}}+\mathfrak{v}(z)*\underbrace{\chi_2(z)*\cdots*\chi_2(z)}_{\mathfrak{m}\text{ times}}. \end{aligned}$$

where

$$\chi_1(z)=\frac{(\delta-1)qz^2+z}{(1-z)(1-qz)}, \quad \chi_2(z)=\frac{(\delta-1)qz^2+(1-2\delta)z^2}{(1-z)(1-qz)}.$$

A function $\mathfrak{f}:\mathbb{E}\to\mathbb{C}$ is said to be subordinate to another function $\mathfrak{g}:\mathbb{E}\to\mathbb{C}$, denoted by $\mathfrak{f}(z)\prec\mathfrak{g}(z)$, if there exists a complex-valued function ω mapping $\mathbb{E}$ into itself with $\omega(0)=0$, such that $\mathfrak{f}(z)=\mathfrak{g}(\omega(z))$ (see [15]).

Denote $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ as the subclass of $\mathcal{SH}_q^0$ consisting of functions $\mathfrak{f}$ of the form (1.1) that satisfy the condition:

$$\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}\prec\frac{1+\eta z}{1+\mu z}, \quad -\mu\leq\eta<\mu\leq 1. \tag{1.10}$$

As $q\to 1^-$, this class converges to $\mathcal{SH}^0(\delta,\mathfrak{m},\eta,\mu)$ introduced by Çakmak et al. [16]. Additionally, for $q\to 1^-$ and with the choices of specific parameters, the following classes are obtained, which have been previously studied:

(i) $\mathcal{SH}^0(1,\delta,\eta,\mu)=H_\delta(\eta,\mu), \quad \delta\in\mathbb{N}_0=\mathbb{N}\cup\{0\}$ ([17]),

(ii) $\mathcal{SH}^0(1,1,\eta,\mu)=S_H^*(\eta,\mu)\cap\mathcal{SH}^0$ ([18]),

(iii) $\mathcal{SH}^0(\delta,\mathfrak{m},2\alpha-1,1)=\mathcal{SH}(\delta,\mathfrak{m},\alpha)\cap\mathcal{SH}^0$ ([6]),

(iv) $\mathcal{SH}^0(1,\mathfrak{m},2\alpha-1,1)=H^0(\mathfrak{m},\alpha)$ ([7]),

(v) $\mathcal{SH}^0(1,0,2\alpha-1,1)=S_{H^0}^*(\alpha)$ ([19], [20], [21]),

(vi) $\mathcal{SH}^0(1,1,2\alpha-1,1)=S_{H^0}^c(\alpha)$ ([19]),

(vii) $\mathcal{SH}^0(\delta,\mathfrak{m},2\alpha-1,1)=\overline{\mathcal{SH}}(\delta,1-\delta,\mathfrak{m},\alpha)$ ([22]).

Further details on these classes and their properties can be found in the works of Jahangiri et al. [23], Murugusundaramoorthy et al. [24] and Canbulat et al. [25].

The aim of this paper is to advance the study of harmonic functions by introducing the Al-Oboudi $q-$differential operator, an extension of the well-known Sălăgean operator. We define a new subclass of harmonic univalent functions using this generalized operator, which allows us to explore several key properties of these functions. Building on techniques and methodologies from Dziok ([18], [26]), and Dziok et al. ([17]), we analyze fundamental aspects of this subclass, including coefficient conditions, extreme points, distortion bounds and radii of convexity. Through these investigations, our aim is to enhance the understanding of harmonic function theory and provide new insights into the geometric behavior of these functions.

2. Main Theorems and Results

First, we establish a necessary and sufficient condition involving convolution for harmonic functions in $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.

Theorem 2.1. *Let $z\in\mathbb{E}\setminus\{0\}$ and suppose that $\mathfrak{f}$ belongs to $\mathcal{SH}_q^0$. The function $\mathfrak{f}$ is an element of $\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if and only if the following condition is satisfied:*

$$D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\chi(z;\zeta)\neq 0 \, \textit{for all} \ (\zeta\in\mathbb{C} \textit{ with } |\zeta|=1),$$

where

$$\chi(z;\zeta)=\frac{[(\eta-\mu)\zeta+\delta(1+\mu\zeta)]qz^2+(\mu-\eta)\zeta z}{(1-qz)(1-z)}-(-1)^{\mathfrak{m}}\frac{[-\delta(1+\mu\zeta)+(\mu-\eta)\zeta]q\bar{z}^2+[2\delta(1+\mu\zeta)-(\mu-\eta)\zeta]\bar{z}}{(1-q\bar{z})(1-\bar{z})}.$$

