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Research Article

A bialgebra theory for compatible Hom-Lie algebras

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Abstract

In this paper, we introduce the notions of matched pairs and Manin triple for compatible Hom-Lie algebras. Then, we give a bialgebra theory of compatible Hom-Lie algebras with emphasis on its compatibility with Manin triple of compatible Hom-Lie algebras associated to a nondegenerate symmetric bilinear form. Moreover, we study coboundary compatible Hom-Lie bialgebras. Finally, we investigate some properties of a representation of a Hom-Nijenhuis Hom-Lie algebra and introduce the notion of a Hom-Nijenhuis Hom-Lie coalgebra. Furthermore, a Hom-Nijenhuis Hom-Lie bialgebra can be established by a Hom-Nijenhuis Hom-Lie algebra and a Hom-Nijenhuis Hom-Lie coalgebra satisfying some compatible conditions.

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1. Introduction

The notion of a Hom-Lie algebra was introduced in [16] in the context of deformations of the Witt algebra and the Virasoro algebra. In a Hom-Lie algebra $\mathfrak{g}$, the Jacobi identity defining a Lie algebra is twisted by a linear map α, called the Hom-Jacobi identity, i.e.

$$[\alpha(x), [y, z]_{\mathfrak{g}}]_{\mathfrak{g}} + [\alpha(y), [z, x]_{\mathfrak{g}}]_{\mathfrak{g}} + [\alpha(z), [x, y]_{\mathfrak{g}}]_{\mathfrak{g}} = 0, \tag{1.1}$$

which recovers the Jacobi identity in the special case when α is the identity map. With this generalization of the Lie algebra, some q-deformations of the Witt and the Virasoro algebras have the structure of a Hom-Lie algebra [16]. Due to their close relationship with discrete and deformed vector fields and differential calculus [16, 19, 20], Hom-Lie algebras are studied in the following contexts: representation and cohomology theory [1, 3, 31], deformation theory [26], bialgebra theory [2, 4, 33]. See [8, 18, 21, 25, 32] for other interesting Hom-algebra structures.

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In some contexts, two algebraic structures of the same kind are said to be compatible if their sum supports the same type of algebraic structure. For instance, the notion of compatibility of two Poisson structures on a manifold was first introduced in the mathematical study of biHamiltonian mechanics [17,22]. Using the correspondence between Lie algebra structures on a vector space $\mathfrak{g}$ and linear Poisson structures on $\mathfrak{g}^*$, one obtains a notion of compatible Lie algebras [17]. See [15,28] for more studies on compatible Lie algebras. Some other compatible structures include compatible associative algebras [27], compatible Lie bialgebras [30], compatible Lie algebroids and Lie bialgebroids [11]. See also [13,29] for the operadic study of compatible algebraic structures. Recently, a cohomology theory for compatible Leibniz algebras and (Hom-)associative algebras introduced in [9,10,24].

In this paper, we consider a bialgebra theory for compatible Hom-Lie algebras introduced by Das in [12]. Explicitly, we consider an analogue of Manin triple of Hom-Lie algebras which is equivalent to a Hom-Lie bialgebra, namely, a Manin triple of compatible HomLie algebras associated to a non-degenerate invariant symmetric bilinear form. Compatible Hom-Lie bialgebras have many similar properties to Hom-Lie bialgebras. For example, there are the so-called coboundary compatible Hom-Lie bialgebras which lead to a construction from the classical Hom-Yang-Baxter equation in compatible Hom-Lie algebras as a combination of two classical Hom-Yang-Baxter equations in Hom-Lie algebras.

The paper is organized as follows. In Section 2, we give some preliminary results on Hom-Lie algebras, compatible Hom-Lie algebras and Hom-Lie bialgebras. In Section 3, we study the matched pairs of compatible Hom-Lie algebras and give the definition of Manin triples of compatible Hom-Lie algebras which are interpreted in terms of matched pairs. Also, we give a notion of compatible Hom-Lie bialgebra which is equivalent to a Manin triple of compatible Hom-Lie algebras. In Section 4, the coboundary compatible Hom-Lie bialgebras are considered, which lead to a construction from the classical Hom-Yang-Baxter equation in compatible Hom-Lie algebras as a combination of two classical Hom-Yang-Baxter equations in Hom-Lie algebras. In the last section, we present the dual representation of a representation of a Hom-Nijenhuis Hom-Lie algebra in order to define Hom-Nijenhuis Hom-Lie bialgebra. This tool will be used to construct a compatible Hom-Lie bialgebra.

Throughout this paper, all vector spaces are finite-dimensional and over a field $\mathbb{K}$ of characteristic zero. In the following, we give some notations which will be used in the sequel.

Notations. Let V and W be two vector spaces:

(1) Denote by $\tau : V \otimes W \to W \otimes V$ the switch isomorphism, $\tau(v \otimes w) = w \otimes v$.
(2) For a linear map $\Delta : V \to V^{\otimes 2}$, we use Sweedler's notation $\Delta(x) = \sum_{(x)} x_1 \otimes x_2$ for $x \in V$. We will often omit the summation sign $\sum_{(x)}$ to simplify the notations.
(3) Denote by $V^* = Hom(V, \mathbb{K})$ the linear dual of V. For $\varphi \in V^*$ and $u \in V$, we write $\langle \varphi, u\rangle := \varphi(u) \in \mathbb{K}$.
(4) For a linear map $\phi : V \to W$, we define the map $\phi^* : W^* \to V^*$ by

$$\langle \phi^*(\xi), v\rangle = \langle \xi, \phi(v)\rangle,\ \forall v \in V, \xi \in W^*. \tag{1.2}$$

(5) For an element x in a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ and $n \geq 2$, define the adjoint map $ad_x : \wedge^n\mathfrak{g} \to \wedge^n\mathfrak{g}$ by

$$ad_x(y_1 \wedge \cdots \wedge y_n) = \sum_{i=1}^{n} \alpha(y_1) \wedge \cdots \wedge \alpha(y_{i-1}) \wedge [x, y_i]_{\mathfrak{g}} \wedge \alpha(y_{i+1}) \wedge \cdots \wedge \alpha(y_n) \tag{1.3}$$

for all $y_1, \ldots, y_n \in \mathfrak{g}$.

2. Preliminaries and basic results

In this preliminary section, we recall some basics on Hom-Lie algebras, compatible Hom-Lie algebras and Hom-Lie bialgebras (see [1, 6, 12, 26] for more details).

A Hom-Lie algebra is a vector space $\mathfrak{g}$ together with a bilinear skew-symmetric bracket $[\cdot,\cdot]_\mathfrak{g} : \mathfrak{g}\times\mathfrak{g} \to \mathfrak{g}$ and a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$ satisfying the identity (1.1). It is called regular if α is invertible and multiplicative if $\alpha \circ [\cdot,\cdot]_\mathfrak{g} = [\cdot,\cdot]_\mathfrak{g} \circ (\alpha \otimes \alpha)$. Throughout the paper, all Hom-Lie algebras are multiplicative and regular. It follows from the above discussion that Hom-Lie algebras are a twisted version of Lie algebras. More specifically, a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ with $\alpha = id$ is nothing but a Lie algebra.

Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}, \alpha')$ be two Hom-Lie algebras. A linear map $f : \mathfrak{g} \to \mathfrak{g}'$ is said to be a Hom-Lie algebra morphism if $f \circ \alpha = \alpha' \circ f$ and

$$f([x,y]_\mathfrak{g}) = [f(x), f(y)]_{\mathfrak{g}'}, \quad \forall x, y \in \mathfrak{g}. \tag{2.1}$$

Example 2.1. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ be a Lie algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. Then the triple $(\mathfrak{g}, \alpha \circ [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is a Hom-Lie algebra.

Example 2.2. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ be a Hom-Lie algebra. Then for any $n \geq 0$, the triple $(\mathfrak{g}, [\cdot,\cdot]^{(n)}_\mathfrak{g} = \alpha^n \circ [\cdot,\cdot]_\mathfrak{g}, \alpha^{n+1})$ is a Hom-Lie algebra, called the n-th derived Hom-Lie algebra.

Definition 2.1. A Hom-Lie coalgebra is a tuple $(\mathfrak{g}, \Delta, \alpha)$ consisting of a vector space $\mathfrak{g}$, a comultiplication $\Delta : \mathfrak{g} \to \mathfrak{g} \otimes \mathfrak{g}$ satisfying

$$\tau\Delta = -\Delta, \qquad \text{Co-skew-symmetric} \tag{2.2}$$

$$(id + \sigma + \sigma^2)(\alpha \otimes \Delta)\Delta = 0, \qquad \text{Co-Hom-Jacobi identity} \tag{2.3}$$

where $\sigma(x \otimes y \otimes z) = y \otimes z \otimes x$, for any $x, y, z \in \mathfrak{g}$.

Definition 2.2. A representation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ is a triple (V, ρ, β) consisting of a vector space V , an element $\beta \in End(V)$ and a linear map $\rho : \mathfrak{g} \to End(V)$, such that for any $x, y \in \mathfrak{g}$, the following equalities hold:

$$\rho(\alpha(x)) \circ \beta = \beta \circ \rho(x), \tag{2.4}$$

$$\rho([x,y]_\mathfrak{g}) \circ \beta = \rho(\alpha(x)) \circ \rho(y) - \rho(\alpha(y)) \circ \rho(x). \tag{2.5}$$

It is straightforward to see that $(\mathfrak{g}, ad, \alpha)$, where $ad(x)(y) := [x,y]_\mathfrak{g}$ for all $x, y \in \mathfrak{g}$, is a representation of $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$, called the adjoint representation. If $\alpha = Id_\mathfrak{g}$ and $\beta = Id_V$, then (V, ρ) is a representation of the classical Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$.

As the classical case, the triple (V, ρ, β) is a representation of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ if and only if the direct sum $\mathfrak{g} \oplus V$ is turned into a Hom-Lie algebra $(\mathfrak{g} \ltimes_\rho V := (\mathfrak{g} \oplus V, [\cdot,\cdot]_{\mathfrak{g}\oplus V}, \alpha \oplus \beta)$) by defining

$$[x+u, y+v]_{\mathfrak{g}\oplus V} = [x,y]_\mathfrak{g} + \rho(x)v - \rho(y)u, \tag{2.6}$$

$$(\alpha \oplus \beta)(x+u) = \alpha(x) + \beta(u). \tag{2.7}$$

for all $x, y \in \mathfrak{g}$ and $u, v \in V$.

Lemma 2.3. Let (V, ρ) be a representation of the Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g})$ and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra morphism. If there exits $\beta \in End(V)$ such that

$$\beta \circ \rho(x) = \rho(\alpha(x)) \circ \beta, \qquad \text{for any } x \in \mathfrak{g}. \tag{2.8}$$

Then $(V, \rho_\beta = \beta \circ \rho, \beta)$ is a representation of the Hom-Lie algebra $(\mathfrak{g}, \alpha \circ [\cdot,\cdot]_\mathfrak{g}, \alpha)$.

Definition 2.3. Let $(\mathfrak{g}, [\cdot,\cdot]_\mathfrak{g}, \alpha)$ and $(\mathfrak{h}, [\cdot,\cdot]_\mathfrak{h}, \alpha')$ be two Hom-Lie algebras, $\rho : \mathfrak{g} \to End(\mathfrak{h})$ and $\mu : \mathfrak{h} \to End(\mathfrak{g})$ be two linear maps such that $(\mathfrak{h}, \rho, \alpha')$ and $(\mathfrak{g}, \mu, \alpha)$ are representations of $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha)$ and $(\mathfrak{h},[\cdot,\cdot]_\mathfrak{h},\alpha')$ respectively. The tuple $(\mathfrak{g},\mathfrak{h};\rho,\mu)$ is called a matched pair of Hom-Lie algebras if

$$\rho(\alpha(x))[a,b]_\mathfrak{h} = [\rho(x)(a),\alpha'(b)]_\mathfrak{h}+[\alpha'(a),\rho(x)(b)]_\mathfrak{h}+\rho(\mu(b)(x))(\alpha'(a))-\rho(\mu(a)(x))(\alpha'(b)), \tag{2.9}$$

$$\mu(\alpha'(a))[x,y]_\mathfrak{g} = [\mu(a)(x),\alpha(y)]_\mathfrak{g}+[\alpha(x),\mu(a)(y)]_\mathfrak{g}+\mu(\rho(y)(a))(\alpha(x))-\mu(\rho(x)(a))(\alpha(y)) \tag{2.10}$$

for any $x,y\in\mathfrak{g}$ and $a,b\in\mathfrak{h}$.

Remark 2.1. In fact, $(\mathfrak{g},\mathfrak{h};\rho,\mu)$ is a matched pair of Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha)$ and $(\mathfrak{h},[\cdot,\cdot]_\mathfrak{h},\alpha')$ if and only if there exists a Hom-Lie algebra structure on the vector space $\mathfrak{g}\oplus\mathfrak{h}$ given by

$$[x+a,y+b]_{\mathfrak{g}\oplus\mathfrak{h}}=[x,y]_\mathfrak{g}+[a,b]_\mathfrak{h}+\mu(a)y-\mu(b)x+\rho(x)b-\rho(y)a,$$
$$\alpha_{\mathfrak{g}\oplus\mathfrak{h}}(x+a)=\alpha(x)+\alpha'(a)$$

for any $x,y\in\mathfrak{g}$ and $a,b\in\mathfrak{h}$.

Remark 2.2. A matched pair $(\mathfrak{g},\mathfrak{h};\rho,\mu)$ of Hom-Lie algebras with $\alpha=Id_\mathfrak{g}$ and $\alpha'=Id_\mathfrak{h}$ is exactly a matched pair of Lie algebras, as defined in [23].

Definition 2.4. Let $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha)$ be a Hom-Lie algebra. A symmetric bilinear form B on $\mathfrak{g}$ is called invariant if for all $x,y,z\in\mathfrak{g}$, we have

$$B([x,y]_\mathfrak{g},\alpha(z))=-B(\alpha(y),[x,z]_\mathfrak{g}), \tag{2.11}$$
$$B(\alpha(x),\alpha(y))=B(x,y). \tag{2.12}$$

The quadruple $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha,B)$ is called a quadratic Hom-Lie algebra.

Definition 2.5. A Manin triple for Hom-Lie algebras is a triple of Hom-Lie algebras $(\mathfrak{g}\oplus\mathfrak{g}';\mathfrak{g},\mathfrak{g}')$ in which $(\mathfrak{g}\oplus\mathfrak{g}',[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'},\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}=\alpha_\mathfrak{g}\oplus\alpha_{\mathfrak{g}'},B)$ is a quadratic Hom-Lie algebra, $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{g}',[\cdot,\cdot]_{\mathfrak{g}'},\alpha_{\mathfrak{g}'})$ are isotropic Hom-Lie sub-algebras of $\mathfrak{g}\oplus\mathfrak{g}'$ with respect to the symmetric bilinear form B, i.e. $B_{|\mathfrak{g}\times\mathfrak{g}}=0,\ B_{|\mathfrak{g}'\times\mathfrak{g}'}=0$.

Now, we recall the definition of compatible Hom-Lie algebras.

Definition 2.6. Two Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},\alpha)$ and $(\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ are said to be compatible if for all $k_1,k_2\in\mathbb{K}$, the triple $(\mathfrak{g},k_1[\cdot,\cdot]^1_\mathfrak{g}+k_2[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ is a Hom-Lie algebra.

The condition in the above definition is equivalent to the following

$$[[x,y]^1_\mathfrak{g},\alpha(z)]^2_\mathfrak{g}+[[y,z]^1_\mathfrak{g},\alpha(x)]^2_\mathfrak{g}+[[z,x]^1_\mathfrak{g},\alpha(y)]^2_\mathfrak{g} + [[x,y]^2_\mathfrak{g},\alpha(z)]^1_\mathfrak{g}+[[y,z]^2_\mathfrak{g},\alpha(x)]^1_\mathfrak{g}+[[z,x]^2_\mathfrak{g},\alpha(y)]^1_\mathfrak{g}=0, \tag{2.13}$$

for any $x,y,z\in\mathfrak{g}$.

Definition 2.7. A compatible Hom-Lie algebra is a quadruple $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ in which $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},\alpha)$ and $(\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ are both Hom-Lie algebras and are compatible.

Example 2.4. Consider the 3-dimensional Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},\alpha)$ and $(\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha)$, where $\mathfrak{g}=<e_1,e_2,e_3>$ and structure maps for the two brackets are given by

$$[e_1,e_2]^1_\mathfrak{g}=e_3,\qquad [e_2,e_3]^1_\mathfrak{g}=e_1,$$
$$[e_1,e_3]^2_\mathfrak{g}=e_2,$$
$$\alpha(e_1)=e_1,\quad \alpha(e_2)=2e_2,\quad \alpha(e_3)=2e_3.$$

It is easy to see that the quadruple $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ is a compatible Hom-Lie algebra.

Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ and $(\mathfrak{g}', [\cdot,\cdot]^1_{\mathfrak{g}'}, [\cdot,\cdot]^2_{\mathfrak{g}'}, \alpha')$ be two compatible Hom-Lie algebras. A morphism between them is a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ which is a Hom-Lie algebra morphism from $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, \alpha)$ to $(\mathfrak{g}', [\cdot,\cdot]^1_{\mathfrak{g}'}, \alpha')$ and a Hom-Lie algebra morphism from $(\mathfrak{g}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ to $(\mathfrak{g}', [\cdot,\cdot]^2_{\mathfrak{g}'}, \alpha')$.

Example 2.5. Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}})$ be a compatible Lie algebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a compatible Hom-Lie algebra morphism, i.e., α is a Lie algebra morphism for both the Lie algebras $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}})$ and $(\mathfrak{g}, [\cdot,\cdot]^2_{\mathfrak{g}})$. Then the quadruple $(\mathfrak{g}, \alpha \circ [\cdot,\cdot]^1_{\mathfrak{g}}, \alpha \circ [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ is a compatible Hom-Lie algebra.

Example 2.6. Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ be a compatible Hom-Lie algebra. Then for each $n \geq 0$, the quadruple $(\mathfrak{g}, [\cdot,\cdot]^{(n)}_1 = \alpha^n \circ [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^{(n)}_2 = \alpha^n \circ [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha^{n+1})$ is a compatible Hom-Lie algebra. This is the n-th derived compatible Hom-Lie algebra.

Now, we introduce the notion of compatible Hom-Lie coalgebras.

Definition 2.8. Two Hom-Lie coalgebras $(\mathfrak{g}, \Delta_1, \alpha)$ and $(\mathfrak{g}, \Delta_2, \alpha)$ are said to be compatible if for all $k_1, k_2 \in \mathbb{K}$, the triple $(\mathfrak{g}, k_1\Delta_1 + k_2\Delta_2, \alpha)$ is a Hom-Lie coalgebra.

The condition in the above definition is equivalent to the following

$$(id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta_1) \circ \Delta_2 + (id + \sigma + \sigma^2) \circ (\alpha \otimes \Delta_2) \circ \Delta_1 = 0. \tag{2.14}$$

Example 2.7. Consider the 3-dimensional Hom-Lie coalgebras $(\mathfrak{g}, \Delta_1, \alpha)$ and $(\mathfrak{g}, \Delta_2, \alpha)$, where $\mathfrak{g} =< e_1, e_2, e_3 >$ and structure maps for the two cobrackets are given by

$$\begin{aligned}
&\Delta_1(e_2) = 4(e_1 \otimes e_3 - e_3 \otimes e_1),\\
&\Delta_2(e_3) = 4(e_1 \otimes e_2 - e_2 \otimes e_1), \qquad \Delta_2(e_1) = e_2 \otimes e_3 - e_3 \otimes e_2,\\
&\alpha(e_1) = e_1, \quad \alpha(e_2) = \frac{1}{2}e_2, \quad \alpha(e_3) = \frac{1}{2}e_3.
\end{aligned}$$

Definition 2.9. Let $(\mathfrak{g}, \Delta_1, \Delta_2, \alpha)$ and $(\mathfrak{g}', \Delta'_1, \Delta'_2, \alpha')$ be two compatible Hom-Lie coalgebras. A morphism between them is a linear map $f : \mathfrak{g} \to \mathfrak{g}'$ which is a Hom-Lie coalgebra morphism from $(\mathfrak{g}, \Delta_1, \alpha)$ to $(\mathfrak{g}', \Delta'_1, \alpha')$, and a Hom-Lie coalgebra morphism from $(\mathfrak{g}, \Delta_2, \alpha)$ to $(\mathfrak{g}', \Delta'_2, \alpha')$.

Example 2.8. Let $(\mathfrak{g}, \Delta^1, \Delta^2)$ be a compatible Lie coalgebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a compatible Lie coalgebra morphism. Then the triple $(\mathfrak{g}, \Delta^1_\alpha = \Delta^1 \circ \alpha, \Delta^2_\alpha = \Delta^2 \circ \alpha, \alpha)$ is a co-multiplicative compatible Hom-Lie coalgebra.

Example 2.9. Let $(\mathfrak{g}, \Delta^1, \Delta^2, \alpha)$ be a co-multiplicative compatible Hom-Lie coalgebra. Then $(\mathfrak{g}, \Delta^1_{\alpha^n} = \Delta^1 \circ \alpha^n, \Delta^2_{\alpha^n} = \Delta^2 \circ \alpha^n, \alpha^{n+1})$ is a co-multiplicative compatible Hom-Lie coalgebra for each integer $n \geq 0$.

Remark 2.3. If $(\mathfrak{g}, \Delta_1, \Delta_2, \alpha)$ is a compatible Hom-Lie coalgebra, then $(\mathfrak{g}^*, [\cdot,\cdot]^1_{\mathfrak{g}^*}, [\cdot,\cdot]^2_{\mathfrak{g}^*}, \alpha^*)$ is a compatible Hom-Lie algebra. Here $[\cdot,\cdot]^1_{\mathfrak{g}^*}$, $[\cdot,\cdot]^2_{\mathfrak{g}^*}$ and α^* in $\mathfrak{g}^*$ are dual to Δ_1, Δ_2 and α, respectively, in $\mathfrak{g}$. Conversely, if $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ is a compatible Hom-Lie algebra, then $(\mathfrak{g}^*, \Delta^*_1, \Delta^*_2, \alpha^*)$ is a compatible Hom-Lie coalgebra, where Δ^*_1, Δ^*_2 and α^* in $\mathfrak{g}^*$ are dual to $[\cdot,\cdot]^1_{\mathfrak{g}}$, $[\cdot,\cdot]^2_{\mathfrak{g}}$ and α, respectively, in $\mathfrak{g}$.

In the following, we recall the definition of a representation of a compatible Hom-Lie algebra [12].

Definition 2.10. A representation of a compatible Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$ on a vector space V is a triple of linear maps $\rho, \mu : \mathfrak{g} \to End(V)$ and $\beta : V \to V$ such that for any $k_1, k_2 \in \mathbb{K}$, $(V, k_1\rho + k_2\mu, \beta)$ is a representation of the Hom-Lie algebra $(\mathfrak{g}, k_1[\cdot,\cdot]^1_{\mathfrak{g}} + k_2[\cdot,\cdot]^2_{\mathfrak{g}}, \alpha)$. We denote it by (V, ρ, μ, β).

Example 2.10. Let $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ be a compatible Hom-Lie algebra. Define $ad^1_{\mathfrak{g}},ad^2_{\mathfrak{g}}:\mathfrak{g}\to End(\mathfrak{g})$ by $ad^1_{\mathfrak{g}}(x)y=[x,y]^1_{\mathfrak{g}}$ and $ad^2_{\mathfrak{g}}(x)y=[x,y]^2_{\mathfrak{g}}$, for all $x,y\in\mathfrak{g}$. Then $(\mathfrak{g},ad^1_{\mathfrak{g}},ad^2_{\mathfrak{g}},\alpha)$ is a representation of $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$, called the adjoint representation.

Proposition 2.1. *Let $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ be a compatible Hom-Lie algebra. Let V be a vector space and $\rho,\mu:\mathfrak{g}\to End(V)$, $\beta:V\to V$ be a triple of linear maps. Then the following conditions are equivalent:*

(1) *(V,ρ,μ,β) is a representation of $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ on the vector space V.*

(2) *(V,ρ,β) is a representation of the Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},\alpha)$, (V,μ,β) is a representation of the Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ and for any $x,y\in\mathfrak{g}$ we have the following compatibility condition*

$$\rho([x,y]^2_{\mathfrak{g}})\circ\beta+\mu([x,y]^1_{\mathfrak{g}})\circ\beta=\rho(\alpha(x))\mu(y)-\mu(\alpha(y))\rho(x)+\mu(\alpha(x))\rho(y)-\rho(\alpha(y))\mu(x). \quad (2.15)$$

(3) *Define two skew-symmetric bilinear operations $[\cdot,\cdot]_1,[\cdot,\cdot]_2:\wedge^2(\mathfrak{g}\oplus V)\to(\mathfrak{g}\oplus V)$ by*

$$[x+u,y+v]_1=[x,y]^1_{\mathfrak{g}}+\rho(x)v-\rho(y)u,$$
$$[x+u,y+v]_2=[x,y]^2_{\mathfrak{g}}+\mu(x)v-\mu(y)u,$$

for any $x,y\in\mathfrak{g}$ and $u,v\in V$. Then $(\mathfrak{g}\oplus V,[\cdot,\cdot]_1,[\cdot,\cdot]_2,\alpha\oplus\beta)$ is a compatible Hom-Lie algebra.

Remark 2.2. We call the above $(\mathfrak{g}\oplus V,[\cdot,\cdot]_1,[\cdot,\cdot]_2,\alpha\oplus\beta)$ the semidirect product of a compatible Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ and a representation (ρ,μ,β,V), and denote it by $\mathfrak{g}\ltimes_{\rho,\mu}V$, or simply $\mathfrak{g}\ltimes V$.

Remark 2.3. A representation (V,ρ,μ,β) of compatible Hom-Lie algebras with $\alpha=Id_{\mathfrak{g}}$ and $\beta=Id_V$ is exactly a representation of compatible Lie algebras, as defined by Bai and Wu [30].

Let (V,β,ρ) be a representation of a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$. In the sequel, we always assume that β is invertible. Define $\rho^*:\mathfrak{g}\to gl(V^*)$ as usual by

$$\langle\rho^*(x)(\xi),v\rangle=-\langle\xi,\rho(x)(v)\rangle,\quad\forall x\in\mathfrak{g},v\in V,\xi\in V^*. \quad (2.16)$$

To ensure that ρ^* is a representation of $\mathfrak{g}$ on V^* with respect to β^*, we necessitate strong conditions. In order to get rid of them, we define a new map $\rho^\star:\mathfrak{g}\to gl(V^*)$ by

$$\rho^\star(x)(\xi):=\rho^*(\alpha(x))((\beta^{-2})^*(\xi)),\quad\forall x\in\mathfrak{g},\xi\in V^*. \quad (2.17)$$

That is

$$\langle\rho^\star(x)(\xi),v\rangle=-\langle\xi,\rho(\alpha^{-1}(x))(\beta^{-2}(v))\rangle. \quad (2.18)$$

Lemma 2.4. *Let (V,ρ,β) be a representation of a Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$. Then $\rho^\star:\mathfrak{g}\to gl(V^*)$ defined above by (2.17) is a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ on V^* with respect to $(\beta^{-1})^*$.*

Corollary 2.5. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ be a Hom-Lie algebra. Then $ad^\star:\mathfrak{g}\to gl(\mathfrak{g}^*)$ defined by*

$$ad^\star(x)(\xi)=ad^*(\alpha(x))(\alpha^{-2})^*(\xi),\quad\forall x\in\mathfrak{g},\xi\in\mathfrak{g}^* \quad (2.19)$$

is a representation of the Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ on $\mathfrak{g}^*$ with respect to $(\alpha^{-1})^*$, which is called the coadjoint representation.

Proposition 2.6. *Let $(\mathfrak{g},\mathfrak{h};\rho,\mu)$ be a matched pair of Lie algebras. Let $\alpha_{\mathfrak{g}}:\mathfrak{g}\to\mathfrak{g}$ and $\alpha_{\mathfrak{h}}:\mathfrak{h}\to\mathfrak{h}$ be two Lie algebra morphisms and set $\rho_{\alpha_{\mathfrak{h}}}=\alpha_{\mathfrak{h}}\circ\rho:\mathfrak{g}\to End(\mathfrak{h})$ and $\mu_{\alpha_{\mathfrak{g}}}=\alpha_{\mathfrak{g}}\circ\mu:\mathfrak{h}\to End(\mathfrak{g})$ be two linear maps such that the following conditions are satisfied*

$$\alpha_{\mathfrak{h}}\circ\rho(x)=\rho(\alpha_{\mathfrak{g}}(x))\circ\alpha_{\mathfrak{h}},\quad\alpha_{\mathfrak{g}}\circ\mu(a)=\mu(\alpha_{\mathfrak{h}}(a))\circ\alpha_{\mathfrak{g}},\ \textit{for any } x\in\mathfrak{g}\textit{ and } a\in\mathfrak{h}. \quad (2.20)$$

Then $(\mathfrak{g},\mathfrak{h};\rho_{\alpha_\mathfrak{h}},\mu_{\alpha_\mathfrak{g}})$ *is a matched pair of Hom-Lie algebras* $(\mathfrak{g},[\cdot,\cdot]_{\alpha_\mathfrak{g}}=\alpha_\mathfrak{g}\circ[\cdot,\cdot],\alpha_\mathfrak{g})$ *and* $(\mathfrak{h},[\cdot,\cdot]_{\alpha_\mathfrak{h}}=\alpha_\mathfrak{h}\circ[\cdot,\cdot],\alpha_\mathfrak{h})$.

Proof. Using Lemma 2.3, then $(\mathfrak{h},\rho_{\alpha_\mathfrak{h}},\alpha_\mathfrak{h})$ resp. $(\mathfrak{g},\mu_{\alpha_\mathfrak{g}},\alpha_\mathfrak{g})$ is a representation of Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\alpha_\mathfrak{g}}=\alpha_\mathfrak{g}\circ[\cdot,\cdot],\alpha_\mathfrak{g})$ resp. $(\mathfrak{h},[\cdot,\cdot]_{\alpha_\mathfrak{h}}=\alpha_\mathfrak{h}\circ[\cdot,\cdot],\alpha_\mathfrak{h})$. Now, we need to show that Eqs.(2.9) and (2.10) hold. For any $x\in\mathfrak{g},a,b\in\mathfrak{h}$, we have

$$\begin{aligned}&\rho_{\alpha_\mathfrak{h}}(\alpha_\mathfrak{g}(x))([a,b]_{\alpha_\mathfrak{h}})-[\rho_{\alpha_\mathfrak{h}}(x)(a),\alpha_\mathfrak{h}(b)]_{\alpha_\mathfrak{h}}-[\alpha_\mathfrak{h}(a),\rho_{\alpha_\mathfrak{h}}(x)(b)]_{\alpha_\mathfrak{h}}\\&\quad-\rho_{\alpha_\mathfrak{h}}(\mu_{\alpha_\mathfrak{g}}(b)(x))(\alpha_\mathfrak{h}(a))+\rho_{\alpha_\mathfrak{h}}(\mu_{\alpha_\mathfrak{g}}(a)(x))(\alpha_\mathfrak{h}(b)\\&=\alpha_\mathfrak{h}\circ\rho(\alpha_\mathfrak{g}(x))(\alpha_\mathfrak{h}\circ[a,b])-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}\circ\rho(x)(a),\alpha_\mathfrak{h}(b)]-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}(a),\alpha_\mathfrak{h}\circ\rho(x)(b)]\\&\quad-\alpha_\mathfrak{h}\circ\rho(\alpha_\mathfrak{g}\circ\mu(b)(x))(\alpha_\mathfrak{h}(a))+\alpha_\mathfrak{h}\circ\rho(\alpha_\mathfrak{g}\circ\mu(a)(x))(\alpha_\mathfrak{h}(b))\\&\overset{(2.20)}{=}\alpha_\mathfrak{h}^2\Big(\rho(x)([a,b])-[\rho(x)a,b]-[a,\rho(x)(b)]-\rho(\mu(b)(x))(a)+\rho(\mu(a)(x))(b)\Big)=0,\end{aligned}$$

then Eq. (2.9) holds. Similarly, we can prove Eq. (2.10). Hence the result. □

Definition 2.11. A pair of Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\alpha)$ and $(\mathfrak{g}^*,[\cdot,\cdot]_{\mathfrak{g}^*},(\alpha^{-1})^*)$ is called a Hom-Lie bialgebra if the following compatiblity condition holds

$$\Delta([x,y]_\mathfrak{g})=ad(\alpha^{-1}(x))\Delta(y)-ad(\alpha^{-1}(y)\Delta(x),\tag{2.21}$$

where $ad_{\alpha^{-1}(x)}\Delta(y)=[\alpha^{-1}(x),\Delta(y)]_\mathfrak{g}=[\alpha^{-1}(x),y_1]_\mathfrak{g}\wedge\alpha(y_2)+\alpha(y_1)\wedge[\alpha^{-1}(x),y_2]_\mathfrak{g}$ and $\Delta:\mathfrak{g}\to\wedge^2\mathfrak{g}$ is the dual of the Hom-Lie algebra structure $[\cdot,\cdot]_{\mathfrak{g}^*}:\wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$, i.e.

$$\langle\Delta(x),\xi\wedge\eta\rangle=\langle x,[\xi,\eta]_{\mathfrak{g}^*}\rangle.$$

We denote the Hom-Lie bialgebra by $(\mathfrak{g},\Delta)$ or $(\mathfrak{g},[\cdot,\cdot]_\mathfrak{g},\Delta,\alpha)$.

Remark 2.4. A Hom-Lie bialgebra $(\mathfrak{g},\Delta)$ with $\alpha=Id_\mathfrak{g}$ is exactly a Lie bialgebra, as defined by Drinfel'd [7, 14].

