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Modification of DTM for Solving Multi-Interval Boundary Value Problems

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Abstract. Although the well-known differential transform method (DTM) is one of the effective methods for solving single-interval boundary value problems (SIBVPs), this method cannot be directly applied to multi-interval boundary value transmission problems (MIBVTPs). In this study, we generalized the classical DTM so that it can be applied to solving not only SIBVPs but also MIBVTPs. To justify the effectiveness of the presented generalization of DTM, we solved two MIBVTPs for the three-interval differential equations and graphically compared the obtained approximate solutions with the corresponding exact solutions.

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1. Introduction

A semi-analytical method called the differential transform method (DTM) was first proposed by Pukhov [9] in 1986 to solve problems arising in the mathematical modeling of some physical processes. In the same year Zhou [11] also applied DTM to solve initial value problems for differential equations arising in the study of electrical circuits. Using DTM, one can obtain approximate, in some cases even analytical solutions to various ordinary and partial differential equations (PDEs) in the form of polynomials or Taylor series expansions. Although DTM is based on the Taylor expansion, it differs from the Taylor series expansion method in that DTM does not require extensive calculation of the highest derivatives of the given functions Chen and Ho [5] modified this method to solve certain types of PDEs and obtained series solutions for initial value problems for such PDEs. Ayaz [4] solved a system of differential equations by applying a new modification of the DTM. In [10] a new generalization of DTM (called random differential transform) is developed to solve random differential equations. In [12], Zou et al. for the first time have applied the differential transform technique to solve differential-difference equations. In [3], this technique is extended to solve difference equations of any type and order. In [1], Al-Amr presents a new version of DTM and calls it the reduced differential transform method (RDTM) for solving certain nonlinear differential equations of mathematical physics, as well as the generalized Drinfeld-Sokolov equation and Kaup-Kupershmidt equation. In [8], by combining the DTM and the Adomian Decomposition Method have been devoted a new method to solve inhomogeneous Dirichlet type boundary value problems (BVPs) for PDEs. In [2], Al-Rozbayani and Qasim have been applied the modified α-parameterized

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differential transform method and the genetic algorithm to solve the general Gardner equation. In [7], Mukhtarov et al. propose a new transformation method that includes an auxiliary parameter α, and call it the α-parameterized DTM (α-P DTM), to obtain approximate eigenvalues of the differential operators generated by the initial and/or boundary value problems(BVPs). In [6], the transformed equation is studied using α -PDTM.

In this work, we present a new generalization of the standard DTM to solve not only single-interval BVPs but also multi-interval BVPs with additional transmission conditions. To illustrate the proposed generalization of the DTM, we solved two multi-interval boundary value transmission problems (MIBVTPs).

2. The Definition and Basic Properties of the Transform Method

Let $p = p(t)$ be any analytic function defined on some neighborhood of the point $t = t_0$ and let

$$p(t) = \sum_{l=0}^{\infty} P_{t_0}(l)(t-t_0)^l$$

be a Taylor expansion of the function $p(t)$, where

$$P_{t_0}(l) := \frac{1}{l!}\left[\frac{d^l}{dt^l}p(t)\right]_{t=t_0}, \quad l = 0, 1, 2, ...$$

are the Taylor coefficients.

Definition 2.1 ([11]). The sequence $(P_{t_0}(0), P_{t_0}(1), ...)$ is called differential transform of the analytic function $p(t)$ at the point $t = t_0$ and is denoted by $K_{t_0}(p)$.

Definition 2.2 ([11]). The differential inverse transform of the sequence $K_{t_0}(p) = (P_{t_0}(0), P_{t_0}(1), ...)$ is said to be the series $\sum_{l=0}^{\infty} P_{t_0}(l)(t-t_0)^l$ and is denoted by $K_{t_0}^{-1}(K_{t_0}(p))$.