Proof. Let $\mathfrak{f}\in\mathcal{SH}_q^0$. The condition $\mathfrak{f}\in\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is satisfied if and only if the condition (1.10) holds, which is equivalent to

$$\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}\neq\frac{1+\eta\zeta}{1+\mu\zeta} \text{ for } (\zeta\in\mathbb{C},|\zeta|=1). \tag{2.1}$$

Consider

$$D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)=D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left(\frac{z}{1-z}+\frac{\bar{z}}{1-\bar{z}}\right),$$

and

$$D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)=D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left(\chi_1(z)+\overline{\chi_2(z)}\right),$$

then the inequality (2.1) leads to

$$\begin{aligned}(1+\mu\zeta)D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)-(1+\eta\zeta)D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z) &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{(1+\mu\zeta)\left[(\delta-1)qz^2+z\right]}{(1-qz)(1-z)}+\frac{(1+\mu\zeta)\left[(\delta-1)q\bar{z}^2+(1-2\delta)\bar{z}\right]}{(1-q\bar{z})(1-\bar{z})}\right\}\\ &\quad -D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{(1+\eta\zeta)z}{1-z}+\frac{(1+\eta\zeta)\bar{z}}{1-\bar{z}}\right\}\\ &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\left\{\frac{[(\eta-\mu)\zeta+\delta(1+\mu\zeta)]qz^2+(\mu-\eta)\zeta z}{(1-qz)(1-z)}\right.\\ &\quad\left.-\frac{[-\delta(1+\mu\zeta)+(\mu-\eta)\zeta]q\bar{z}^2+[2\delta(1+\mu\zeta)-(\mu-\eta)\zeta]\bar{z}}{(1-q\bar{z})(1-\bar{z})}\right\}\\ &= D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)*\chi(z;\zeta)\neq 0.\end{aligned}$$

□

Theorem 2.2. *Let $\mathfrak{f}=\mathfrak{u}+\bar{\mathfrak{v}}\in\mathcal{SH}_q^0$, where $\mathfrak{u}$ and $\mathfrak{v}$ are represented as in (1.1). Then, $\mathfrak{f}\in\mathcal{SH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ if the following inequality is satisfied:*

$$\sum_{k=2}^{\infty}\left(P_k|a_k|+Q_k|b_k|\right)\leq\mu-\eta, \tag{2.2}$$

where the sequences P_s and Q_s are given by:

$$P_s=\left[1+\delta\left([s]_q-1\right)\right]^{\mathfrak{m}}\left[\delta\left([s]_q-1\right)(\mu+1)+\mu-\eta\right], \tag{2.3}$$

and

$$Q_s=\left[-1+\delta\left([s]_q+1\right)\right]^{\mathfrak{m}}\left[\delta\left([s]_q+1\right)(\mu+1)+\eta-\mu\right]. \tag{2.4}$$

Proof. The theorem is evidently valid for $\mathfrak{f}(z) = z$. Now, consider the case where $a_s \neq 0$ or $b_s \neq 0$ for $s \geq 2$. Since $P_s \geq [s]_q(\mu - \eta)$ and $Q_s \geq [s]_q(\mu - \eta)$, from (2.2), we obtain:

$$\begin{aligned} \left|D_q\mathfrak{u}(z)\right| - \left|D_q\mathfrak{v}(z)\right| &\geq 1 - \sum_{s=2}^{\infty}[s]_q\,|a_s|\,|z|^{s-1} - \sum_{s=2}^{\infty}[s]_q\,|b_s|\,|z|^{s-1} \\ &\geq 1 - |z|\sum_{s=2}^{\infty}[s]_q\,(|a_s| + |b_s|) \\ &\geq 1 - \frac{|z|}{\mu - \eta}\sum_{s=2}^{\infty}(P_s\,|a_s| + Q_s\,|b_s|) \\ &\geq 1 - |z| > 0. \end{aligned}$$

Thus, $\mathfrak{f}$ belongs to $\mathcal{SH}_q^0$.