Example 2.11. Consider the 3-dimensional vector space $\mathfrak{g}=<e_1,e_2,e_3>$ and we define the structure maps by

$$\begin{aligned}&[e_1,e_2]^1_\mathfrak{g}=e_3,\qquad [e_2,e_3]^1_\mathfrak{g}=e_1,\\&\Delta_1(e_2)=4(e_1\otimes e_3-e_3\otimes e_1),\\&\alpha(e_1)=e_1,\quad\alpha(e_2)=2e_2,\quad\alpha(e_3)=2e_3.\end{aligned}$$

Then, $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},\Delta_1,\alpha)$ is a Hom-Lie bialgebra.

Example 2.12. Consider the 3-dimensional vector space $\mathfrak{g}=<e_1,e_2,e_3>$ and we define the structure maps by

$$\begin{aligned}&[e_1,e_3]^2_\mathfrak{g}=e_2,\\&\Delta_2(e_3)=4(e_1\otimes e_2-e_2\otimes e_1),\qquad\Delta_2(e_1)=e_2\otimes e_3-e_3\otimes e_2,\\&\alpha(e_1)=e_1,\quad\alpha(e_2)=2e_2,\quad\alpha(e_3)=2e_3.\end{aligned}$$

Then, $(\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\Delta_2,\alpha)$ is a Hom-Lie bialgebra.

Definition 2.12. Let $(\mathfrak{g},\Delta_\mathfrak{g})$ and $(\mathfrak{h},\Delta_\mathfrak{h})$ be two Hom-Lie bialgebras. A linear map $\psi:(\mathfrak{g},\Delta_\mathfrak{g})\to(\mathfrak{h},\Delta_\mathfrak{h})$ is a Hom-Lie bialgebra morphism if ψ satisfies, for any $x,y\in\mathfrak{g}$

$$\psi([x,y]_\mathfrak{g})=[\psi(x),\psi(y)]_\mathfrak{h},\qquad\psi\circ\alpha_\mathfrak{g}=\alpha_\mathfrak{h}\circ\psi,\qquad(\psi\otimes\psi)\circ\Delta_\mathfrak{g}=\Delta_\mathfrak{h}\circ\psi.$$

Theorem 2.7. *Let* $(\mathfrak{g},\Delta)$ *be a Lie bialgebra and* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *be an invertible Lie bialgebra morphism. Define a linear map* $\Delta_{\alpha^{-1}}=\Delta\circ\alpha^{-1}:\mathfrak{g}\to\wedge^2\mathfrak{g}$ *by*

$$\langle\Delta\circ\alpha^{-1}(x),\xi\wedge\eta\rangle=\langle x,(\alpha^{-1})^*\circ[\xi,\eta]_{\mathfrak{g}^*}\rangle,\tag{2.22}$$

i.e., $\Delta_{\alpha^{-1}}$ *is the dual of the Hom-Lie algebra structure* $[\cdot,\cdot]_{(\alpha^{-1})^*}=(\alpha^{-1})^*\circ[\xi,\eta]_{\mathfrak{g}^*}:\wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$. *Then* $(\mathfrak{g},\Delta_{\alpha^{-1}})$ *is a Hom-Lie bialgebra.*

Proof. Let $(\mathfrak{g},\Delta)$ be a Lie bialgebra and $\alpha:\mathfrak{g}\to\mathfrak{g}$ be an invertible Lie bialgebra morphism. Then $(\mathfrak{g},[\cdot,\cdot]_\alpha,\alpha)$ is a Hom-Lie algebra and it is easy to show that $\alpha^{-1}:\mathfrak{g}\to\mathfrak{g}$ is also a Lie bialgebra morphism. For any $x,y\in\mathfrak{g}$, we have

$$\begin{aligned}
&\Delta_{\alpha^{-1}}([x,y]_\alpha)-ad_{\alpha^{-1}(x)}\Delta_{\alpha^{-1}}(y)+ad_{\alpha^{-1}(y)}\Delta_{\alpha^{-1}}(x)\\
=&\Delta\circ\alpha^{-1}(\alpha\circ[x,y])-[\alpha^{-1}(x),\Delta\circ\alpha^{-1}(y)]_\alpha+[\alpha^{-1}(y),\Delta\circ\alpha^{-1}(x)]_\alpha\\
=&\Delta([x,y])-\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_1)]\wedge y_2-y_1\wedge\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_2)]\\
&+\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_1)]\wedge x_2+x_1\wedge\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_2)]\\
=&\Delta([x,y])-ad_x\Delta(y)+ad_y\Delta(x)=0.
\end{aligned}$$

Then $(\mathfrak{g},\Delta_{\alpha^{-1}})$ is a Hom-Lie bialgebra. □

The following result shows that a Hom-Lie bialgebra deforms into another Hom-Lie bialgebra along an invertible morphism.

Proposition 2.8. *Let $(\mathfrak{g},\Delta)$ be a Hom-Lie bialgebra and $\beta:\mathfrak{g}\to\mathfrak{g}$ be an invertible Hom-Lie bialgebra morphism. Then $(\mathfrak{g},\Delta_{\beta^{-1}}=\Delta\circ\beta^{-1})$ is a Hom-Lie bialgebra, where $\Delta_{\beta^{-1}}:\mathfrak{g}\to\wedge^2\mathfrak{g}$ is the dual of the Hom-Lie algebra structure $[\cdot,\cdot]_{(\beta^{-1})^*}:\wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$, i.e.*

$$\langle\Delta_{\beta^{-1}}(x),\xi\wedge\eta\rangle=\langle x,(\beta^{-1})^*\circ[\xi,\eta]_{\mathfrak{g}^*}\rangle.$$

Proof. Let $(\mathfrak{g},[\cdot,\cdot],\alpha)$ and $(\mathfrak{g}^*,[\cdot,\cdot]_{\mathfrak{g}^*},(\alpha^{-1})^*)$ be two Hom-Lie algebras and $\beta:\mathfrak{g}\to\mathfrak{g}$ be an invertible morphism of Hom-Lie bialgebras, it is easy to show that $(\mathfrak{g},[\cdot,\cdot]_\beta,\beta\circ\alpha)$ and $(\mathfrak{g}^*,(\beta^{-1})^*\circ[\cdot,\cdot]_{\mathfrak{g}^*},(\beta^{-1})^*\circ(\alpha^{-1})^*)$ are two Hom-Lie algebras, and for any $x,y\in\mathfrak{g}$, we have

$$\begin{aligned}
&\Delta_{\beta^{-1}}([x,y]_\beta)-ad_{\alpha^{-1}\circ\beta^{-1}(x)}\Delta_{\beta^{-1}}(y)+ad_{\alpha^{-1}\circ\beta^{-1}(y)}\Delta_{\beta^{-1}}(x)\\
=&\Delta([x,y])-[\beta^{-1}\circ\alpha^{-1}(x),\Delta\circ\beta^{-1}(y)]_\beta+[\beta^{-1}\circ\alpha^{-1}(y),\Delta\circ\beta^{-1}(x)]_\beta\\
=&\Delta([x,y])-\beta\circ[\beta^{-1}\circ\alpha^{-1}(x),\beta^{-1}(y_1)]\wedge\alpha(y)-\alpha(y_1)\wedge\beta\circ[\beta^{-1}\circ\alpha^{-1}(x),\beta^{-1}(y_2)]\\
&+\beta\circ[\beta^{-1}\circ\alpha^{-1}(y),\beta^{-1}(x_1)]\wedge\alpha(x_2)+\alpha(x_1)\wedge\beta\circ[\beta^{-1}\circ\alpha^{-1}(y),\beta^{-1}(x_2)]\\
=&\Delta([x,y])-ad_{\alpha^{-1}(x)}\Delta(y)+ad_{\alpha^{-1}(y)}\Delta(x)=0.
\end{aligned}$$

Then $(\mathfrak{g},\Delta_{\beta^{-1}}=\Delta\circ\beta^{-1})$ is a Hom-Lie bialgebra. □

Definition 2.13. A Hom-Lie bialgebra $(\mathfrak{g},\Delta)$ is said to be coboundary if

$$\Delta(x)=ad_{\mathfrak{g}}(\alpha^{-2}(x))r=[\alpha^{-2}(x),r_1]_{\mathfrak{g}}\wedge\alpha(r_2)+\alpha(r_1)\wedge[\alpha^{-2}(x),r_2]_{\mathfrak{g}},\tag{2.23}$$

for some $r=r_1\wedge r_2\in\wedge^2\mathfrak{g}$.

Remark 2.9. for some $r=r_1\wedge r_2\in\wedge^2\mathfrak{g}$. A coboundary Hom-Lie bialgebra in which $\alpha=Id_{\mathfrak{g}}$ is exactly a coboundary Lie bialgebra, as defined by Drinfel'd [7, 14].

Theorem 2.10. *Let $\alpha:\mathfrak{g}\to\mathfrak{g}$ be an invertible Lie bialgebra morphism and $r\in\wedge^2\mathfrak{g}$ makes $(\mathfrak{g},\Delta)$ a coboundary Lie bialgebra. Define $\tilde{r}=\alpha^{-1}(r_1)\wedge\alpha^{-1}(r_2)$, then $\tilde{r}$ makes $(\mathfrak{g},\Delta_{\alpha^{-1}})$ a coubandary Hom-Lie bialgebra.*

Proof. Let $(\mathfrak{g},\Delta)$ be a Lie bialgebra and $\alpha:\mathfrak{g}\to\mathfrak{g}$ be an invertible Lie bialgebra morphism. Then $(\mathfrak{g},[\cdot,\cdot]_\alpha,\alpha)$ is a Hom-Lie algebra and for any $x\in\mathfrak{g}$, we have

$$\begin{aligned}
ad(\alpha^{-2}(x))\tilde{r}&=[\alpha^{-2}(x),\alpha^{-1}(r_1)]_\alpha\wedge r_2+r_1\wedge[\alpha^{-2}(x),\alpha^{-1}(r_2)]_\alpha\\
&=[\alpha^{-1}(x),r_1]\wedge r_2+r_1\wedge[\alpha^{-1}(x),r_2]\\
&=\Delta_{\alpha^{-1}}(x).
\end{aligned}$$

Hence the result. □

3. Matched pairs, Manin triples and compatible Hom-Lie bialgebras

Let $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ and $(\mathfrak{h},[\cdot,\cdot]^1_{\mathfrak{h}},[\cdot,\cdot]^2_{\mathfrak{h}},\alpha_{\mathfrak{h}})$ be two compatible Hom-Lie algebras. Let $\rho_{\mathfrak{g}},\mu_{\mathfrak{g}}:\mathfrak{g}\to End(\mathfrak{h})$ and $\rho_{\mathfrak{h}},\mu_{\mathfrak{h}}:\mathfrak{h}\to End(\mathfrak{g})$ be four linear maps. On the direct sum of the underlying vector spaces, $\mathfrak{g}\oplus\mathfrak{h}$, we define $\alpha_{\mathfrak{g}\oplus\mathfrak{h}}:\mathfrak{g}\oplus\mathfrak{h}\to\mathfrak{g}\oplus\mathfrak{h}$ by

$$\alpha_{\mathfrak{g}\oplus\mathfrak{h}}(x+a)=\alpha_{\mathfrak{g}}(x)+\alpha_{\mathfrak{h}}(a),$$

and define two skew-symmetric bilinear operations $[\cdot,\cdot]^1_{\bowtie},[\cdot,\cdot]^2_{\bowtie}:\wedge^2(\mathfrak{g}\oplus\mathfrak{h})\to(\mathfrak{g}\oplus\mathfrak{h})$ by

$$[x+a,y+b]^1_{\bowtie}=[x,y]^1_{\mathfrak{g}}+[a,b]^1_{\mathfrak{h}}+\rho_{\mathfrak{g}}(x)b-\rho_{\mathfrak{h}}(b)x+\rho_{\mathfrak{h}}(a)y-\rho_{\mathfrak{g}}(y)a, \tag{3.1}$$

$$[x+a,y+b]^2_{\bowtie}=[x,y]^2_{\mathfrak{g}}+[a,b]^2_{\mathfrak{h}}+\mu_{\mathfrak{g}}(x)b-\mu_{\mathfrak{h}}(b)x+\mu_{\mathfrak{h}}(a)y-\mu_{\mathfrak{g}}(y)a, \tag{3.2}$$

for any $x,y\in\mathfrak{g},\ a,b\in\mathfrak{h}$.
With the above notations, we have the following.

Theorem 3.1. $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]^1_{\bowtie},[\cdot,\cdot]^2_{\bowtie},\alpha_{\mathfrak{g}\oplus\mathfrak{h}})$ *is a compatible Hom-Lie algebra if and only if the following three conditions hold:*
(i) $((\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}}),(\mathfrak{h},[\cdot,\cdot]^1_{\mathfrak{h}}),\rho_{\mathfrak{g}},\rho_{\mathfrak{h}})$ *and* $((\mathfrak{g},[\cdot,\cdot]^2_{\mathfrak{g}}),(\mathfrak{h},[\cdot,\cdot]^2_{\mathfrak{h}}),\mu_{\mathfrak{g}},\mu_{\mathfrak{h}})$ *are two matched pairs of Hom-Lie algebras.*
(ii) $(\mathfrak{h},\rho_{\mathfrak{g}},\mu_{\mathfrak{g}},\alpha_{\mathfrak{h}})$ *is a representation of the compatible Hom-Lie algebra* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g},\rho_{\mathfrak{h}},\mu_{\mathfrak{h}},\alpha_{\mathfrak{g}})$ *is a representation of the compatible Hom-Lie algebra* $(\mathfrak{h},[\cdot,\cdot]^1_{\mathfrak{h}},[\cdot,\cdot]^2_{\mathfrak{h}},\alpha_{\mathfrak{h}})$.
(iii) For any $x,y\in\mathfrak{g}$ *and* $a,b\in\mathfrak{h}$, *the following two equations hold:*

$$\begin{aligned}\mu_{\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))[a,b]^1_{\mathfrak{h}}+\rho_{\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))[a,b]^2_{\mathfrak{h}}\ &=\ [\alpha_{\mathfrak{h}}(a),\mu_{\mathfrak{g}}(x)b]^1_{\mathfrak{h}}+[\mu_{\mathfrak{g}}(x)(a),\alpha_{\mathfrak{h}}(b)]^1_{\mathfrak{h}}+\mu_{\mathfrak{g}}(\rho_{\mathfrak{h}}(b)x)\alpha_{\mathfrak{h}}(a)\\ -\mu_{\mathfrak{g}}(\rho_{\mathfrak{h}}(a)x)\alpha_{\mathfrak{h}}(b)+[\alpha_{\mathfrak{h}}(a),\rho_{\mathfrak{g}}(x)(b)]^2_{\mathfrak{h}}&+[\rho_{\mathfrak{g}}(x)a,\alpha_{\mathfrak{h}}(b)]^2_{\mathfrak{h}}+\rho_{\mathfrak{g}}(\mu_{\mathfrak{h}}(b)x)\alpha_{\mathfrak{h}}(a)-\rho_{\mathfrak{g}}(\mu_{\mathfrak{h}}(a)x)\alpha_{\mathfrak{h}}(b),\end{aligned} \tag{3.3}$$

$$\begin{aligned}\mu_{\mathfrak{h}}(\alpha_{\mathfrak{h}}(a))[x,y]^1_{\mathfrak{g}}+\rho_{\mathfrak{h}}(\alpha_{\mathfrak{h}}(a))[x,y]^2_{\mathfrak{g}}\ &=\ [\alpha_{\mathfrak{g}}(x),\mu_{\mathfrak{h}}(a)y]^1_{\mathfrak{g}}+[\mu_{\mathfrak{h}}(a)(x),\alpha(y)]^1_{\mathfrak{g}}+\mu_{\mathfrak{h}}(\rho_{\mathfrak{g}}(y)a)\alpha_{\mathfrak{g}}(x)\\ -\mu_{\mathfrak{h}}(\rho_{\mathfrak{g}}(x)a)\alpha_{\mathfrak{g}}(y)+[\alpha_{\mathfrak{g}}(x),\rho_{\mathfrak{h}}(a)(y)]^2_{\mathfrak{g}}&+[\rho_{\mathfrak{h}}(a)x,\alpha_{\mathfrak{g}}(y)]^2_{\mathfrak{g}}+\rho_{\mathfrak{h}}(\mu_{\mathfrak{g}}(y)a)\alpha_{\mathfrak{g}}(x)-\rho_{\mathfrak{h}}(\mu_{\mathfrak{g}}(x)a)\alpha_{\mathfrak{g}}(y).\end{aligned} \tag{3.4}$$