It is clear that $K_{t_0}^{-1}(K_{t_0}(p)) = p(t)$. Here, $p(t)$ is called the original function and the sequence $K_{t_0}(p)$ is called the K-transform of $p(t)$. From the definition of the K-transform it follows easily the following properties:

i. $K_{t_0}(a_1 + a_2) = K_{t_0}(a_1) + K_{t_0}(a_2)$
ii. $K_{t_0}(\gamma b) = \gamma K_{t_0}(b)$, for any $\gamma \in R$
iii. If $K_{t_0}(p) = (P_{t_0}(l))$, then $K_{t_0}(\frac{dp}{dt}) = ((l+1)P_{t_0}(l+1))$
iv. If $K_{t_0}(a) = (A_{t_0}(l))$, $K_{t_0}(b) = (B_{t_0}(l))$ and $K_{t_0}(ab) = (C_{t_0}(l))$, then $C_{t_0}(l) = (A_{t_0}(l)) * (B_{t_0}(l))$,

where $(A_{t_0}(l)) * (B_{t_0}(l))$ is denoted the convolution of the sequences $(A_{t_0}(l))$ and $(B_{t_0}(l))$.
In a real application, the differential inverse transform $K_{t_0}^{-1}(K_{t_0}(p))$ is defined by a finite sum

$$K_{t_0}^{-1}(K_{t_0}(p)) = \sum_{l=0}^{s} P_l(t_0)(t-t_0)^l$$

for sufficiently large s.

3. The Algoritm of the Generalized DTM for Solving Multi-Interval Transmission Problems

A differential equation defined on two or more disjoint intervals with comman endpoints (i.e. on the domain of definition which has the form $[t_0, t_1) \cup (t_1, t_2) \cup ... \cup (t_{n-1}, t_n]$ is called a multi-interval differential equation (MIDE). Boundary value problems consisting of a MIDE together with the boundary and transmission conditions are called multi-interval boundary value transmission problems (MIBVTPs). It is not clear how to apply the classical DTM to MIBVTPs. Based on classical DTM we developed a new modification of DTM, which we called as generalized differential transformation method (GDTM) to solve MIBVTPs. The algoritm of the GDTM consists of the following steps. First we apply the differential transformation on the first interval $[t_0, t_1)$. Then using the obtained approximate solution defined on the first interval $[t_0, t_1)$ and applying the tranmission conditions at the interaction point $t = t_1$) we construct the initial conditions for the second interval $[t_1, t_2)$ to find approximate solution at the second interval, and so on. As a results, we have an approximate solution defined on the whole multi-interval $[t_0, t_1) \cup (t_1, t_2) \cup ... \cup (t_{n-1}, t_n]$. To demostrate the applicability and effectiveness of the proposed modification of DTM, we will solve two examples for MIBVTPs using the GDTM.

Example 3.1. (**Application of the GDTM**) Let us consider the differential equation, which is defined on three separated intervals and given by

$$y''(t) - \frac{4(t-1)}{2t-3}y'(t) + \frac{4}{2t-3}y(t) = 0, \quad t \in [0,1) \cup (1, \frac{3}{2}) \cup (\frac{3}{2}, 2] \cup (2,3] \tag{3.1}$$

subject to the initial conditions, given by

$$y(0) = -3, \quad y'(0) = 3$$

and additional transmission conditions specified at the common end-points $t = 1$ and $t = 2$, given by

$$y(1+0) = y(1-0) + 5e^3, \quad y'(1+0) = y(1-0) + 2 + 10e^3$$

and

$$y(2+0) = \frac{4}{5}y(2-0) - \frac{8}{5}, \quad y'(2+0) = \frac{8}{5}y(2-0) - \frac{16}{5},$$

respectively. Let $K_0(l)$, $K_1(l)$, and $K_2(l)$, be the differential transform of the original function $y(t)$ at the points $t = 0$, $t = 1$ and $t = 2$, respectively. If we apply differential transform to the equation (3.1) in the interval $[0,1)$ with $t_0 = 0$, then we find that

$$K_0(y_1, l+2) = \frac{1}{3(l+1)(l+2)}[(2l+4)(l+1)K_0(y_1, l+1) + 4(1-l)K_0(y_1, l)] \tag{3.2}$$