A function $\mathfrak{f}$ belongs to the class $\mathcal{SH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$ if there exists a complex-valued function ω such that $\omega(0) = 0$ and $|\omega(z)| < 1$ for all $z \in U$. This condition is met if and only if the following holds:

$$\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z)}{D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)} = \frac{1 + \eta\omega(z)}{1 + \mu\omega(z)},$$

which is equivalent to the inequality:

$$\left|\frac{D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z) - D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}{\mu D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z) - \eta D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)}\right| < 1, \quad z \in \mathbb{E}. \tag{2.5}$$

The inequality in (2.5) holds because for $|z| = r$ with $0 < r < 1$, we have:

$$\begin{aligned} & \left|D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z) - D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)\right| - \left|\mu D_{\delta,q}^{\mathfrak{m}+1}\mathfrak{f}(z) - \eta D_{\delta,q}^{\mathfrak{m}}\mathfrak{f}(z)\right| \\ = & \left|\sum_{s=2}^{\infty}[\delta([s]_q - 1) + 1]^{\mathfrak{m}}\,\delta([s]_q - 1)a_s z^s - (-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}[\delta([s]_q + 1) - 1]^{\mathfrak{m}}\,\delta([s]_q + 1)\overline{b_s z^s}\right| \\ - & \left|(\mu - \eta)z + \sum_{s=2}^{\infty}[\delta([s]_q - 1) + 1]^{\mathfrak{m}}\,[\delta\mu([s]_q - 1) + \mu - \eta]\,a_s z^s\right. \\ & \left. -(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}[\delta([s]_q + 1) - 1]^{\mathfrak{m}}\,[\delta\mu([s]_q + 1) - \mu + \eta]\,\overline{b_s z^s}\right| \\ \leq & \sum_{s=2}^{\infty}[\delta([s]_q - 1) + 1]^{\mathfrak{m}}\,\delta([s]_q - 1)\,|a_s|\,r^s + \sum_{s=2}^{\infty}[\delta([s]_q + 1) - 1]^{\mathfrak{m}}\,\delta([s]_q + 1)\,|b_s|\,r^s \\ - & (\mu - \eta)r + \sum_{s=2}^{\infty}[\delta([s]_q - 1) + 1]^{\mathfrak{m}}\,[\delta\mu([s]_q - 1) + \mu - \eta]\,|a_s|\,r^s \\ + & \sum_{s=2}^{\infty}[\delta([s]_q + 1) - 1]^{\mathfrak{m}}\,[\delta\mu([s]_q + 1) - \mu + \eta]\,|b_s|\,r^s \\ < & 1. \end{aligned}$$

Therefore, $\mathfrak{f} \in \mathcal{SH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$, completing the proof. □

Next, we demonstrate that the condition given in (2.2) is also a necessary criterion for a function $\mathfrak{f} \in \mathcal{SH}_q^0$ to belong to the class $\mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu) = \mathcal{T}^{\mathfrak{m}} \cap \mathcal{SH}^0(\delta, \mathfrak{m}, \eta, \mu)$, where $\mathcal{T}^{\mathfrak{m}}$ represents the set of functions $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}} \in \mathcal{SH}_q^0$ such that

$$\mathfrak{f}(z) = \mathfrak{u}(z) + \overline{\mathfrak{v}(z)} = z - \sum_{s=2}^{\infty}|a_s|\,z^s + (-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}|b_s|\,\overline{z}^s, \quad z \in \mathbb{E}. \tag{2.6}$$

Theorem 2.3. *Consider the definition of $\mathfrak{f} = \mathfrak{u} + \overline{\mathfrak{v}}$ in (2.6). Then, $\mathfrak{f} \in \mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$ if and only if condition (2.2) is satisfied.*

Proof. The sufficiency of this condition follows directly from Theorem 2.2. To prove necessity, suppose $\mathfrak{f} \in \mathcal{TSH}_q^0(\delta, \mathfrak{m}, \eta, \mu)$. Using (2.5), we can write

$$\left|\frac{\sum_{s=2}^{\infty}([s]_q-1)\,\delta\,[([s]_q-1)\,\delta+1]^{\mathfrak{m}}\,|a_s|\,z^s+([s]_q+1)\,\delta\,[([s]_q+1)\,\delta-1]^{\mathfrak{m}}\,|b_s|\,\bar{z}^s}{(\mu-\eta)z-\sum_{s=2}^{\infty}[([s]_q-1)\,\delta\mu+\mu-\eta]\,[([s]_q-1)\,\delta+1]^{\mathfrak{m}}\,|a_s|\,z^s+[([s]_q+1)\,\delta\mu+\eta-\mu]\,[([s]_q+1)\,\delta-1]^{\mathfrak{m}}\,|b_s|\,\bar{z}^s}\right|<1.$$