Proof. Let $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]^1_{\bowtie},[\cdot,\cdot]^2_{\bowtie},\alpha_{\mathfrak{g}\oplus\mathfrak{h}})$ be a compatible Hom-Lie algebra, by the fact that $\alpha_{\mathfrak{g}\oplus\mathfrak{h}}$ is an algebra homomorphism with respect to $[\cdot,\cdot]^1_{\bowtie}$ and $[\cdot,\cdot]^2_{\bowtie}$ respectively, we have $\alpha_{\mathfrak{g}\oplus\mathfrak{h}}([x,a]^1_{\bowtie})=[\alpha_{\mathfrak{g}}(x),\alpha_{\mathfrak{h}}(a)]^1_{\bowtie}$ and $\alpha_{\mathfrak{g}\oplus\mathfrak{h}}([x,a]^2_{\bowtie})=[\alpha_{\mathfrak{g}}(x),\alpha_{\mathfrak{h}}(a)]^2_{\bowtie}$ respectively, which implies that

$$\rho_{\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))\circ\alpha_{\mathfrak{h}}(a)=\alpha_{\mathfrak{h}}\circ\rho_{\mathfrak{g}}(x)a,\quad \rho_{\mathfrak{h}}(\alpha_{\mathfrak{h}}(a))\circ\alpha_{\mathfrak{g}}(x)=\alpha_{\mathfrak{g}}\circ\rho_{\mathfrak{h}}(a)x,$$

and

$$\mu_{\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))\circ\alpha_{\mathfrak{h}}(a)=\alpha_{\mathfrak{h}}\circ\mu_{\mathfrak{g}}(x)a,\quad \mu_{\mathfrak{h}}(\alpha_{\mathfrak{h}}(a))\circ\alpha_{\mathfrak{g}}(x)=\alpha_{\mathfrak{g}}\circ\mu_{\mathfrak{h}}(a)x,$$

respectively. Then Condition (i) holds if and only if $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]^1_{\bowtie},\alpha_{\mathfrak{g}\oplus\mathfrak{h}})$ and $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]^2_{\bowtie},\alpha_{\mathfrak{g}\oplus\mathfrak{h}})$ are Hom-Lie algebras respectively. Moreover, computing the compatiblity condition for compatible Hom-Lie algebra $(\mathfrak{g}\oplus\mathfrak{h},[\cdot,\cdot]^1_{\bowtie},[\cdot,\cdot]^2_{\bowtie},\alpha_{\mathfrak{g}\oplus\mathfrak{h}})$ for $x,y\in\mathfrak{g}$ and $a\in\mathfrak{h}$, we have

$$\begin{aligned}&[[x,y]^1_{\bowtie},\alpha_{\mathfrak{h}}(a)]^2_{\bowtie}+[[y,a]^1_{\bowtie},\alpha_{\mathfrak{g}}(x)]^2_{\bowtie}+[[a,x]^1_{\bowtie},\alpha_{\mathfrak{g}}(y)]^2_{\bowtie}+[[x,y]^2_{\bowtie},\alpha_{\mathfrak{h}}(a)]^1_{\bowtie}\\ &+[[y,a]^2_{\bowtie},\alpha_{\mathfrak{g}}(x)]^1_{\bowtie}+[[a,x]^2_{\bowtie},\alpha_{\mathfrak{g}}(y)]^1_{\bowtie}=0.\end{aligned} \tag{3.5}$$

Similarly, computing the compatiblity condition for $x\in\mathfrak{g}$ and $a,b\in\mathfrak{h}$, we get

$$\begin{aligned}&[[a,b]^1_{\bowtie},\alpha_{\mathfrak{g}}(x)]^2_{\bowtie}+[[b,x]^1_{\bowtie},\alpha_{\mathfrak{h}}(a)]^2_{\bowtie}+[[x,a]^1_{\bowtie},\alpha_{\mathfrak{h}}(b)]^2_{\bowtie}+[[a,b]^2_{\bowtie},\alpha_{\mathfrak{g}}(x)]^1_{\bowtie}\\ &+[[b,x]^2_{\bowtie},\alpha_{\mathfrak{h}}(a)]^1_{\bowtie}+[[x,a]^2_{\bowtie},\alpha_{\mathfrak{h}}(b)]^1_{\bowtie}=0,\end{aligned} \tag{3.6}$$

which implies that we have the following correspondences: Eq. (3.5)$\Longleftrightarrow$ $(\mathfrak{h},\rho_{\mathfrak{g}},\mu_{\mathfrak{g}},\alpha_{\mathfrak{h}})$ is a representation of $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ and Eq. (3.4) holds; Eq. (3.6)$\Longleftrightarrow$ $(\mathfrak{g},\rho_{\mathfrak{h}},\mu_{\mathfrak{h}},\alpha_{\mathfrak{g}})$ is a representation of $(\mathfrak{h},[\cdot,\cdot]^1_{\mathfrak{h}},[\cdot,\cdot]^2_{\mathfrak{h}},\alpha_{\mathfrak{h}})$ and Eq. (3.3) holds. □

Definition 3.1. A matched pair of compatible Hom-Lie algebras, which we denote by $(\mathfrak{g},\mathfrak{h};\rho_\mathfrak{g},\mu_\mathfrak{g},\rho_\mathfrak{h},\mu_\mathfrak{h})$, consists of two compatible Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{h},[\cdot,\cdot]^1_\mathfrak{h},[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$, together with three conditions given by (i), (ii) and (iii) respectively in Theorem 3.1.

Proposition 3.2. *Let $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{h},[\cdot,\cdot]^1_\mathfrak{h},[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$ be two compatible Hom-Lie algebras. Then $(\mathfrak{g},\mathfrak{h};\rho_\mathfrak{g},\mu_\mathfrak{g},\rho_\mathfrak{h},\mu_\mathfrak{h})$ is a matched pair of compatible Hom-Lie algebras if and only if for any $k_1,k_2\in\mathbb{K}$, $((\mathfrak{g},k_1[\cdot,\cdot]^1_\mathfrak{g}+k_2[\cdot,\cdot]^2_\mathfrak{g}),(\mathfrak{h},k_1[\cdot,\cdot]^1_\mathfrak{h}+k_2[\cdot,\cdot]^2_\mathfrak{h}),k_1\rho_\mathfrak{g}+k_2\mu_\mathfrak{g},k_1\rho_\mathfrak{h}+k_2\mu_\mathfrak{h})$ is a matched pair of Hom-Lie algebras $(\mathfrak{g},k_1[\cdot,\cdot]^1_\mathfrak{g}+k_2[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{h},k_1[\cdot,\cdot]^1_\mathfrak{h}+k_2[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$.*

Remark 3.3. We denote the compatible Hom-Lie algebra defined by Eqs. (3.1)-(3.2) by $\mathfrak{g}\bowtie^{\rho_\mathfrak{g},\mu_\mathfrak{g}}_{\rho_\mathfrak{h},\mu_\mathfrak{h}}\mathfrak{h}$ or simply $\mathfrak{g}\bowtie\mathfrak{h}$.

Remark 3.4. A matched pair $(\mathfrak{g},\mathfrak{h};\rho_\mathfrak{g},\mu_\mathfrak{g},\rho_\mathfrak{h},\mu_\mathfrak{h})$ of compatible Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{h},[\cdot,\cdot]^1_\mathfrak{h},[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$ with $\alpha_\mathfrak{g}=Id_\mathfrak{g}$ and $\alpha_\mathfrak{h}=Id_\mathfrak{h}$ is exactly a matched pair of compatible Lie algebras, as defined in [30].

Proposition 3.5. *Let (V,ρ,μ) be a representation of compatible Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g})$. If there exits a morphism $\alpha:\mathfrak{g}\to\mathfrak{g}$ and $\beta\in End(V)$ such that*

$$\beta\circ\rho(x)=\rho(\alpha(x))\circ\beta,\qquad \beta\circ\mu(x)=\mu(\alpha(x))\circ\beta,\ \textit{for any } x\in\mathfrak{g}. \tag{3.7}$$

Then $(V,\rho_\beta=\beta\circ\rho,\mu_\beta=\beta\circ\mu,\beta)$ is a representation of the compatible Hom-Lie algebra $(\mathfrak{g},\alpha\circ[\cdot,\cdot]^1_\mathfrak{g},\alpha\circ[\cdot,\cdot]^2_\mathfrak{g},\alpha)$.

Proof. Using Lemma 2.3, then (V,ρ_β,β) and (V,μ_β,β) are two representations of $(\mathfrak{g},\alpha\circ[\cdot,\cdot]^1_\mathfrak{g},\alpha)$ and $(\mathfrak{g},\alpha\circ[\cdot,\cdot]^2_\mathfrak{g},\alpha)$ respectively. Next, we need to show that Eq.(2.15) holds, then for any $x,y\in\mathfrak{g}$, we have

$$\begin{aligned}
&\rho_\beta(\alpha\circ[x,y]^2_\mathfrak{g})\circ\beta+\mu_\beta(\alpha\circ[x,y]^1_\mathfrak{g})\circ\beta\\
&-\rho_\beta(\alpha(x))\circ\mu_\beta(y)+\mu_\beta(\alpha(y))\circ\rho_\beta(x)-\mu_\beta(\alpha(x))\circ\rho_\beta(y)+\rho_\beta(\alpha(y))\circ\mu_\beta(x)\\
=&\beta\circ\rho(\alpha\circ[x,y]^2_\mathfrak{g})\circ\beta+\beta\circ\mu(\alpha\circ[x,y]^1_\mathfrak{g})\circ\beta-\beta\circ\rho(\alpha(x))\circ\beta\mu(y)+\beta\circ\mu(\alpha(y))\circ(\beta\circ\rho(x))\\
&-\beta\circ\mu(\alpha(x))\circ(\beta\circ\rho(y))+\beta\circ\rho(\alpha(y))\circ(\beta\circ\mu(x))\\
\overset{(3.7)}{=}&\beta^2\Big(\rho([x,y]^2_\mathfrak{g})+\mu([x,y]^1_\mathfrak{g})-\rho(x)\circ\mu(y)+\mu(y)\circ\rho(x)-\mu(x)\circ\rho(y)+\rho(y)\circ\mu(x)\Big)=0.
\end{aligned}$$

Hence the result. □

Theorem 3.6. *Let $(\mathfrak{g},\mathfrak{h};\rho_\mathfrak{g},\mu_\mathfrak{g},\rho_\mathfrak{h},\mu_\mathfrak{h})$ be a matched pair of compatible Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_\mathfrak{g},[\cdot,\cdot]^2_\mathfrak{g})$ and $(\mathfrak{h},[\cdot,\cdot]^1_\mathfrak{h},[\cdot,\cdot]^2_\mathfrak{h})$. If there exist two Lie algebra morphisms $\alpha_\mathfrak{g}\in End(\mathfrak{g})$ and $\alpha_\mathfrak{h}\in End(\mathfrak{h})$ such that*

$$\rho_\mathfrak{g}(\alpha_\mathfrak{g}(x))\circ\alpha_\mathfrak{h}=\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(x),\quad \mu_\mathfrak{g}(\alpha_\mathfrak{g}(x))\circ\alpha_\mathfrak{h}=\alpha_\mathfrak{h}\circ\mu_\mathfrak{g}(x), \tag{3.8}$$

$$\rho_\mathfrak{h}(\alpha_\mathfrak{h}(a))\circ\alpha_\mathfrak{g}=\alpha_\mathfrak{g}\circ\rho_\mathfrak{h}(a),\quad \mu_\mathfrak{h}(\alpha_\mathfrak{h}(a))\circ\alpha_\mathfrak{g}=\alpha_\mathfrak{g}\circ\mu_\mathfrak{h}(a). \tag{3.9}$$

Then $(\mathfrak{g},\mathfrak{h};\alpha_\mathfrak{h}\circ\rho_\mathfrak{g},\alpha_\mathfrak{h}\circ\mu_\mathfrak{g},\alpha_\mathfrak{g}\circ\rho_\mathfrak{h},\alpha_\mathfrak{g}\circ\mu_\mathfrak{h})$ is a matched pair of compatible Hom-Lie algebras $(\mathfrak{g},\alpha_\mathfrak{g}\circ[\cdot,\cdot]^1_\mathfrak{g},\alpha_\mathfrak{g}\circ[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ and $(\mathfrak{h},\alpha_\mathfrak{h}\circ[\cdot,\cdot]^1_\mathfrak{h},\alpha_\mathfrak{h}\circ[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$.

Proof. Using Proposition 3.5, then $(\mathfrak{h},\alpha_\mathfrak{h}\circ\rho_\mathfrak{g},\alpha_\mathfrak{h}\circ\mu_\mathfrak{g},\alpha_\mathfrak{h})$ (resp. $(\mathfrak{g},\alpha_\mathfrak{g}\circ\rho_\mathfrak{h},\alpha_\mathfrak{g}\circ\mu_\mathfrak{h},\alpha_\mathfrak{g})$) is a representation of compatible Hom-Lie algebra $(\mathfrak{g},\alpha_\mathfrak{g}\circ[\cdot,\cdot]^1_\mathfrak{g},\alpha_\mathfrak{g}\circ[\cdot,\cdot]^2_\mathfrak{g},\alpha_\mathfrak{g})$ (resp. $(\mathfrak{h},\alpha_\mathfrak{h}\circ[\cdot,\cdot]^1_\mathfrak{h},\alpha_\mathfrak{h}\circ[\cdot,\cdot]^2_\mathfrak{h},\alpha_\mathfrak{h})$). Now, we need to show that Eqs.(3.3) and (3.4) hold. For any $x\in\mathfrak{g},a,b\in\mathfrak{h}$, we have

$$\begin{aligned}
&\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(\alpha_\mathfrak{g}(x))(\alpha_\mathfrak{h}\circ[a,b]^2_\mathfrak{h})+\alpha_\mathfrak{h}\circ\mu_\mathfrak{g}(\alpha_\mathfrak{g}(x))(\alpha_\mathfrak{h}\circ[a,b]^1_\mathfrak{h})\\
&-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(x)(a),\alpha_\mathfrak{h}(b)]^2_\mathfrak{h}-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}(a),\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(x)(b)]^2_\mathfrak{h}-\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(\alpha_\mathfrak{g}\circ\mu_\mathfrak{h}(b)(x))(\alpha_\mathfrak{h}(a))\\
&+\alpha_\mathfrak{h}\circ\rho_\mathfrak{g}(\alpha_\mathfrak{g}\circ\mu_\mathfrak{h}(a)(x))(\alpha_\mathfrak{h}(b))-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}\circ\mu_\mathfrak{g}(x)(a),\alpha_\mathfrak{h}(b)]^1_\mathfrak{h}-\alpha_\mathfrak{h}\circ[\alpha_\mathfrak{h}(a),\alpha_\mathfrak{h}\circ\mu_\mathfrak{g}(x)(b)]^1_\mathfrak{h}
\end{aligned}$$

$$- \alpha_{\mathfrak{h}} \circ \mu_{\mathfrak{g}}(\alpha_{\mathfrak{g}} \circ \rho_{\mathfrak{h}}(b)(x))(\alpha_{\mathfrak{h}}(a)) + \alpha_{\mathfrak{h}} \circ \mu_{\mathfrak{g}}(\alpha_{\mathfrak{g}} \circ \rho_{\mathfrak{h}}(a)(x))(\alpha_{\mathfrak{h}}(b))$$
$$\overset{(3.8)}{=} \alpha_{\mathfrak{h}}^2 \Big(\rho_{\mathfrak{g}}(x)([a,b]_{\mathfrak{h}}^2) + \mu_{\mathfrak{g}}(x)([a,b]_{\mathfrak{h}}^1) - [\rho_{\mathfrak{g}}(x)a,b]_{\mathfrak{h}}^2 - [a,\rho_{\mathfrak{g}}(x)(b)]_{\mathfrak{h}}^2 - \rho_{\mathfrak{g}}(\mu_{\mathfrak{h}}(b)(x))(a)$$
$$+ \rho_{\mathfrak{g}}(\mu_{\mathfrak{h}}(a)(x))(b) - [\mu_{\mathfrak{g}}(x)(a),b]_{\mathfrak{h}}^1 - [a,\mu_{\mathfrak{g}}(x)(b)]_{\mathfrak{h}}^1 - \mu_{\mathfrak{g}}(\rho_{\mathfrak{h}}(b)(x))(a) + \mu_{\mathfrak{g}}(\rho_{\mathfrak{h}}(a)(x))(b) \Big) = 0,$$

then Eq. (3.3) holds. Similarly, we can prove that Eq. (2.10) using Eq.(3.9). Hence the result. □

Definition 3.2. Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha)$ be a compatible Hom-Lie algebra. A symmetric bilinear form B on $\mathfrak{g}$ is called invariant if for all $x, y, z \in \mathfrak{g}$, we have

$$B(k_1[x,y]_{\mathfrak{g}}^1 + k_2[x,y]_{\mathfrak{g}}^2, \alpha(z)) = -B(\alpha(y), k_1[x,z]_{\mathfrak{g}}^1 + k_2[x,z]_{\mathfrak{g}}^2), \tag{3.10}$$
$$B(\alpha(x), \alpha(y)) = B(x,y). \tag{3.11}$$

The quadruple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha, B)$ is called a quadratic compatible Hom-Lie algebra.

Now we give the closely related notion of Manin triples of compatible Hom-Lie algebras.