Now, applying the differential inverse transform we have

$$y_1(t) = K_0(y_1, 0) + K_0(y_1, 1)t + K_0(y_1, 2)t^2 + K_0(y_1, 3)t^3 + ...,$$

where $y_1(t)$ denotes the restriction of $y(t)$ on the left interval $[0,1)$. The initial conditions $y(0) = -3$ and $y'(0) = 3$ yields

$$K_0(y_1, 0) = -3$$

and

$$K_0(y_1, 1) = 3,$$

respectively. By using the recurrence formula (3.2), we can calculate the other terms of the sequence $(K_0(y_1, l))$ as follows.

$$K_0(y_1, 2) = \frac{2}{3}(K_0(y_1, 1) - K_0(y_1, 0)), \quad K_0(y_1, 3) = \frac{2}{3}K_0(y_1, 2), \quad K_0(y_1, 4) = \frac{1}{3}K_0(y_1, 2),$$

$$K_0(y_1, 5) = \frac{2}{15}K_0(y_1, 2), \quad K_0(y_1, 6) = \frac{2}{45}K_0(y_1, 2), \quad K_0(y_1, 7) = \frac{4}{315}K_0(y_1, 2), ...$$

Thus, we have the following formula for the solution that is defined on the first interval $[0,1)$.

$$\begin{aligned} y_1(t) &= K_0(y_1, 0) + K_0(y_1, 1)t + K_0(y_1, 2)t^2 + K_0(y_1, 3)t^3 + ... \\ &= -3 + 3t. \end{aligned}$$

Secondly, let us get the solution defined on the second interval $(1,2)$. If the differential transform method is applied to the differential equation (3.1) in the around of the point $t_0 = 1$, we have

$$K_1(y_2, l+2) = \frac{1}{(l+1)(l+2)}[2l(l+1)K_1(y_2, l+1) + 4(1-l)K_1(y_2, l)], \tag{3.3}$$

where $y_2(t)$ denotes the restriction of $y(t)$ on the second interval $(1,2)$. By applying the differential inverse transform in the second interval $(1,2)$ we have

$$y_2(t) = K_1(y_2, 0) + K_1(y_2, 1)(t-1) + K_1(y_2, 2)(t-1)^2 + K_1(y_2, 3)(t-1)^3 + ...$$

Using transmission conditions $y(1+0) = y(1-0)5e^3$ and $y'(1+0) = y(1-0) + 2 + 10e^3$, we get

$$K_1(y_2, 0) = 5e^3$$

and

$$K_1(y_2, 1) = 2 + 10e^3,$$

respectively. By using the recurrence formula (3.3) we have

$$K_1(y_2,2) = 2K_1(y_2,0), \quad K_1(y_2,3) = \frac{2}{3}K_1(y_2,2), \quad K_1(y_2,4) = \frac{1}{3}K_0(y_2,2),$$

$$K_1(y_2,5) = \frac{2}{15}K_0(y_2,2), \quad K_1(y_2,6) = \frac{2}{45}K_1(y_2,2), \quad K_1(y_2,7) = \frac{4}{315}K_1(y_2,2), ...$$

Consequently,

$$\begin{aligned} y_2(t) &= K_1(y_2,0) + K_1(y_2,1)(t-1) + K_1(y_2,2)(t-1)^2 + K_1(y_2,3)(t-1)^3 + ... \\ &= K_1(y_2,0) + K_1(y_2,1)(t-1) + K_1(y_2,2)(\frac{1}{2}e^{2t-2} - t + \frac{1}{2}) \\ &= 5e^3 + (2+10e^3)(t-1) + 10e^3(\frac{1}{2}e^{2t-2} - t + \frac{1}{2}). \end{aligned}$$

Finally, let us get the solution in the third interval $(2,3]$. If the transform method is applied to the differential equation (3.1) in the interval $(2,3]$ at the point $t_0 = 2$, then we have the following recurrence formula

$$K_2(y_3, l+2) = \frac{-1}{(l+1)(l+2)}[(2l-4)(l+1)K_2(y_3, l+1) + 4(1-l)K_2(y_3, l)], \tag{3.4}$$

where $y_3(t)$ denotes the restriction of $y(t)$ on the interval $(2,3]$. Applying the differential inverse transform gives

$$y_3(t) = K_2(y_3,0) + K_2(y_3,1)(t-2) + K_2(y_3,2)(t-2)^2 + K_2(y_3,3)(t-2)^3 + ...$$