For $z=r<1$, this simplifies to

$$\frac{\sum_{s=2}^{\infty}\left\{([s]_q-1)\,\delta\,[([s]_q-1)\,\delta+1]^{\mathfrak{m}}\,|a_s|+([s]_q+1)\,\delta\,[([s]_q+1)\,\delta-1]^{\mathfrak{m}}\,|b_s|\right\}r^{s-1}}{\mu-\eta-\sum_{s=2}^{\infty}\left\{[([s]_q-1)\,\delta\mu+\mu-\eta]\,[([s]_q-1)\,\delta+1]^{\mathfrak{m}}\,|a_s|+[([s]_q+1)\,\delta\mu+\eta-\mu]\,[([s]_q+1)\,\delta-1]^{\mathfrak{m}}\,|b_s|\right\}r^{s-1}}<1.$$

Therefore, for the terms P_s and Q_s as defined in (2.3) and (2.4), we have the inequality

$$\sum_{s=2}^{\infty}\left[P_s\,|a_s|+Q_s\,|b_s|\right]r^{s-1}<\mu-\eta\quad(0\le r<1).\tag{2.7}$$

Let $\{\sigma_s\}$ be the sequence defined by the partial sums of the series given by

$$\sum_{s=2}^{\infty}\left[P_s\,|a_s|+Q_s\,|b_s|\right].$$

Since $\{\sigma_s\}$ is non-decreasing and bounded above by $\mu-\eta$, it must converge, and hence

$$\sum_{s=2}^{\infty}\left[P_s\,|a_s|+Q_s\,|b_s|\right]=\lim_{s\to\infty}\sigma_s\le\mu-\eta.$$

This establishes condition (2.2). □

In the following, we demonstrate that the function class given in equation (2.6) is both convex and compact.

Theorem 2.4. *The class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is convex and compact within the space $\mathcal{SH}_q^0$.*

Proof. Consider a sequence $\mathfrak{f}_k\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, where

$$\mathfrak{f}_k(z)=z-\sum_{s=2}^{\infty}\left|a_{k,s}\right|z^s+(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}\left|b_{k,s}\right|\overline{z^s},\quad z\in\mathbb{E},\ k\in\mathbb{N}.\tag{2.8}$$

To prove convexity, let $0\le\lambda\le1$, and suppose $\mathfrak{f}_1$ and $\mathfrak{f}_2$ belong to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, with each defined as in (2.8). Define a new function as

$$\begin{aligned}\kappa(z)&=\lambda\mathfrak{f}_1(z)+(1-\lambda)\mathfrak{f}_2(z)\\&=z-\sum_{s=2}^{\infty}(\lambda\,|a_{1,s}|+(1-\lambda)\,|a_{2,s}|)\,z^s+(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}(\lambda\,|b_{1,s}|+(1-\lambda)\,|b_{2,s}|)\,\overline{z^s}.\end{aligned}$$

Next, we verify that $\kappa(z)$ also belongs to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$. To achieve this, we examine the following condition:

$$\begin{aligned}\sum_{s=2}^{\infty}\{P_s\,[\lambda\,|a_{1,s}|+(1-\lambda)\,|a_{2,s}|]+Q_s\,[\lambda\,|b_{1,s}|+(1-\lambda)\,|b_{2,s}|]\}&=\lambda\sum_{s=2}^{\infty}\{P_s\,|a_{1,s}|+Q_s\,|b_{1,s}|\}+(1-\lambda)\sum_{s=2}^{\infty}\{P_s\,|a_{2,s}|+Q_s\,|b_{2,s}|\}\\&\le\lambda(\mu-\eta)+(1-\lambda)(\mu-\eta)=\mu-\eta.\end{aligned}$$

Hence, $\kappa(z)$ remains within the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, establishing the convexity of the class.

To demonstrate compactness, consider any function $\mathfrak{f}_k\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$. We derive the following inequality for $|z|\le r$ $(0<r<1)$:

$$\begin{aligned}|\mathfrak{f}_k(z)|&\le r+\sum_{s=2}^{\infty}\left\{\left|a_{k,s}\right|+\left|b_{k,s}\right|\right\}r^s\\&\le r+\sum_{s=2}^{\infty}\left\{P_s\left|a_{k,s}\right|+Q_s\left|b_{k,s}\right|\right\}r^s\\&\le r+(\mu-\eta)r^2.\end{aligned}$$

This confirms that the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is locally uniformly bounded.