Definition 3.3. Let $\Big((\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}); (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}}), (\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^1, [\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'})\Big)$ be a triple of compatible Hom-Lie algebras. Then it will be called a Manin triple of compatible Hom-Lie algebras if for any $k_1, k_2 \in \mathbb{K}$, $\Big((\mathfrak{g}\oplus\mathfrak{g}', k_1[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}); (\mathfrak{g}, k_1[\cdot,\cdot]_{\mathfrak{g}}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}}), (\mathfrak{g}', k_1[\cdot,\cdot]_{\mathfrak{g}'}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'})\Big)$ is a Manin triple of Hom-Lie algebras with respect to the symmetric bilinear form B. Therefore in the above sense, we denote a Manin triple of compatible Hom-Lie algebras $\Big((\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}); (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}}), (\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^1, [\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'})\Big)$ simply by $\Big(\mathfrak{g}\oplus\mathfrak{g}'; \mathfrak{g}, \mathfrak{g}'\Big)$.

Lemma 3.7. $\Big(\mathfrak{g}\oplus\mathfrak{g}'; \mathfrak{g}, \mathfrak{g}'\Big)$ *is a Manin triple of compatible Hom-Lie algebras associated to the symmetric bilinear form B if and only if both* $((\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}); (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, \alpha_{\mathfrak{g}}), (\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^1, \alpha_{\mathfrak{g}'}))$ *and* $((\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}); (\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}}), (\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'}))$ *are Manin triples of Hom-Lie algebras with respect to the same symmetric bilinear form B.*

Proof. The "only if" part is obvious. For the "if" part, we show that

$$B([x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(z)) = -B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y), [x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^1),$$
$$B([x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(z)) = -B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y), [x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^2),$$

and $B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(x), \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y)) = B(x,y)$ for any $x, y, z \in \mathfrak{g}\oplus\mathfrak{g}'$. Hence for any $k_1, k_2 \in \mathbb{K}$, we have

$$B\Big(k_1[x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^1 + k_2[x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(z)\Big) = -B\Big(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y), k_1[x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^1 + k_2[x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^2\Big),$$
$$B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(x), \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y)) = B(x,y).$$

Since $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^1, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^1, \alpha_{\mathfrak{g}'})$ are Hom-Lie subalgebras of $(\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'})$, also $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'})$ are Hom-Lie subalgebras of $(\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'})$, we show that $(\mathfrak{g}, k_1[\cdot,\cdot]_{\mathfrak{g}}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}}^2, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}', k_1[\cdot,\cdot]_{\mathfrak{g}'}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}'}^2, \alpha_{\mathfrak{g}'})$ are subalgebras of $(\mathfrak{g}\oplus\mathfrak{g}', k_1[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^1 + k_2[\cdot,\cdot]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'})$. Hence the conclusion follows. □

Remark 3.8. By the above Lemma, we can rewrite the definition of a Manin triple of compatible Hom-Lie algebras as follows: $\Big(\mathfrak{g}\oplus\mathfrak{g}'; \mathfrak{g}, \mathfrak{g}'\Big)$ is a Manin triple of compatible Hom-Lie algebras with respect to a symmetric bilinear form B if the following conditions are satisfied

(1) $B([x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^1, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(z)) = -B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y), [x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^1)$,
$B([x,y]_{\mathfrak{g}\oplus\mathfrak{g}'}^2, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(z)) = -B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y), [x,z]_{\mathfrak{g}\oplus\mathfrak{g}'}^2)$ and $B(\alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(x), \alpha_{\mathfrak{g}\oplus\mathfrak{g}'}(y)) = B(x,y)$ for any $x, y, z \in \mathfrak{g}\oplus\mathfrak{g}'$,

(2) $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}', [\cdot,\cdot]^1_{\mathfrak{g}'}, [\cdot,\cdot]^2_{\mathfrak{g}'}, \alpha_{\mathfrak{g}'})$ are compatible Hom-Lie subalgebras of $(\mathfrak{g}\oplus\mathfrak{g}', [\cdot,\cdot]^1_{\mathfrak{g}\oplus\mathfrak{g}'}, [\cdot,\cdot]^2_{\mathfrak{g}\oplus\mathfrak{g}'}, \alpha_{\mathfrak{g}\oplus\mathfrak{g}'})$,
(3) $\alpha_{\mathfrak{g}\oplus\mathfrak{g}'} = \alpha_{\mathfrak{g}} \oplus \alpha_{\mathfrak{g}'}$,
(4) $\mathfrak{g}$ and $\mathfrak{g}'$ are isotropic with respect to B, i.e. $B_{|\mathfrak{g}\times\mathfrak{g}} = 0,\ B_{|\mathfrak{g}'\times\mathfrak{g}'} = 0$.

Next, we introduce the dual representation of compatible Hom-Lie algebras which will be useful in the study of compatible Hom-Lie bialgebras.

Proposition 3.9. *Let (V,ρ,μ,β) be a representation of a compatible Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. Then $(V^*, \rho^\star, \mu^\star, (\beta^{-1})^*)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$, which is called the dual representation of (ρ,μ), where $\rho^\star$ and $\mu^\star$ are given by Eq.(2.17). In particular, $(\mathfrak{g}^*, ad^\star_{1\mathfrak{g}}, ad^\star_{2\mathfrak{g}}, (\alpha_{\mathfrak{g}}^{-1})^*)$ is the dual rep- resentation of the representation $(\mathfrak{g}, ad_{1\mathfrak{g}}, ad_{2\mathfrak{g}}, \alpha_{\mathfrak{g}})$.*

Proof. Let (V,ρ,μ,β) be a representation of a compatible Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. By Lemma 2.4, $\rho^\star$ and $\mu^\star$ given by Eq.(2.17) are two representations of $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ on V^* with respect to $(\beta^{-1})^*$ and $(\mathfrak{g}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ on V^* with respect to $(\beta^{-1})^*$ respectively. Moreover, for all $x,y\in\mathfrak{g}, \xi\in V^*$ and $u\in V$, we have

$$\begin{aligned}
&\langle \rho^\star([x,y]^2_{\mathfrak{g}})((\beta^{-1})^*(\xi)) + \mu^\star([x,y]^1_{\mathfrak{g}}((\beta^{-1})^*(\xi)), u\rangle\\
=&\langle \rho^*(\alpha_{\mathfrak{g}}([x,y]^2_{\mathfrak{g}}))((\beta^{-3})^*(\xi)) + \mu^*(\alpha_{\mathfrak{g}}([x,y]^1_{\mathfrak{g}}))((\beta^{-3})^*(\xi)), u\rangle\\
=&-\langle (\beta^{-3})^*(\xi), \rho(\alpha_{\mathfrak{g}}([x,y]^2_{\mathfrak{g}}))(u) + \mu(\alpha_{\mathfrak{g}}([x,y]^1_{\mathfrak{g}}))(u)\rangle\\
=&-\langle (\beta^{-3})^*(\xi), \rho(\alpha^2_{\mathfrak{g}}(x))\mu(\alpha_{\mathfrak{g}}(y))((\beta^{-1})(u)) - \mu(\alpha^2_{\mathfrak{g}}(y))\rho(\alpha_{\mathfrak{g}}(x))((\beta^{-1})(u))\\
&+\mu(\alpha^2_{\mathfrak{g}}(x))\rho(\alpha_{\mathfrak{g}}(y))((\beta^{-1})(u)) - \rho(\alpha^2_{\mathfrak{g}}(y))\mu(\alpha_{\mathfrak{g}}(x))((\beta^{-1})(u))\rangle\\
=&-\langle (\beta^{-4})^*(\xi), \rho(\alpha^3_{\mathfrak{g}}(x))\mu(\alpha^2_{\mathfrak{g}}(y))(u) - \mu(\alpha^3_{\mathfrak{g}}(y))\rho(\alpha^2_{\mathfrak{g}}(x))(u)\\
&+\mu(\alpha^3_{\mathfrak{g}}(x))\rho(\alpha^2_{\mathfrak{g}}(y))(u) - \rho(\alpha^3_{\mathfrak{g}}(y))\mu(\alpha^2_{\mathfrak{g}}(x))(u)\rangle\\
=&-\langle \mu^*(\alpha^2_{\mathfrak{g}}(y))\rho^*(\alpha^3_{\mathfrak{g}}(x))(\beta^{-4})^*(\xi) - \rho^*(\alpha^2_{\mathfrak{g}}(x))\mu^*(\alpha^3_{\mathfrak{g}}(y))(\beta^{-4})^*(\xi)\\
&+\rho^*(\alpha^2_{\mathfrak{g}}(y))\mu^*(\alpha^3_{\mathfrak{g}}(x))(\beta^{-4})^*(\xi) - \mu^*(\alpha^2_{\mathfrak{g}}(x))\rho^*(\alpha^3_{\mathfrak{g}}(y))(\beta^{-4})^*(\xi), u\rangle\\
=&-\langle \mu^\star(\alpha_{\mathfrak{g}}(y))\rho^\star(x)(\xi) - \rho^\star(\alpha_{\mathfrak{g}}(x))\mu^\star(y)(\xi) + \rho^\star(\alpha_{\mathfrak{g}}(y)\mu^\star(x)(\xi) - \mu^\star(\alpha_{\mathfrak{g}}(x))\rho^\star(y)(\xi), u\rangle,
\end{aligned}$$

which implies that

$$\begin{aligned}
&\rho^\star([x,y]^2_{\mathfrak{g}})((\beta^{-1})^*(\xi)) + \mu^\star([x,y]^1_{\mathfrak{g}})((\beta^{-1})^*(\xi))\\
=&-\mu^\star(\alpha_{\mathfrak{g}}(y))\rho^\star(x)(\xi) + \rho^\star(\alpha_{\mathfrak{g}}(x))\mu^\star(y)(\xi) - \rho^\star(\alpha_{\mathfrak{g}}(y)\mu^\star(x)(\xi) + \mu^\star(\alpha_{\mathfrak{g}}(x))\rho^\star(y)(\xi).
\end{aligned}$$

Therefore, $(V^*, \rho^\star, \mu^\star, (\beta^{-1})^*)$ is a representation of $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$. □

Theorem 3.10. *Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]^1_{\mathfrak{g}^*}, [\cdot,\cdot]^2_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}^*})$ be two compatible Hom-Lie algebras. Then $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$ is a Manin triple of compatible Hom-Lie algebras associated to the nondegenerate symmetric invariant bilinear form on $\mathfrak{g}\oplus\mathfrak{g}^*$ defined by*

$$B(x+\xi, y+\eta) = \langle x,\eta\rangle + \langle y,\xi\rangle, \forall x,y\in\mathfrak{g},\ \xi,\eta\in\mathfrak{g}^*. \tag{3.12}$$

if and only if $\alpha_{\mathfrak{g}^*} = (\alpha_{\mathfrak{g}}^{-1})^*$ and $(\mathfrak{g},\mathfrak{g}^*, ad^\star_{1\mathfrak{g}}, ad^\star_{2\mathfrak{g}}, ad^\star_{1\mathfrak{g}^*}, ad^\star_{2\mathfrak{g}^*})$ is a matched pair of compatible Hom-Lie algebras.

Proof. $\Longleftarrow$ Suppose that $\alpha_{\mathfrak{g}^*} = (\alpha_{\mathfrak{g}}^{-1})^*$ and $(\mathfrak{g},\mathfrak{g}^*, ad^\star_{1\mathfrak{g}}, ad^\star_{2\mathfrak{g}}, ad^\star_{1\mathfrak{g}^*}, ad^\star_{2\mathfrak{g}^*})$ is a matched pair of compatible Hom-Lie algebras, then there exists a compatible Hom-Lie algebra structure $\mathfrak{g}\bowtie^{ad^\star_{1\mathfrak{g}}, ad^\star_{2\mathfrak{g}}}_{ad^\star_{1\mathfrak{g}^*}, ad^\star_{2\mathfrak{g}^*}}\mathfrak{g}^*$ on $\mathfrak{g}\oplus\mathfrak{g}^*$ given by

$$[x+\xi, y+\eta]^1_{\bowtie} = [x,y]^1_{\mathfrak{g}} + [\xi,\eta]^1_{\mathfrak{g}^*} + ad^\star_{1\mathfrak{g}}(x)\eta - ad^\star_{1\mathfrak{g}^*}(\eta)x + ad^\star_{1\mathfrak{g}^*}(\xi)y - ad^\star_{1\mathfrak{g}}(y)\xi, \tag{3.13}$$

$$[x+\xi, y+\eta]^2_{\bowtie} = [x,y]^2_{\mathfrak{g}} + [\xi,\eta]^2_{\mathfrak{g}^*} + ad^\star_{2\mathfrak{g}}(x)\eta - ad^\star_{2\mathfrak{g}^*}(\eta)x + ad^\star_{2\mathfrak{g}^*}(\xi)y - ad^\star_{2\mathfrak{g}}(y)\xi, \tag{3.14}$$

with $\alpha_{\mathfrak{g}\oplus\mathfrak{g}^*} = \alpha_{\mathfrak{g}} \oplus (\alpha_{\mathfrak{g}}^{-1})^*$. It is straightforward to show that the bilinear form B given by Eq.(3.12) is invariant in the sense of Remark 3.8 **(1)**. Hence $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$ is a Manin

triple of compatible Hom-Lie algebras.
$\Longrightarrow$ Conversely, let $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$ be a Manin triple of compatible Hom-Lie algebras associated with to the invariant bilinear form B given by Eq.(3.12). Then for any $x,y\in\mathfrak{g}$ and $\xi,\eta\in\mathfrak{g}^*$, due to the invariance of B given in Remark.3.8 **(1)**, we obtain $\alpha_{\mathfrak{g}^*}=(\alpha_{\mathfrak{g}}^{-1})^*$ and the compatible Hom-Lie algebra brackets on $\mathfrak{g}\oplus\mathfrak{g}^*$ are given by Eqs.(3.13) and (3.14). Therefore, $(\mathfrak{g},\mathfrak{g}^*,ad^{\star}_{1\mathfrak{g}},ad^{\star}_{2\mathfrak{g}},ad^{\star}_{1\mathfrak{g}^*},ad^{\star}_{2\mathfrak{g}^*})$ is a matched pair of compatible Hom-Lie algebras. □

The bilinear form B given by Eq.(3.12) is called the standard bilinear form on $\mathfrak{g}\oplus\mathfrak{g}^*$, and the hom-Lie brackets given by Eqs.(3.13) and (3.14) is called the standard compatible hom-Lie bracket on $\mathfrak{g}\oplus\mathfrak{g}^*$. The Manin triple $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$ is called the standard Manin triple. Obviously, a standard Manin triple of compatible Hom-Lie algebras is a Manin triple of compatible Hom-Lie algebras. Conversely, we have

Proposition 3.11. *Every Manin triple of compatible Hom-Lie algebras* $(\mathcal{G};\mathfrak{g},\mathfrak{g}')$ *is isomorphic to the standard one* $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$.

Proof. First of all, since $\mathfrak{g}$ and $\mathfrak{g}'$ are isotropic under the nondegenerate bilinear form B on $\mathcal{G}=\mathfrak{g}\oplus\mathfrak{g}'$, we deduce that $\mathfrak{g}'$ is isomorphic to $\mathfrak{g}^*$. Thus, $\mathcal{G}$ is isomorphic to $\mathfrak{g}\oplus\mathfrak{g}^*$ as vector spaces. Also, we can transfer the nondegenerate bilinear form B to $\mathfrak{g}\oplus\mathfrak{g}^*$, then we obtain the standard bilinear form given by Eq.(3.12). Denote by $\alpha_{\mathfrak{g}^*}:\mathfrak{g}^*\to\mathfrak{g}^*$ the induced map from $\alpha_{\mathfrak{g}'}=\alpha_{\mathcal{G}|\mathfrak{g}'}$. Due to the invariance of B, we check that $\alpha_{\mathfrak{g}^*}=(\alpha_{\mathfrak{g}}^{-1})^*$. It is straightforward to check that the compatible Hom-Lie algebra brackets on $\mathcal{G}$ can be transferred into $\mathfrak{g}\oplus\mathfrak{g}^*$, which is the standard compatible Hom-Lie algebra brackets given by Eqs.(3.13) and (3.14). Thus, $(\mathcal{G};\mathfrak{g},\mathfrak{g}')$ is isomorphic to the standard Manin triple $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$. □

For a compatible Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ (resp $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha_{\mathfrak{g}}^{-1})^*)$), let $\Delta^1_{\mathfrak{g}^*},\Delta^2_{\mathfrak{g}^*}:\mathfrak{g}^*\to\wedge^2\mathfrak{g}^*$ (resp $\Delta_1,\Delta_2:\mathfrak{g}\to\wedge^2\mathfrak{g}$) are the dual of the Hom-Lie algebra structures $[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}}:\wedge^2\mathfrak{g}\to\mathfrak{g}$ (resp $[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*}:\wedge^2\mathfrak{g}^*\to\mathfrak{g}^*$) i.e.,

$$\begin{aligned}&\langle\Delta^1_{\mathfrak{g}^*}(\xi),x\wedge y\rangle=\langle\xi,[x,y]^1_{\mathfrak{g}}\rangle,\quad\langle\Delta^2_{\mathfrak{g}^*}(\xi),x\wedge y\rangle=\langle\xi,[x,y]^2_{\mathfrak{g}}\rangle,\\&\langle\Delta^1_{\mathfrak{g}}(x),\xi\wedge\eta\rangle=\langle x,[\xi,\eta]^1_{\mathfrak{g}^*}\rangle,\quad\langle\Delta^2_{\mathfrak{g}}(x),\xi\wedge\eta\rangle=\langle x,[\xi,\eta]^2_{\mathfrak{g}^*}\rangle.\end{aligned}\tag{3.15}$$

In particular, we set $\Delta_1\ resp(\Delta_2)=\Delta^1_{\mathfrak{g}}\ resp(\Delta^2_{\mathfrak{g}})$

Theorem 3.12. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha_{\mathfrak{g}}^{-1})^*)$ *be two compatible Hom-Lie algebras. Then* $(\mathfrak{g},\mathfrak{g}^*,ad^{\star}_{1\mathfrak{g}},ad^{\star}_{2\mathfrak{g}},ad^{\star}_{1\mathfrak{g}^*},ad^{\star}_{2\mathfrak{g}^*})$ *is a matched pair if and only if for any* $x,y\in\mathfrak{g}$

$$\Delta_1([x,y]^1_{\mathfrak{g}})=ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_1(y)-ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_1(x),\tag{3.16}$$

$$\Delta_2([x,y]^2_{\mathfrak{g}})=ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_2(y)-ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_2(x),\tag{3.17}$$

$$\begin{aligned}&\Delta_1([x,y]^2_{\mathfrak{g}})+\Delta_2([x,y]^1_{\mathfrak{g}})\\&=ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_1(y)-ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_1(x)+ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_2(y)-ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_2(x),\end{aligned}\tag{3.18}$$

where $\Delta_1\ resp(\Delta_2)=\Delta^1_{\mathfrak{g}}\ resp(\Delta^2_{\mathfrak{g}})$ *are given by* (3.15).