Using transmission conditions $y(2+0) = \frac{4}{5}y(2-0) - \frac{8}{5}$ and $y'(2+0) = \frac{8}{5}y(2-0) - \frac{16}{5}$, we get

$$K_2(y_3,0) = 4e^5$$

and

$$K_2(y_3,1) = 8e^5,$$

respectively. By applying the recurrence formula (3.4) we have

$$K_2(y_3,2) = 2K_2(y_3,1) - 2K_2(y_3,0), \quad K_2(y_3,3) = \frac{2}{3}K_2(y_3,2), \quad K_2(y_3,4) = \frac{1}{3}K_2(y_3,2),$$

$$K_2(y_3,5) = \frac{2}{15}K_2(y_3,2), \quad K_2(y_3,6) = \frac{2}{45}K_2(y_3,2), \quad K_2(y_3,7) = \frac{4}{315}K_2(y_3,2), ...$$

So,

$$y_3(t) = K_2(y_3,0) + K_2(y_3,1)(t-2) + K_2(y_3,2)(\frac{1}{2}e^{2t-4} - t + \frac{3}{2}).$$

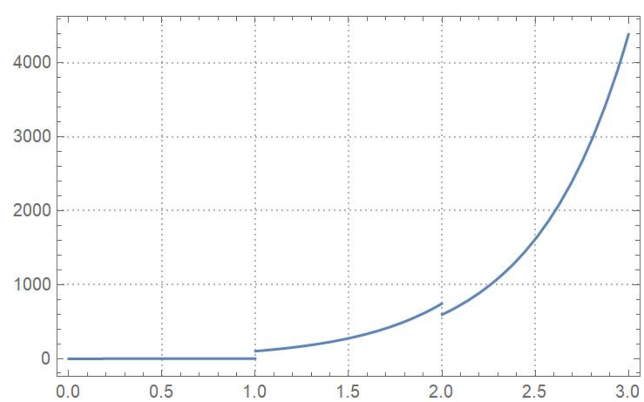

Figure 1. Graph of the approximate solution

We can show that the exact solution of this MIBVTP is

$$y(t) = \begin{cases} 3t-3 \quad for\ t \in [0,1) \\ 2t + 5e^{2t+1} - 2 \quad for\ t \in (1,2) \\ 4e^{2t+1} \quad for\ t \in (2,3]. \end{cases}$$

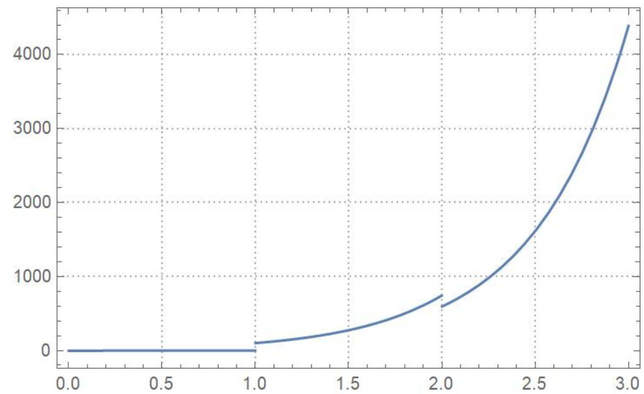

FIGURE 2. Graph of the exact solution

Example 3.2. (**Application of the DTM**) Let us consider the differential equation, which is defined on three separated intervals and given by

$$(t+1)^2 y''(t) - (t+1)y'(t) + y(t) = 0, \quad t \in [0,1) \cup (1,2) \cup (2,3] \tag{3.5}$$

subject to the initial conditions, given by

$$y(0) = 2, \quad y'(0) = 2$$

and additional transmission conditions specified at the common end-points $t = 1$ and $t = 2$, given by

$$y(1+0) = \frac{3}{2}y(1-0), \quad y'(1+0) = \frac{5}{4}y(1-0)$$

and

$$y(2+0) = \frac{9}{4}y(2-0), \quad y'(2+0) = \frac{3}{2}y(2-0),$$

respectively. Let $K_0(l)$, $K_1(l)$, and $K_2(l)$, be the differential transform of the original function $y(t)$ at the points $t = 0$, $t = 1$ and $t = 2$, respectively. If we apply differential transform to the equation (3.5) in the interval $[0, 1)$ with $t_0 = 0$, then we find that

$$K_0(y_1, l+2) = \frac{-1}{(l+2)(l+1)}[(l^2 - 2l + 1)K_0(y_1, l) + (2l-1)(l+1)K_0(y_1, l+1)]. \tag{3.6}$$