Now, consider the sequence $\mathfrak{f}_k(z)$ given by $z-\sum\limits_{s=2}^{\infty}\left|a_{k,s}\right|z^s+(-1)^{\mathfrak{m}}\sum\limits_{s=2}^{\infty}\left|b_{k,s}\right|\overline{z^s}$. Let $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, where $\mathfrak{u}$ and $\mathfrak{v}$ are as described in equation (1.1). From Theorem 2.3, we have the inequality

$$\sum_{s=2}^{\infty}\left\{P_s\left|a_s\right|+Q_s\left|b_s\right|\right\}\leq\mu-\eta. \tag{2.9}$$

If $\mathfrak{f}_k\to\mathfrak{f}$, it follows that $\left|a_{k,s}\right|\to\left|a_s\right|$ and $\left|b_{k,s}\right|\to\left|b_s\right|$ as $k\to\infty$. The sequence $\{\sigma_s\}$, which represents the partial sums of the series $\sum_{s=2}^{\infty}\left\{P_s\left|a_s\right|+Q_s\left|b_s\right|\right\}$, is both monotonic and upper-bounded by $\mu-\eta$. Consequently, it is convergent. Therefore, we have

$$\sum_{s=2}^{\infty}\left\{P_s\left|a_s\right|+Q_s\left|b_s\right|\right\}=\lim_{s\to\infty}\sigma_s\leq\mu-\eta.$$

Thus, $\mathfrak{f}$ belongs to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, and it follows that this class is closed. Consequently, the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ is compact within $\mathcal{SH}_q^0$. □

We now present the following result, originally established by Jahangiri [5].

Lemma 2.5 ([5])**.** *Consider the $q-$harmonic mapping $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, where $\mathfrak{u}$ and $\mathfrak{v}$ are defined as in (1.1). Suppose that the following condition is satisfied:*

$$\sum_{s=2}^{\infty}\left\{\frac{[s]_q-\alpha}{1-\alpha}\left|a_s\right|+\frac{[s]_q+\alpha}{1-\alpha}\left|b_s\right|\right\}\leq 1\quad(z\in\mathbb{E}),$$

where $0\leq\alpha<1$. Consequently, the function $\mathfrak{f}$ belongs to the class $\mathcal{SH}_q^{0,*}(\alpha)$.

For functions belonging to the class $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, the radii of starlikeness and convexity are given by the following theorems.

Theorem 2.6. *Let $0\leq\alpha<1$, and let P_s and Q_s be defined by equations (2.3) and (2.4), respectively. Then*

$$r_\alpha^*(\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu))=\inf_{k\geq 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha},\frac{Q_s}{[s]_q+\alpha}\right\}\right]^{\frac{1}{s-1}}. \tag{2.10}$$

Proof. Let $\mathfrak{f}\in\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$ be represented by the form in (2.6). For $|z|=r<1$, the following holds:

$$\begin{aligned}\left|\frac{D_{1,q}\mathfrak{f}(z)-(1+\alpha)\mathfrak{f}(z)}{D_{1,q}\mathfrak{f}(z)+(1-\alpha)\mathfrak{f}(z)}\right| &= \left|\frac{-\alpha z-\sum_{s=2}^{\infty}([s]_q-1-\alpha)\left|a_s\right|z^s-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}([s]_q+1+\alpha)\left|b_s\right|\overline{z}^s}{(2-\alpha)z-\sum_{s=2}^{\infty}([s]_q+1-\alpha)\left|a_s\right|z^s-(-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}([s]_q-1+\alpha)\left|b_s\right|\overline{z}^s}\right|\\ &\leq \frac{\alpha+\sum_{s=2}^{\infty}\left\{([s]_q-1-\alpha)|a_s|+([s]_q+1+\alpha)|b_s|\right\}r^{s-1}}{2-\alpha-\sum_{s=2}^{\infty}\left\{([s]_q+1-\alpha)|a_s|+([s]_q-1+\alpha)|b_s|\right\}r^{s-1}}.\end{aligned}$$