Proof. Eqs. (3.16) and (3.17) are equivalent to the facts that $(\mathfrak{g},\mathfrak{g}^*,ad^{\star}_{1\mathfrak{g}},ad^{\star}_{1\mathfrak{g}^*})$ and $(\mathfrak{g},\mathfrak{g}^*,ad^{\star}_{2\mathfrak{g}},ad^{\star}_{2\mathfrak{g}^*})$ are matched pairs of Hom-Lie algebras respectively. In fact, $(\mathfrak{g},\mathfrak{g}^*,ad^{\star}_{1\mathfrak{g}},ad^{\star}_{1\mathfrak{g}^*})$ is a matched pair of Hom-Lie algebras $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},(\alpha_{\mathfrak{g}}^{-1})^*$, then for any $x,y\in\mathfrak{g},\ \xi\in\mathfrak{g}^*$, we have

$$\begin{aligned}&-ad^{\star}_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))([x,y]^1_{\mathfrak{g}})+[ad^{\star}_{1\mathfrak{g}^*}(\xi)(x),\alpha_{\mathfrak{g}}(y)]^1_{\mathfrak{g}}-[ad^{\star}_{1\mathfrak{g}^*}(\xi)(y),\alpha_{\mathfrak{g}}(x)]^1_{\mathfrak{g}}\\&+ad^{\star}_{1\mathfrak{g}^*}(ad^{\star}_{1\mathfrak{g}}(y)\xi)\alpha_{\mathfrak{g}}(x)-ad^{\star}_{1\mathfrak{g}^*}(ad^{\star}_{1\mathfrak{g}}(x)\xi)\alpha_{\mathfrak{g}}(y)=0.\end{aligned}$$

Note that, for any $x \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$, by Corollary 2.5, $ad^\star_{\mathfrak{g}}(x)(\xi) = ad^*_{\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))((\alpha_{\mathfrak{g}}^{-2})^*(\xi))$ is a representation of $\mathfrak{g}$ on $\mathfrak{g}^*$ with respect to $(\alpha_{\mathfrak{g}}^{-1})^*$ and $ad^\star_{\mathfrak{g}^*}(\xi)(x) = ad^*_{\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(x))$ is a representation of $\mathfrak{g}^*$ on $\mathfrak{g}$ with respect to $\alpha_{\mathfrak{g}}$. Then for any $\eta \in \mathfrak{g}^*$, we have

$$\begin{aligned}
&\langle \eta, -ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-2})^*(\xi))(\alpha_{\mathfrak{g}}^2([x,y]^1_{\mathfrak{g}}) + [ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(x)), \alpha_{\mathfrak{g}}(y)]^1_{\mathfrak{g}}\\
&\quad - [ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(y)), \alpha_{\mathfrak{g}}(x)]^1_{\mathfrak{g}} + ad^*_{1\mathfrak{g}^*}\Big(ad^*_{\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(x)\\
&\quad - ad^*_{1\mathfrak{g}^*}\Big(ad^*_{\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(y)\rangle\\
=&\langle [(\alpha_{\mathfrak{g}}^{-2})^*(\xi), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^2([x,y]^1_{\mathfrak{g}})\rangle - \langle ad_{1\mathfrak{g}^*}(((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}(y))(\eta), \alpha_{\mathfrak{g}}^2(x)\rangle\\
&\quad + \langle ad_{1\mathfrak{g}^*}(((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))(\eta), \alpha_{\mathfrak{g}}^2(y)\rangle - \langle [ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(x)\rangle\\
&\quad + \langle [ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(y)\rangle\\
=&\langle \xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta), \Delta_1([x,y]^1_{\mathfrak{g}})\rangle - \langle ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y))(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_1(x)\rangle\\
&\quad + \langle ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x))(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_1(y)\rangle\\
=&\langle \Delta_1([x,y]^1_{\mathfrak{g}}) + ad_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y))\Delta_1(x) - ad_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x))\Delta_1(y), \xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)\rangle = 0,
\end{aligned}$$

which implies that (3.16) holds. Similarly, we can prove that $(\mathfrak{g}, \mathfrak{g}^*, ad^\star_{2\mathfrak{g}}, ad^\star_{2\mathfrak{g}^*})$ is a matched pair is equivalent to Eq. (3.17). Now, we need to show that Eqs. (3.3) and (3.4) are equivalent to Eq. (3.18) in the case that $\rho_{\mathfrak{g}} = ad^\star_{1\mathfrak{g}}$, $\mu_{\mathfrak{g}} = ad^\star_{2\mathfrak{g}}$, $\rho_{\mathfrak{h}} = ad^\star_{1\mathfrak{g}^*}$, $\mu_{\mathfrak{h}} = ad^\star_{2\mathfrak{g}^*}$ and $\alpha_{\mathfrak{h}} = (\alpha_{\mathfrak{g}}^{-1})^*$. As an example, we give an explicit proof of the case that (for any $x, y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$)

$$\begin{aligned}
&ad^\star_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))([x,y]^1_{\mathfrak{g}}) + ad^\star_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))([x,y]^2_{\mathfrak{g}})\\
&- [\alpha_{\mathfrak{g}}(x), ad^\star_{2\mathfrak{g}^*}(\xi)(y)]^1_{\mathfrak{g}} - [ad^\star_{2\mathfrak{g}^*}(\xi)(x), \alpha_{\mathfrak{g}}(y)]^1_{\mathfrak{g}} + ad^\star_{2\mathfrak{g}^*}(ad^\star_{1\mathfrak{g}}(x)\xi)\alpha_{\mathfrak{g}}(y) - ad^\star_{2\mathfrak{g}^*}(ad^\star_{1\mathfrak{g}}(y)\xi)\alpha_{\mathfrak{g}}(x)\\
&- [\alpha_{\mathfrak{g}}(x), ad^\star_{1\mathfrak{g}^*}(\xi)(y)]^2_{\mathfrak{g}} - [ad^\star_{1\mathfrak{g}^*}(\xi)(x), \alpha_{\mathfrak{g}}(y)]^2_{\mathfrak{g}} + ad^\star_{1\mathfrak{g}^*}(ad^\star_{2\mathfrak{g}}(x)\xi)\alpha_{\mathfrak{g}}(y) - ad^\star_{1\mathfrak{g}^*}(ad^\star_{2\mathfrak{g}}(y)\xi)\alpha_{\mathfrak{g}}(x) = 0,
\end{aligned}$$

is equivalent to Eq.(3.18). A similar argument applies to the other case. Indeed, if the left-hand side of the above equation acts on an arbitrary element $\eta \in \mathfrak{g}^*$, then we obtain

$$\begin{aligned}
&\langle \eta, ad^*_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-2})^*(\xi))(\alpha_{\mathfrak{g}}^2([x,y]^1_{\mathfrak{g}})) + ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-2})^*(\xi))(\alpha_{\mathfrak{g}}^2([x,y]^2_{\mathfrak{g}}))\\
&\quad - [\alpha_{\mathfrak{g}}(x), ad^*_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(y))]^1_{\mathfrak{g}} + [\alpha_{\mathfrak{g}}(y), ad^*_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(x))]^1_{\mathfrak{g}}\\
&\quad + ad^*_{2\mathfrak{g}^*}\Big(ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(y) - ad^*_{2\mathfrak{g}^*}\Big(ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(x)\\
&\quad - [\alpha_{\mathfrak{g}}(x), ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(y))]^2_{\mathfrak{g}} + [\alpha_{\mathfrak{g}}(y), ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi))(\alpha_{\mathfrak{g}}^2(x))]^2_{\mathfrak{g}}\\
&\quad + ad^*_{1\mathfrak{g}^*}\Big(ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(y) - ad^*_{1\mathfrak{g}^*}\Big(ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi))\Big)\alpha_{\mathfrak{g}}^3(x)\rangle\\
=& - \langle [(\alpha_{\mathfrak{g}}^{-2})^*(\xi), \eta]^2_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^2([x,y]^1_{\mathfrak{g}})\rangle - \langle [(\alpha_{\mathfrak{g}}^{-2})^*(\xi), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^2([x,y]^2_{\mathfrak{g}})\rangle\\
&\quad - \langle ad^*_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))(\eta), \alpha_{\mathfrak{g}}^2(y)\rangle + \langle ad^*_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}(y))(\eta), \alpha_{\mathfrak{g}}^2(x)\rangle\\
&\quad - \langle [ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^2_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(y)\rangle + \langle [ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^2_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(x)\rangle\\
&\quad - \langle ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))(\eta), \alpha_{\mathfrak{g}}^2(y)\rangle + \langle ad^*_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}}^{-1})^*(\xi)) \circ ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}(y))(\eta), \alpha_{\mathfrak{g}}^2(x)\rangle\\
&\quad - \langle [ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(x))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(y)\rangle + \langle [ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^2(y))((\alpha_{\mathfrak{g}}^{-3})^*(\xi)), \eta]^1_{\mathfrak{g}^*}, \alpha_{\mathfrak{g}}^3(x)\rangle\\
=& - \langle \Delta_2([x,y]^1_{\mathfrak{g}}), \xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)\rangle - \langle \Delta_1([x,y]^2_{\mathfrak{g}}), \xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)\rangle - \langle ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x)(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_2(y)\rangle\\
&\quad + \langle ad^*_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y)(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_2(x)\rangle - \langle ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x)(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_1(y)\rangle\\
&\quad + \langle ad^*_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y)(\xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)), \Delta_1(x)\rangle\\
=&\langle -\Delta_2([x,y]^1_{\mathfrak{g}}) - \Delta_1([x,y]^2_{\mathfrak{g}}) + ad_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x))\Delta_2(y)\\
&\quad - ad_{1\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y))\Delta_2(x) + ad_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(x))\Delta_1(y) - ad_{2\mathfrak{g}}(\alpha_{\mathfrak{g}}^{-1}(y))\Delta_1(x), \xi \wedge (\alpha_{\mathfrak{g}}^2)^*(\eta)\rangle.
\end{aligned}$$

Hence the conclusion holds. □

Definition 3.4. Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]^1_{\mathfrak{g}^*}, [\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$ be two compatible Hom-Lie algebras. The tuple $(\mathfrak{g}, \Delta_1, \Delta_2)$ is a compatible Hom-Lie bialgebra if for any $k_1, k_2 \in \mathbb{K}$, $(\mathfrak{g}, k_1\Delta_1 + k_2\Delta_2)$ is a Hom-Lie bialgebra defined by the two Hom-Lie algebras $(\mathfrak{g}, k_1[\cdot,\cdot]^1_{\mathfrak{g}} + k_2[\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, k_1[\cdot,\cdot]^1_{\mathfrak{g}^*} + k_2[\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$.

Proposition 3.13. *Let* $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g}^*, [\cdot,\cdot]^1_{\mathfrak{g}^*}, [\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$ *be two compatible Hom-Lie algebras. Then* $(\mathfrak{g}, \Delta_1, \Delta_2)$ *is a compatible Hom-Lie bialgebra if and only if* Δ_1 *and* Δ_2 *defined by* (3.15) *satisfy Eqs.* (3.16),(3.17) *and* (3.18).

Proof. $\Longrightarrow$ For any $k_1, k_2 \in \mathbb{K}$, it is straightforward to show that Eqs. (3.16), (3.17) and (3.18) correspond to the facts that $(\mathfrak{g}, k_1\Delta_1 + k_2\Delta_2)$ is a Hom-Lie bialgebra defined by the two Hom-Lie algebras structure $(\mathfrak{g}, k_1[\cdot,\cdot]^1_{\mathfrak{g}} + k_2[\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, k_1[\cdot,\cdot]^1_{\mathfrak{g}^*} + k_2[\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$ for the cases $k_1 = 1$ *and* $k_2 = 0$; $k_1 = 0$ *and* $k_2 = 1$ and finaly $k_1 = 1$ *and* $k_2 = 1$ respectively.
$\Longleftarrow$ For any $k_1, k_2 \in \mathbb{K}$, let $(\mathfrak{g}, k_1[\cdot,\cdot]^1_{\mathfrak{g}} + k_2[\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, k_1[\cdot,\cdot]^1_{\mathfrak{g}^*} + k_2[\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$ be two Hom-Lie algebras. Then we have

$$\begin{aligned}
&(k_1\Delta_1 + k_2\Delta_2)(k_1[x,y]^1_{\mathfrak{g}} + k_2[x,y]^2_{\mathfrak{g}})\\
=&k_1^2\Delta_1([x,y]^1_{\mathfrak{g}}) + k_2^2\Delta_2([x,y]^2_{\mathfrak{g}}) + k_1k_2\Delta_1([x,y]^2_{\mathfrak{g}}) + k_1k_2\Delta_2([x,y]^1_{\mathfrak{g}})\\
=&k_1^2\Big(ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_1(y) - ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_1(x)\Big) + k_2^2\Big(ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_2(y)\\
&- ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_2(x)\Big) + k_1k_2\Big(ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_1(y) - ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_1(x)\\
&+ ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta_2(y) - ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta_2(x)\Big)\\
=&\Big(k_1ad_{1\mathfrak{g}} + k_2ad_{2\mathfrak{g}}\Big)((\alpha_{\mathfrak{g}})^{-1}(x))\Big(k_1\Delta_1 + k_2\Delta_2\Big)(y) - \Big(k_1ad_{1\mathfrak{g}} + k_2ad_{2\mathfrak{g}}\Big)((\alpha_{\mathfrak{g}})^{-1}(y))\Big(k_1\Delta_1 + k_2\Delta_2\Big)(x).
\end{aligned}$$

Therefore $(\mathfrak{g}, k_1\Delta_1 + k_2\Delta_2)$ is a Hom-Lie bialgebra structure defined by the two Hom-Lie algebras $(\mathfrak{g}, k_1[\cdot,\cdot]^1_{\mathfrak{g}} + k_2[\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, k_1[\cdot,\cdot]^1_{\mathfrak{g}^*} + k_2[\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$. □

Remark 3.14. A compatible Hom-Lie bialgebra $(\mathfrak{g}, \Delta_1, \Delta_2)$ of compatible Hom-Lie algebras $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{g}^*, [\cdot,\cdot]^1_{\mathfrak{g}^*}, [\cdot,\cdot]^2_{\mathfrak{g}^*}, (\alpha_{\mathfrak{g}}^{-1})^*)$ with $\alpha_{\mathfrak{g}} = Id_{\mathfrak{g}}$ is exactly a compatible Lie bialgebra, as defined in [30].

Example 3.1. Combining Example (2.11) and Example (2.12), then $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \Delta_1, \Delta_2, \alpha)$ is a compatible Hom-Lie bialgebra.