Now applying the differential inverse transform we have

$$y_1(t) = K_0(y_1, 0) + K_0(y_1, 1)t + K_0(y_1, 2)t^2 + K_0(y_1, 3)t^3 + ...,$$

where $y_1(t)$ denotes the restriction of $y(t)$ on the left interval $[0, 1)$. The initial conditions $y(0) = 2$ and $y'(0) = 2$ yields

$$K_0(y_1, 0) = 2$$

and

$$K_0(y_1, 1) = 2,$$

respectively. By using the recurrence formula (3.6), we can calculate the other terms of the sequence $(K_0(y_1, l))$ as follows.

$$K_0(y_1, 2) = 0, \quad K_0(y_1, 3) = 0, \quad K_0(y_1, 4) = 0, ...$$

Thus, we have the following formula for the solution that is defined on the first interval $[0, 1)$.

$$\begin{aligned} y_1(t) &= K_0(y_1, 0) + K_0(y_1, 1)t + K_0(y_1, 2)t^2 + K_0(y_1, 3)t^3 + ... \\ &= 2t. \end{aligned}$$

Secondly, let us get the solution defined on the second interval $(1, 2)$. If the differential transform method is applied to the differential equation (3.5) in the around of the point $t_0 = 1$, then we have

$$K_1(y_2, l+2) = \frac{-(4l-2)}{5(l+2)}K_1(y_2, l+1) - \frac{-(l-1)^2}{5(l+2)(l+1)}K_1(y_2, l), \tag{3.7}$$

where $y_2(t)$ denotes the restriction of $y(t)$ on the second interval $(1, 2)$. By applying the differential inverse transform in the second interval $(1, 2)$ we have

$$y_2(t) = K_1(y_2, 0) + K_1(y_2, 1)(t-1) + K_1(y_2, 2)(t-1)^2 + K_1(y_2, 3)(t-1)^3 + ...$$

Using transmission conditions $y(1+0) = \frac{3}{2}y(1-0)$ and $y'(1+0) = \frac{5}{4}y(1-0)$, we get

$$K_1(y_2, 0) = 6$$

and

$$K_1(y_2, 1) = 5,$$

respectively. By applying the recurrence formula (3.7) we have

$$K_1(y_2, 2) = \frac{-8}{5}, \quad K_1(y_2, 3) = \frac{16}{75}, \quad K_1(y_2, 4) = \frac{-14}{375}, ...$$

Consequently,

$$\begin{aligned} y_2(t) &= K_1(y_2, 0) + K_1(y_2, 1)(t-1) + K_1(y_2, 2)(t-1)^2 + K_1(y_2, 3)(t-1)^3 + ... \\ &= t^2 + t. \end{aligned}$$

Finally, let us get the solution in the third interval $(2, 3]$. If the transform method is applied to the differential equation, (3.5) in the interval $(2, 3]$ at the point $t_0 = 2$, then we have the following recurrence formula

$$K_2(y_3, l+2) = \frac{-1}{13(l+1)(l+2)}[(6l-3)K_2(y_3, l+1) + (l^2 - 2l + 1)K_2(y_3, l)], \tag{3.8}$$

where $y_3(t)$ denotes the restriction of $y(t)$ on the interval $(2, 3]$. Applying the differential inverse transform gives

$$y_3(t) = K_2(y_3, 0) + K_2(y_3, 1)(t-2) + K_2(y_3, 2)(t-2)^2 + K_2(y_3, 3)(t-2)^3 + ...$$