According to Lemma 2.5, the function $\mathfrak{f}$ is $q-$starlike of order α in $\mathbb{E}_r$ if and only if

$$\left|\frac{D_{1,q}\mathfrak{f}(z)-(1+\alpha)\mathfrak{f}(z)}{D_{1,q}\mathfrak{f}(z)+(1-\alpha)\mathfrak{f}(z)}\right|<1,\ z\in\mathbb{E}_r$$

which is equivalent to:

$$\sum_{s=2}^{\infty}\left\{\frac{[s]_q-\alpha}{1-\alpha}|a_s|+\frac{[s]_q+\alpha}{1-\alpha}|b_s|\right\}r^{s-1}\leq 1. \tag{2.11}$$

Furthermore, by Theorem 2.2, the following condition must be satisfied:

$$\sum_{s=2}^{\infty}\left\{\frac{P_s}{\mu-\eta}|a_s|+\frac{Q_s}{\mu-\eta}|b_s|\right\}r^{s-1}\leq 1.$$

The inequality in (2.11) holds if:

$$\frac{[s]_q-\alpha}{1-\alpha}r^{s-1}\leq\frac{P_s}{\mu-\eta}r^{s-1},$$

$$\frac{[s]_q+\alpha}{1-\alpha}r^{s-1} \le \frac{Q_s}{\mu-\eta}r^{s-1} \quad (s=2,3,\ldots),$$

or equivalently:

$$r \le \frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha}, \frac{Q_s}{[s]_q+\alpha}\right\}^{\frac{1}{s-1}} \quad (s=2,3,\ldots).$$

Therefore, the function $\mathfrak{f}$ is $q-$starlike of order α in the disk $\mathbb{E}_{r^*_\alpha}$, where:

$$r^*_\alpha := \inf_{s\ge 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q-\alpha}, \frac{Q_s}{[s]_q+\alpha}\right\}\right]^{\frac{1}{s-1}}.$$

Finally, the extremal function:

$$\mathfrak{f}_s(z) = \mathfrak{u}_s(z) + \overline{\mathfrak{v}_s(z)} = z - \frac{\mu-\eta}{P_s}z^s + (-1)^{\mathfrak{m}}\frac{\mu-\eta}{Q_s}\overline{z}^s$$

shows that the radius r^*_α cannot be increased. Thus, we obtain the result (2.10). □

Using a similar approach, we derive the following result.

Theorem 2.7. *Let $0 \le \alpha < 1$, and let P_s and Q_s be defined as in (2.3) and (2.4). Then, we have*

$$r^c_\alpha(\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)) = \inf_{s\ge 2}\left[\frac{1-\alpha}{\mu-\eta}\min\left\{\frac{P_s}{[s]_q([s]_q-\alpha)}, \frac{Q_s}{[s]_q([s]_q+\alpha)}\right\}\right]^{\frac{1}{s-1}}.$$

Our next result concerns the extreme points of the class $\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)$.

Theorem 2.8. *The functions $\mathfrak{u}=\mathfrak{u}_s$ and $\mathfrak{v}=\mathfrak{v}_s$ are defined as follows:*

$$\mathfrak{u}_1(z) = z,$$

$$\mathfrak{u}_s(z) = z - \tfrac{\mu-\eta}{P_s}z^s, \tag{2.12}$$

$$\mathfrak{v}_s(z) = (-1)^{\mathfrak{m}}\tfrac{\mu-\eta}{Q_s}\overline{z^s} \quad (z\in\mathbb{E},\ s\ge 2).$$

The functions $\mathfrak{f}=\mathfrak{u}+\overline{\mathfrak{v}}$, which are represented by the series expansion given in 1.1, are the extreme points of class $\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)$.

Proof. Consider the function $\mathfrak{v}_s$ defined by

$$\mathfrak{v}_s = \lambda\mathfrak{f}_1 + (1-\lambda)\mathfrak{f}_2,$$

where $0<\lambda<1$ and $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are functions in the class $\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)$. Each function $\mathfrak{f}_k$ is given by

$$\mathfrak{f}_k(z) = z - \sum_{s=2}^{\infty}|a_{k,s}|z^s + (-1)^{\mathfrak{m}}\sum_{s=2}^{\infty}|b_{k,s}|\overline{z^s},$$