Definition 3.5. Let $(\mathfrak{g}, [\cdot,\cdot]^1_{\mathfrak{g}}, [\cdot,\cdot]^2_{\mathfrak{g}}, \alpha_{\mathfrak{g}})$ and $(\mathfrak{h}, [\cdot,\cdot]^1_{\mathfrak{h}}, [\cdot,\cdot]^2_{\mathfrak{h}}, \alpha_{\mathfrak{h}})$ be two compatible Hom-Lie algebras and let $(\mathfrak{g}, \Delta_{1\mathfrak{g}}, \Delta_{2\mathfrak{g}})$ and $(\mathfrak{h}, \Delta_{1\mathfrak{h}}, \Delta_{2\mathfrak{h}})$ be two compatible Hom-Lie bialgebras. A linear map $\psi : (\mathfrak{g}, \Delta_{1\mathfrak{g}}, \Delta_{2\mathfrak{g}}) \to (\mathfrak{h}, \Delta_{1\mathfrak{h}}, \Delta_{2\mathfrak{h}})$ is a homomorphism of compatible Hom-Lie bialgebras if ψ satisfies, for any $x, y \in \mathfrak{g}$

$$\begin{aligned}
&\psi([x,y]^1_{\mathfrak{g}}) = [\psi(x),\psi(y)]^1_{\mathfrak{h}}, \quad \psi([x,y]^2_{\mathfrak{g}}) = [\psi(x),\psi(y)]^2_{\mathfrak{h}}, \quad \psi\circ\alpha_{\mathfrak{g}} = \alpha_{\mathfrak{h}}\circ\psi,\\
&(\psi\otimes\psi)\circ\Delta_{1\mathfrak{g}} = \Delta_{1\mathfrak{h}}\circ\psi, \qquad (\psi\otimes\psi)\circ\Delta_{2\mathfrak{g}} = \Delta_{2\mathfrak{h}}\circ\psi.
\end{aligned}$$

Theorem 3.15. *Let* $(\mathfrak{g}, \Delta_1, \Delta_2)$ *be a compatible Lie bialgebra and* $\alpha : \mathfrak{g} \to \mathfrak{g}$ *be an invertible compatible Lie bialgebra morphism. Define two linear maps* $\Delta_1 \circ \alpha^{-1}, \Delta_2 \circ \alpha^{-1} : \mathfrak{g} \to \wedge^2\mathfrak{g}$ *by*

$$\langle \Delta_{1(2)} \circ \alpha^{-1}(x), \xi \wedge \eta\rangle = \langle x, (\alpha^{-1})^* \circ [\xi,\eta]^{1(2)}_{\mathfrak{g}^*}\rangle, \tag{3.19}$$

i.e., $\Delta_{1(2)} \circ \alpha^{-1}$ *is the dual of the Hom-Lie algebra structure* $[\cdot,\cdot]^{1(2)}_{(\alpha^{-1})^*} = (\alpha^{-1})^* \circ [\xi,\eta]^{1(2)}_{\mathfrak{g}^*} : \wedge^2\mathfrak{g}^* \to \mathfrak{g}^*$*. Then* $(\mathfrak{g}, \Delta_1 \circ \alpha^{-1}, \Delta_2 \circ \alpha^{-1})$ *is a compatible Hom-Lie bialgebra.*

Proof. Let $(\mathfrak{g}, \Delta_1, \Delta_2)$ be a compatible Lie bialgebra and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be an invertible compatible Lie bialgebra morphism. Then $(\mathfrak{g}, \alpha\circ[\cdot,\cdot]^1, \alpha\circ[\cdot,\cdot]^2, \alpha)$ is a compatible Hom-Lie algebra and it is easy to show that $\alpha^{-1} : \mathfrak{g} \to \mathfrak{g}$ is also a compatible Lie bialgebra morphism. Using Theorem.2.7, we can show that Eqs. (3.16), (3.17) hold. Next, we need to show that Eq. (3.18) holds, then for any $x, y \in \mathfrak{g}$, we have

$$\begin{aligned}
&\Delta_1\circ\alpha^{-1}(\alpha\circ[x,y]^2) + \Delta_2\circ\alpha^{-1}(\alpha\circ[x,y]^1) - ad_{2\mathfrak{g}}(\alpha^{-1}(x))\Delta_1\circ\alpha^{-1}(y)\\
&+ ad_{2\mathfrak{g}}(\alpha^{-1}(y))\Delta_1\circ\alpha^{-1}(x) - ad_{1\mathfrak{g}}(\alpha^{-1}(x))\Delta_2\circ\alpha^{-1}(y) + ad_{1\mathfrak{g}}(\alpha^{-1}(y))\Delta_2\circ\alpha^{-1}(x)\\
=&\Delta_1([x,y]^2) + \Delta_2([x,y]^1) - \alpha\circ[\alpha^{-1}(x), \Delta_1\circ\alpha^{-1}(y)]^2\\
&+ \alpha\circ[\alpha^{-1}(y), \Delta_1\circ\alpha^{-1}(x)]^2 - \alpha\circ[\alpha^{-1}(x), \Delta_2\circ\alpha^{-1}(y)]^1 + \alpha\circ[\alpha^{-1}(y), \Delta_2\circ\alpha^{-1}(x)]^1
\end{aligned}$$

$$
\begin{aligned}
=&\Delta_1([x,y]^2)+\Delta_2([x,y]^1)-\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_1^1)]^2\wedge y_2^1-y_1^1\wedge\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_2^1)]\\
&+\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_1^1)]^2\wedge x_2^1+x_1^1\wedge\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_2^1)]^2\\
&-\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_1^2)]^1\wedge y_2^2-y_1^2\wedge\alpha\circ[\alpha^{-1}(x),\alpha^{-1}(y_2^2)]^1\\
&+\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_1^2)]^1\wedge x_2^2+x_1^2\wedge\alpha\circ[\alpha^{-1}(y),\alpha^{-1}(x_2^2)]^1\\
=&\Delta_1([x,y]^2)+\Delta_2([x,y]^1)-ad_{2\mathfrak{g}}(x)\Delta_1(y)+ad_{2\mathfrak{g}}(y)\Delta_1(x)-ad_{1\mathfrak{g}}(x)\Delta_2(y)+ad_{1\mathfrak{g}}(y)\Delta_2(x)=0.
\end{aligned}
$$

Hence the result. □

Proposition 3.16. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha_{\mathfrak{g}}^{-1})^*)$ *be two compatible Hom-Lie algebras. Then Eqs.* (3.16)-(3.18) *are equivalent, respectively, to*

$$
\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})=ad_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\xi))\Delta^1_{\mathfrak{g}^*}(\eta)-ad_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\eta))\Delta^1_{\mathfrak{g}^*}(\xi), \tag{3.20}
$$

$$
\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})=ad_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\xi))\Delta^2_{\mathfrak{g}^*}(\eta)-ad_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\eta))\Delta^2_{\mathfrak{g}^*}(\xi), \tag{3.21}
$$

$$
\begin{aligned}
\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})+\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})=&\,ad_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\xi))\Delta^1_{\mathfrak{g}^*}(\eta)-ad_{2\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\eta))\Delta^1_{\mathfrak{g}^*}(\xi)\\
&+ad_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\xi))\Delta^2_{\mathfrak{g}^*}(\eta)-ad_{1\mathfrak{g}^*}((\alpha_{\mathfrak{g}})^*(\eta))\Delta^2_{\mathfrak{g}^*}(\xi),
\end{aligned} \tag{3.22}
$$

where $ad_{1\mathfrak{g}^*}(\xi)(\eta)=[\xi,\eta]^1_{\mathfrak{g}^*}$, $ad_{2\mathfrak{g}^*}(\xi)(\eta)=[\xi,\eta]^2_{\mathfrak{g}^*}$ *and* $\Delta^1_{\mathfrak{g}^*}$, $\Delta^2_{\mathfrak{g}^*}$ *are given by Eq.*(3.15).

Proof. We can easily show that Eq.(3.16) (resp Eq.(3.17)) are equivalent to Eq. (3.20) (resp Eq.(3.21)). Now, we need to show that Eq.(3.18) is equivalent to Eq.(3.22). In fact, first for all $x\in\mathfrak{g}$, $\xi_1,\cdots,\xi_k\in\mathfrak{g}^*$, $\theta=\xi_1\wedge\cdots\wedge\xi_k$, we have

$$
(\alpha_{\mathfrak{g}}^{-1})^*ad^*_{1,2\mathfrak{g}}(x)(\theta)=ad^*_{1,2\mathfrak{g}}(\alpha_{\mathfrak{g}}(x))(\theta)(\alpha_{\mathfrak{g}}^{-1})^*(\theta), \tag{3.23}
$$

$$
ad^*_{1,2\mathfrak{g}}(x)(\xi_1\wedge\cdots\wedge\xi_k)=\sum_{k=1}^{n}\alpha^*_{\mathfrak{g}}(\xi_1)\wedge\cdots\wedge ad^*_{1,2\mathfrak{g}}(x)(\xi_i)\wedge\cdots\wedge\alpha^*_{\mathfrak{g}}(\xi_k). \tag{3.24}
$$

Then for all $x,y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$, by $\langle ad^*_{1,2\mathfrak{g}^*}(\xi)(x),\eta\rangle=\langle x,ad_{1,2\mathfrak{g}^*}(\xi)(\eta)\rangle$, we have

$$
\begin{aligned}
&\langle-\Delta^1_{\mathfrak{g}}([x,y]^2_{\mathfrak{g}})-\Delta^2_{\mathfrak{g}}([x,y]^1_{\mathfrak{g}})+ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta^1_{\mathfrak{g}}(y)\\
&-ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta^1_{\mathfrak{g}}(x)+ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\Delta^2_{\mathfrak{g}}(y)-ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\Delta^2_{\mathfrak{g}}(x),\xi\wedge\eta\rangle\\
&=-\langle[x,y]^2_{\mathfrak{g}},[\xi,\eta]^1_{\mathfrak{g}^*}\rangle-\langle[x,y]^1_{\mathfrak{g}},[\xi,\eta]^2_{\mathfrak{g}^*}\rangle-\langle\Delta^1_{\mathfrak{g}}(y),ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi\wedge\eta)\rangle\\
&+\langle\Delta^1_{\mathfrak{g}}(x),ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi\wedge\eta)\rangle-\langle\Delta^2_{\mathfrak{g}}(y),ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi\wedge\eta)\rangle\\
&+\langle\Delta^2_{\mathfrak{g}}(x),ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi\wedge\eta)\rangle\\
&=-\langle x\wedge y,\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})\rangle-\langle x\wedge y,\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})\rangle\\
&-\langle\Delta^1_{\mathfrak{g}}(y),ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi)\wedge\alpha^*_{\mathfrak{g}}(\eta)+\alpha^*(\xi)\wedge ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\eta)\rangle\\
&+\langle\Delta^1_{\mathfrak{g}}(x),ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi)\wedge\alpha^*_{\mathfrak{g}}(\eta)+\alpha^*_{\mathfrak{g}}(\xi)\wedge ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\eta)\rangle\\
&-\langle\Delta^2_{\mathfrak{g}}(y),ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi)\wedge\alpha^*_{\mathfrak{g}}(\eta)+\alpha^*_{\mathfrak{g}}(\xi)\wedge ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\eta)\rangle\\
&+\langle\Delta^2_{\mathfrak{g}}(x),ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi)\wedge\alpha^*_{\mathfrak{g}}(\eta)+\alpha^*_{\mathfrak{g}}(\xi)\wedge ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\eta)\rangle\\
&=-\langle x\wedge y,\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})\rangle-\langle x\wedge y,\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})\rangle\\
&+\langle y,ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\circ ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi)\rangle-\langle y,ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\circ ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\eta)\rangle\\
&-\langle x,ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\circ ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi)\rangle+\langle x,ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\circ ad^*_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\eta)\rangle\\
&+\langle y,ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\circ ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\xi)\rangle-\langle y,ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\circ ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))(\eta)\rangle\\
&-\langle x,ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\circ ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\xi)\rangle+\langle x,ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\circ ad^*_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))(\eta)\rangle\\
&=-\langle x\wedge y,\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})\rangle-\langle x\wedge y,\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})\rangle\\
&+\langle ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\circ ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(y),\xi\rangle-\langle ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\circ ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(y),\eta\rangle\\
&-\langle ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\circ ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(x),\xi\rangle+\langle ad_{2\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\circ ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(x),\eta\rangle
\end{aligned}
$$

$$+\langle ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\circ ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(y),\xi\rangle-\langle ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(x))\circ ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(y),\eta\rangle$$
$$-\langle ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\circ ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(x),\xi\rangle+\langle ad_{1\mathfrak{g}}((\alpha_{\mathfrak{g}})^{-1}(y))\circ ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(x),\eta\rangle$$
$$=-\langle x\wedge y,\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})\rangle-\langle x\wedge y,\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})\rangle$$
$$+\langle ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(x\wedge y),\Delta^2_{\mathfrak{g}^*}(\xi)\rangle-\langle ad^*_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(x\wedge y),\Delta^2_{\mathfrak{g}^*}(\eta)\rangle$$
$$+\langle ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))(x\wedge y),\Delta^1_{\mathfrak{g}^*}(\xi)\rangle-\langle ad^*_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))(x\wedge y),\Delta^1_{\mathfrak{g}^*}(\eta)\rangle$$
$$=\langle x\wedge y,-\Delta^2_{\mathfrak{g}^*}([\xi,\eta]^1_{\mathfrak{g}^*})-\Delta^1_{\mathfrak{g}^*}([\xi,\eta]^2_{\mathfrak{g}^*})-ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\Delta^2_{\mathfrak{g}^*}(\xi)$$
$$+ad_{1\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\Delta^2_{\mathfrak{g}^*}(\eta)-ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\eta))\Delta^1_{\mathfrak{g}^*}(\xi)+ad_{2\mathfrak{g}^*}(\alpha^*_{\mathfrak{g}}(\xi))\Delta^1_{\mathfrak{g}^*}(\eta)\rangle,$$

which implies that (3.18) and (3.22) are equivalent. □

Combining Theorems 3.10 and 3.12 and Proposition 3.13, we have the following conclusion:

Theorem 3.17. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha_{\mathfrak{g}})$ *and* $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha_{\mathfrak{g}}^{-1})^*)$ *be two compatible Hom-Lie algebras. Then the following conditions are equivalent*

1. $(\mathfrak{g},\Delta_1,\Delta_2)$ *is a compatible Hom-Lie bialgebra.*
2. $(\mathfrak{g}\oplus\mathfrak{g}^*;\mathfrak{g},\mathfrak{g}^*)$ *is a standard Manin triple of compatible Hom-Lie algebra associated to the invariant bilinear form given by Eq.*(3.12)*.*
3. $(\mathfrak{g},\mathfrak{g}^*,ad^\star_{1\mathfrak{g}},ad^\star_{2\mathfrak{g}},ad^\star_{1\mathfrak{g}^*},ad^\star_{2\mathfrak{g}^*})$ *is a matched pair of compatible Hom-Lie algebras.*

4. Coboundary and triangular compatible Hom-Lie bialgebras

For any $r=r_1\wedge r_2\in\wedge^2\mathfrak{g}$, the induced skew-symmetric linear map $r^\sharp:\mathfrak{g}^*\to\mathfrak{g}$ is defined by

$$\langle r^\sharp(\xi),\eta\rangle=\langle r,\xi\wedge\eta\rangle. \tag{4.1}$$

Definition 4.1. A compatible Hom-Lie bialgebra $(\mathfrak{g},\Delta_1,\Delta_2)$ is called coboundary if there exists an element $r\in\wedge^2\mathfrak{g}$ such that for any $x\in\mathfrak{g}$,

$$\Delta_1(x)=ad_{1\mathfrak{g}}(\alpha^{-2}(x))r=[\alpha^{-2}(x),r_1]^1_{\mathfrak{g}}\wedge\alpha(r_2)+\alpha(r_1)\wedge[\alpha^{-2}(x),r_2]^1_{\mathfrak{g}}, \tag{4.2}$$
$$\Delta_2(x)=ad_{2\mathfrak{g}}(\alpha^{-2}(x))r=[\alpha^{-2}(x),r_1]^2_{\mathfrak{g}}\wedge\alpha(r_2)+\alpha(r_1)\wedge[\alpha^{-2}(x),r_2]^2_{\mathfrak{g}}. \tag{4.3}$$

Remark 4.1. It is straightforward to show that a compatible Hom-Lie bialgebra $(\mathfrak{g},\Delta_1,\Delta_2)$ is coboundary if and only if for any $k_1,k_2\in\mathbb{K}$, the Hom-Lie bialgebra structure $(\mathfrak{g},k_1\Delta_1+k_2\Delta_2)$ defined by the two Hom-Lie algebras $(\mathfrak{g},k_1[\cdot,\cdot]^1_{\mathfrak{g}}+k_2[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ and $(\mathfrak{g}^*,k_1[\cdot,\cdot]^1_{\mathfrak{g}^*}+k_2[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ is coboundary, that is,

$$(k_1\Delta_1+k_2\Delta_2)(x)=(k_1ad_{1\mathfrak{g}}+k_2ad_{2\mathfrak{g}})(\alpha^{-2}(x))r.$$

Remark 4.2. A coboundary compatible Hom-Lie bialgebra in which $\alpha=Id_{\mathfrak{g}}$ is exactly a coboundary compatible Lie bialgebra, as defined in [30].