Using transmission conditions $y(2+0) = \frac{9}{4}y(2-0)$ and $y'(2+0) = \frac{3}{2}y(2-0)$, we get

$$K_2(y_3, 0) = \frac{21546}{1000}$$

and

$$K_2(y_3, 1) = \frac{14364}{1000},$$

respectively. By applying the recurrence formula (3.8) we have

$$K_2(y_3, 2) = \frac{21546}{26000}, \quad K_2(y_3, 3) = \frac{-21546}{676000}, \quad K_2(y_3, 4) = \frac{61047}{17576000}, ...$$

So,

$$\begin{aligned} y_3(t) &= K_2(y_3, 0) + K_2(y_3, 1)(t-2) + ... \\ &= 3t^2. \end{aligned}$$

We can show that the exact solution of this MIBVTP is

$$y(t) = \begin{cases} 2 + 2t & for\ t \in [0, 1) \\ t^2 + 3t + 2 & for\ t \in (1, 2) \\ 3(t+1)^2 & for\ t \in (2, 3]. \end{cases}$$

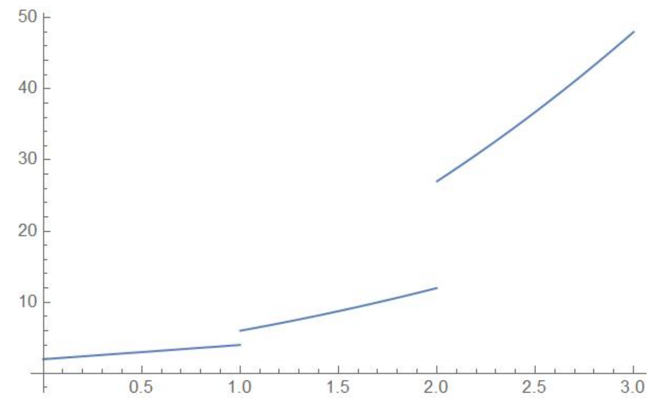

Figure 3. Graph of the approximate solution

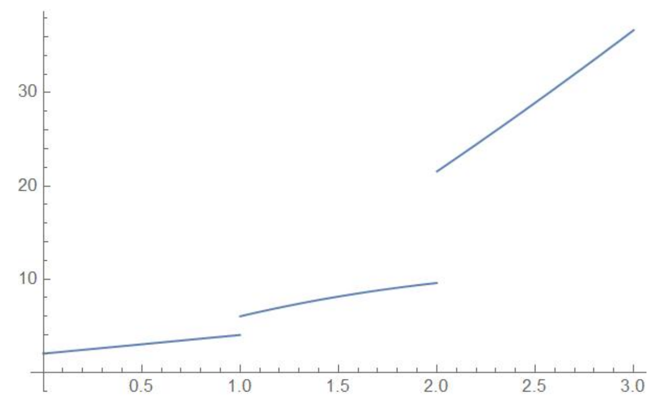

Figure 4. Graph of the exact solution

Conclusion

The differential transformation method (DTM) is an efficient semi-analytical method for solving various kinds of initial and/or boundary value problems for various kinds of differential equations. We know that this method can not be directly applied to MIBVTPs. In this paper, we proposed a new generalization of the classical DTM such that it can be applied not only to single-interval initial and/or boundary value problems but also to multi-interval problems with additional transmission conditions at the points of interaction. The algoritm of the proposed generalization of DTM consists of the following steps. First, we apply the differential transformation to the differential equation on the first subinterval of the finite number of non-intersecting intervals under consideration. Using the approximate solution defined on the first interval and applying the transmission conditions, we construct the initial conditions for the second interval to find the approximate solution on the second interval, and so on. As a result, we have an approximate solution defined on the whole multi-interval. To justify the presented generalization of DTM, we solved two multi-interval problems with additional transmission conditions. Exact solutions are also presented for these multi-interval problems. Then we compared the obtained approximate solutions with the exact solutions graphically. The obtained results show that the proposed generalization of DTM is an efficient and robust method for solving multi-interval problems. The accuracy of the obtained approximate solutions can be increased by calculating more terms of the recurrence formula.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

Each author contributed equally to the paper.

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