where z is in $\mathbb{E}$ and k is either 1 or 2.
By (2.12), it follows that

$$|b_{1,s}| = |b_{2,s}| = \frac{\mu-\eta}{Q_s},$$

which implies $a_{1,k}=a_{2,k}=0$ for $k\in\{2,3,\ldots\}$ and $b_{1,k}=b_{2,k}=0$ for $k\in\{2,3,\ldots\}\setminus\{s\}$. Consequently, $\mathfrak{v}_s(z)=\mathfrak{f}_1(z)=\mathfrak{f}_2(z)$, and $\mathfrak{v}_s$ lies in the class of extreme points of $\mathcal{SH}^0_T(\delta,n,\eta,\mu)$. Similarly, the functions $\mathfrak{u}_s(z)$ can be verified as the extreme points of $\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)$.
Now, assume that a function $\mathfrak{f}$ of the form (1.1) is an extreme point of $\mathcal{TSH}^0_q(\delta,\mathfrak{m},\eta,\mu)$ and that $\mathfrak{f}$ does not match the form (2.12). Then, there exists $n\in\{2,3,\ldots\}$ such that

$$0<|u_n|<\frac{\mu-\eta}{\left[([n]_q-1)\,\delta+1\right]^{\mathfrak{m}}\left[\delta\,([n]_q-1)\,(\mu+1)+\mu-\eta\right]}$$

or

$$0<|v_n|<\frac{\mu-\eta}{\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]}.$$

If

$$0<|u_n|<\frac{\mu-\eta}{\left[\left([n]_q-1\right)\delta+1\right]^{\mathfrak{m}}\left[\delta\left([n]_q-1\right)(\mu+1)+\mu-\eta\right]},$$

then setting

$$\lambda=\frac{|u_n|\left[\left([n]_q-1\right)\delta+1\right]^{\mathfrak{m}}\left[\delta\left([n]_q-1\right)(\mu+1)+\mu-\eta\right]}{\mu-\eta}$$

and

$$\psi=\frac{\mathfrak{f}-\lambda\mathfrak{u}_n}{1-\lambda},$$

we obtain $0<\lambda<1$ and $\mathfrak{u}_s\neq\psi$. Hence, $\mathfrak{f}$ is not an extreme point of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$.
Similarly, if

$$0<|v_n|<\frac{\mu-\eta}{\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]},$$

then setting

$$\lambda=\frac{|v_n|\left[\left([n]_q+1\right)\delta-1\right]^{\mathfrak{m}}\left[\delta\left([n]_q+1\right)(\mu+1)+\eta-\mu\right]}{\mu-\eta}$$

and

$$\psi=\frac{\mathfrak{f}-\lambda\mathfrak{v}_n}{1-\lambda},$$

results in $0<\lambda<1$ and $\mathfrak{v}_n\neq\psi$.
Therefore, $\mathfrak{f}$ is not an element of the set of extreme points in $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, thereby completing the proof.

□

Consequently, according to Theorem 2.8, we obtain the following corollary.

Corollary 2.9. *Let $\mathfrak{f}$ be an element of $\mathcal{TSH}_q^0(\delta,\mathfrak{m},\eta,\mu)$, and let $|z|=r<1$. Then*

$$r-\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}r^2\leq|\mathfrak{f}(z)|\leq r+\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}r^2.$$

From Corollary 2.9, we can derive the following covering result.

Corollary 2.10. *If $\mathfrak{f}$ belongs to $\mathcal{TSH}_q^0(\delta,n,\eta,\mu)$, then $\mathbb{E}_r\subset\mathfrak{f}(\mathbb{E})$, where*

$$r=1-\frac{\mu-\eta}{(q\delta+1)^{\mathfrak{m}}\left[q\delta(\mu+1)+\eta-\mu\right]}.$$

3. Conclusion

In this paper, we introduced a new subclass of harmonic univalent functions by utilizing the generalized Al-Oboudi $q-$differential operator, which extends the classical Sălăgean operator within the framework of $q-$calculus. We derived several important results concerning the analytic and geometric properties of this subclass, including coefficient bounds, subordination conditions, extreme points, convolution characterizations, distortion theorems, and radii of starlikeness and convexity.

Furthermore, we demonstrate the compactness and convexity of the subclass $TSH_q^0(\delta,m,\eta,\mu)$, and established sharp bounds using extremal functions. The operator-theoretic approach adopted in this study not only generalizes many existing results in the literature but also provides a flexible framework for investigating broader families of harmonic mappings.

These findings contribute to the geometric function theory by enriching the structure of harmonic univalent function classes via $q-$calculus and highlight the potential of differential operators in unifying various known subclasses. Future work may focus on applying other classes of quantum differential operators or extending the current results to more general domains and functional settings.

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