Theorem 4.3. *Let* $\alpha:\mathfrak{g}\to\mathfrak{g}$ *be a bijective compatible Lie bialgebra morphism and* $r\in\wedge^2\mathfrak{g}$ *makes* $(\mathfrak{g},\Delta_1,\Delta_2)$ *a coboundary compatible Lie bialgebra. Then* $(\mathfrak{g},\Delta_1\circ\alpha^{-1},\Delta_2\circ\alpha^{-1})$ *is a coubandary compatible Hom-Lie bialgebra with respect to* $\tilde{r}=\alpha^{-1}(r_1)\wedge\alpha^{-1}(r_2)$*.*

Proof. Straightforward by using Theorem.2.10. □

Lemma 4.4. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ *be a compatible Hom-Lie algebra and* $r\in\wedge^2\mathfrak{g}$*. Suppose the linear maps* $\Delta_1,\Delta_2:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$ *are defined by Eqs.* (4.2) *and* (4.3) *respectively. Then* Δ_1 *and* Δ_2 *satisfy Eqs.* (3.16), (3.17) *and* (3.18)*.*

Lemma 4.5. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ *be a compatible Hom-Lie algebra. For some* $r\in\wedge^2\mathfrak{g}$*, let us consider* $\Delta_1,\Delta_2:\mathfrak{g}\to\mathfrak{g}\wedge\mathfrak{g}$ *defined by Eqs.* (4.2) *and* (4.3) *respectively satisfying*

$$r^\sharp\circ(\alpha^{-1})^*=\alpha\circ r^\sharp. \tag{4.4}$$

Then for all $\xi,\eta\in\mathfrak{g}^*$*, we have*

$$\begin{aligned}[\xi,\eta]^1_{\mathfrak{g}^*}&=ad^\star_{1\mathfrak{g}}(r^\sharp(\xi))(\eta)-ad^\star_{1\mathfrak{g}}(r^\sharp(\eta))(\xi)\\&=ad^*_{1\mathfrak{g}}(\alpha(r^\sharp(\xi)))((\alpha^{-2})^*(\eta))-ad^*_{1\mathfrak{g}}(\alpha(r^\sharp(\eta)))((\alpha^{-2})^*(\xi)),\end{aligned}\tag{4.5}$$

$$\begin{aligned}[\xi,\eta]^2_{\mathfrak{g}^*}&=ad^\star_{2\mathfrak{g}}(r^\sharp(\xi))(\eta)-ad^\star_{2\mathfrak{g}}(r^\sharp(\eta))(\xi)\\&=ad^*_{2\mathfrak{g}}(\alpha(r^\sharp(\xi)))((\alpha^{-2})^*(\eta))-ad^*_{2\mathfrak{g}}(\alpha(r^\sharp(\eta)))((\alpha^{-2})^*(\xi)),\end{aligned}\tag{4.6}$$

where $[\cdot,\cdot]^1_{\mathfrak{g}^*}$*,* $[\cdot,\cdot]^2_{\mathfrak{g}^*}$ *are defined by* (3.15)*. Furthermore, we have*

$$[r^\sharp\circ\alpha^*(\xi),r^\sharp\circ\alpha^*(\eta)]^1_{\mathfrak{g}}-r^\sharp\circ\alpha^*([\xi,\eta]^1_{\mathfrak{g}^*})=\frac{1}{2}[r,r]^1_{\mathfrak{g}}(\xi,\eta).\tag{4.7}$$

$$[r^\sharp\circ\alpha^*(\xi),r^\sharp\circ\alpha^*(\eta)]^2_{\mathfrak{g}}-r^\sharp\circ\alpha^*([\xi,\eta]^2_{\mathfrak{g}^*})=\frac{1}{2}[r,r]^2_{\mathfrak{g}}(\xi,\eta).\tag{4.8}$$

Theorem 4.6. *Let* $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ *be a compatible Hom-Lie algebra and* $r\in\wedge^2\mathfrak{g}$ *satisfying* (4.4)*. Then* $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ *is a compatible Hom-Lie algebra, where* $[\cdot,\cdot]^1_{\mathfrak{g}^*}$ *and* $[\cdot,\cdot]^2_{\mathfrak{g}^*}$ *are defined respectively by* (4.5) *and* (4.6)*, if and only if*

$$ad_{1\mathfrak{g}}(x)[r,r]^1_{\mathfrak{g}}=0,\tag{4.9}$$

$$ad_{2\mathfrak{g}}(x)[r,r]^2_{\mathfrak{g}}=0,\tag{4.10}$$

$$ad_{1\mathfrak{g}}(x)[r,r]^2_{\mathfrak{g}}+ad_{2\mathfrak{g}}(x)[r,r]^1_{\mathfrak{g}}=0.\tag{4.11}$$

Under these conditions, $(\mathfrak{g},\Delta_1,\Delta_2)$ *is a coboundary compatible Hom-Lie bialgebra.*

Proof. By a direct computation, $(\alpha^{-1})^*$ is a compatible Hom-Lie algebra automorphism of $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ if and only if (4.4) holds. In fact:

$$\begin{aligned}[(\alpha^{-1})^*(\xi),(\alpha^{-1})^*(\eta)]^1_{\mathfrak{g}^*}&=ad^\star_{1\mathfrak{g}}(r^\sharp\circ(\alpha^{-1})^*(\xi)((\alpha^{-1})^*(\eta)-ad^\star_{1\mathfrak{g}}(r^\sharp\circ(\alpha^{-1})^*(\eta)((\alpha^{-1})^*(\xi)\\&=ad^\star_{1\mathfrak{g}}(\alpha\circ r^\sharp(\xi)((\alpha^{-1})^*(\eta)-ad^\star_{1\mathfrak{g}}(\alpha\circ r^\sharp(\eta)((\alpha^{-1})^*(\xi)\\&=(\alpha^{-1})^*ad^\star_{1\mathfrak{g}}(r^\sharp(\xi)(\eta)-(\alpha^{-1})^*ad^\star_{1\mathfrak{g}}(r^\sharp(\eta)(\xi)\\&=(\alpha^{-1})^*([\xi,\eta]^1_{\mathfrak{g}^*}),\end{aligned}$$

and similarly we can check that $[(\alpha^{-1})^*(\xi),(\alpha^{-1})^*(\eta)]^2_{\mathfrak{g}^*}=(\alpha^{-1})^*([\xi,\eta]^2_{\mathfrak{g}^*})$. Next, we can easily prove that $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},(\alpha^{-1})^*)$ and $(\mathfrak{g}^*,[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ are two Hom-Lie algebras respectively if and only if (4.9) and (4.10) hold respectively. Now, we need to show that the compatibility condition of the compatible Hom-Lie algebra $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ hold if and only if (4.11) holds. In fact, by Lemma 4.5 and the fact that $(\mathfrak{g}^*,ad^\star_{1\mathfrak{g}},ad^\star_{2\mathfrak{g}},(\alpha^{-1})^*)$ is the dual representation of the representation $(\mathfrak{g},ad_{1\mathfrak{g}},ad_{2\mathfrak{g}},\alpha)$, we have

$$\begin{aligned}&[[\xi,\eta]^1_{\mathfrak{g}^*},(\alpha^{-1})^*(\delta)]^2_{\mathfrak{g}^*}+c.p.+[[\xi,\eta]^2_{\mathfrak{g}^*},(\alpha^{-1})^*(\delta)]^1_{\mathfrak{g}^*}+c.p.(\xi,\eta,\gamma).\\=&ad^\star_{2\mathfrak{g}}(r^\sharp([\xi,\eta]^1_{\mathfrak{g}^*})(\alpha^{-1})^*(\delta)-ad^\star_{2\mathfrak{g}}(\alpha\circ r^\sharp(\xi))ad^\star_{1\mathfrak{g}}(r^\sharp(\eta))(\delta)+ad^\star_{2\mathfrak{g}}(\alpha\circ r^\sharp(\eta))ad^\star_{1\mathfrak{g}}(r^\sharp(\xi))(\delta)\\&+ad^\star_{1\mathfrak{g}}(r^\sharp([\xi,\eta]^2_{\mathfrak{g}^*}(\alpha^{-1})^*)(\delta)-ad^\star_{1\mathfrak{g}}(\alpha\circ r^\sharp(\xi))ad^\star_{2\mathfrak{g}}(r^\sharp(\eta))(\delta)+ad^\star_{1\mathfrak{g}}(\alpha\circ r^\sharp(\eta))ad^\star_{2\mathfrak{g}}(r^\sharp(\xi))(\delta)+c.p.\\=&-ad^\star_{1\mathfrak{g}}\Big([r^\sharp(\xi),r^\sharp(\eta)]^2_{\mathfrak{g}}-r^\sharp([\xi,\eta]^2_{\mathfrak{g}^*})\Big)(\alpha^{-1})^*(\delta)-ad^\star_{2\mathfrak{g}}\Big([r^\sharp(\xi),r^\sharp(\eta)]^1_{\mathfrak{g}}-r^\sharp([\xi,\eta]^1_{\mathfrak{g}^*})\Big)(\alpha^{-1})^*(\delta)+c.p.\end{aligned}$$

For all $x\in\mathfrak{g}$, we have

$$\begin{aligned}&\langle[[\xi,\eta]^1_{\mathfrak{g}},(\alpha^{-1})^*(\delta)]^2_{\mathfrak{g}^*}+c.p+[[\xi,\eta]^2_{\mathfrak{g}^*},(\alpha^{-1})^*(\delta)]^1_{\mathfrak{g}^*}+c.p.,x\rangle\\=&-\langle ad^*_{1\mathfrak{g}}\Big([\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^2_{\mathfrak{g}}-\alpha(r^\sharp([\xi,\eta]^2_{\mathfrak{g}^*}))\Big)(\alpha^{-3})^*(\delta)\\&+ad^*_{2\mathfrak{g}}\Big([\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^1_{\mathfrak{g}}-\alpha(r^\sharp([\xi,\eta]^1_{\mathfrak{g}^*}))\Big)(\alpha^{-3})^*(\delta),x\rangle+c.p.\\=&\langle(\alpha^{-3})^*(\delta),ad^1_{\mathfrak{g}}\Big([\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^2_{\mathfrak{g}}-\alpha(r^\sharp([\xi,\eta]^2_{\mathfrak{g}^*}))\Big)(x)\end{aligned}$$

$$
\begin{aligned}
&+ ad^2_{\mathfrak{g}}\Big([\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^1_{\mathfrak{g}} - \alpha(r^\sharp([\xi,\eta]^1_{\mathfrak{g}^*}))\Big)(x)\rangle + c.p.\\
=&\langle ad^*_{1\mathfrak{g}}(x)((\alpha^{-3})^*(\delta)), [\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^2_{\mathfrak{g}} - \alpha(r^\sharp([\xi,\eta]^2_{\mathfrak{g}^*}))\rangle + c.p.\\
&+ \langle ad^*_{2\mathfrak{g}}(x)((\alpha^{-3})^*(\delta)), [\alpha(r^\sharp(\xi)),\alpha(r^\sharp(\eta))]^1_{\mathfrak{g}} - \alpha(r^\sharp([\xi,\eta]^1_{\mathfrak{g}^*}))\rangle + c.p.
\end{aligned}
$$

Thus, by Lemma 4.5 and (3.24), we have

$$
\begin{aligned}
&\langle [[\xi,\eta]^1_{\mathfrak{g}},(\alpha^{-1})^*(\delta)]^2_{\mathfrak{g}^*} + c.p + [[\xi,\eta]^2_{\mathfrak{g}^*},(\alpha^{-1})^*(\delta)]^1_{\mathfrak{g}^*} + c.p., x\rangle\\
=&\langle ad^*_{1\mathfrak{g}}(x)((\alpha^{-3})^*(\delta), \frac{1}{2}[r,r]^2_{\mathfrak{g}}((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\eta))\rangle + c.p.\\
&+ \langle ad^*_{2\mathfrak{g}}(x)((\alpha^{-3})^*(\delta), \frac{1}{2}[r,r]^1_{\mathfrak{g}}((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\eta))\rangle + c.p.\\
=&\frac{1}{2}[r,r]^2_{\mathfrak{g}}\Big((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\eta), ad^*_{1\mathfrak{g}}(x)(\alpha^{-3})^*(\delta)\Big) + c.p.\\
&+ \frac{1}{2}[r,r]^1_{\mathfrak{g}}\Big((\alpha^{-2})^*(\xi),(\alpha^{-2})^*(\eta), ad^*_{2\mathfrak{g}}(x)(\alpha^{-3})^*(\delta)\Big) + c.p.\\
=&\frac{1}{2}\langle [r,r]^2_{\mathfrak{g}}, ad^*_{1\mathfrak{g}}(x)\Big((\alpha^{-3})^*(\xi)\wedge(\alpha^{-3})^*(\eta)\wedge(\alpha^{-3})^*(\delta)\Big)\\
&+ \frac{1}{2}\langle [r,r]^1_{\mathfrak{g}}, ad^*_{2\mathfrak{g}}(x)\Big((\alpha^{-3})^*(\xi)\wedge(\alpha^{-3})^*(\eta)\wedge(\alpha^{-3})^*(\delta)\Big)\\
=&\frac{1}{2}\langle ad_{1\mathfrak{g}}(x)[r,r]^2_{\mathfrak{g}} + ad_{2\mathfrak{g}}(x)[r,r]^1_{\mathfrak{g}}, (\alpha^{-3})^*(\xi\wedge\eta\wedge\delta)\rangle,
\end{aligned}
$$

which implies that the compatibility condition is satisfied if and only if $ad_{1\mathfrak{g}}(x)[r,r]^2_{\mathfrak{g}} + ad_{2\mathfrak{g}}(x)[r,r]^1_{\mathfrak{g}} = 0$, for all $x\in\mathfrak{g}$. □

Definition 4.2. Let $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ be a compatible Hom-Lie algebra and $r\in\wedge^2\mathfrak{g}$ satisfying (4.4). The set of the following equations

$$[r,r]^1_{\mathfrak{g}} = 0, \qquad [r,r]^2_{\mathfrak{g}} = 0 \tag{4.12}$$

is called the classical Hom-Yang-Baxter equation in the compatible Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$. If r is a solution of the classical Hom-Yang-Baxter equation, a coboundary compatible Hom-Lie bialgebra is called triangular compatible Hom-Lie bialgebra.

Remark 4.7. Let $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ be a compatible Hom-Lie algebra and $r\in\wedge^2\mathfrak{g}$ satisfying (4.4). Then r is a solution of the classical Hom-Yang-Baxter equation in $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$ if and only if r is a solution classical Hom-Yang-Baxter equation in $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},\alpha)$ and $(\mathfrak{g},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$.

Corollary 4.8. *If r is a solution of the classical Hom-Yang-Baxter equation in $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$, then $r^\sharp$ is a morphism from the compatible Hom-Lie algebra $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ to the compatible Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$. That is, $r^\sharp:\mathfrak{g}^*\to\mathfrak{g}$ is a Hom-Lie algebra morphism from $(\mathfrak{g}^*,[\cdot,\cdot]^1_{\mathfrak{g}^*},(\alpha^{-1})^*)$ to $(\mathfrak{g},[\cdot,\cdot]^1_{\mathfrak{g}},\alpha)$, and a Hom-Lie algebra morphism from $(\mathfrak{g}^*,[\cdot,\cdot]^2_{\mathfrak{g}^*},(\alpha^{-1})^*)$ to $(\mathfrak{g},[\cdot,\cdot]^2_{\mathfrak{g}},\alpha)$.*

5. Hom-Nijenhuis operators and compatible Hom-Lie bialgebras

In this section, we recall the notion of a Hom-Nijenhuis Hom-Lie algebra (see [5]) and we introduce the concept of a Hom-Nijenhuis Hom-Lie coalgebra by duality. We also present some properties and the dual representation of a representation of a Hom-Nijenhuis Hom-Lie algebra in order to define Hom-Nijenhuis Hom-Lie bialgebra. A Hom-Nijenhuis operator is a linear map $N:\mathfrak{g}\to\mathfrak{g}$ satisfying $\alpha\circ N = N\circ\alpha$ and the following twisted integrability condition

$$[Nx,Ny]_{\mathfrak{g}} + N^2[\alpha^{-1}(x),\alpha^{-1}(y)]_{\mathfrak{g}} = N([N\alpha^{-1}(x),y]_{\mathfrak{g}}) + N([x,N\alpha^{-1}(y)]_{\mathfrak{g}}), \quad \text{for } x,y\in\mathfrak{g}.$$

Definition 5.1. A Hom-Nijenhuis Hom-Lie algebra is a tuple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha, N)$ consisting of a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ and a Hom-Nijenhuis operator N on $\mathfrak{g}$.

Example 5.1. Consider the 3-dimensional Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ with basis $\{e_1, e_2, e_3\}$, where the bracket and the structure map α are given by

$$[e_1, e_2]_{\mathfrak{g}} = 2e_2, \quad [e_1, e_3]_{\mathfrak{g}} = 2e_3$$
$$\alpha(e_1) = -e_1, \quad \alpha(e_2) = e_2, \text{and} \quad \alpha(e_3) = -e_3.$$

Consider the map $N : \mathfrak{g} \to \mathfrak{g}$ defined by

$$N(e_1) = e_1, \quad N(e_2) = e_2, \quad N(e_3) = -e_3.$$

Then N is a Hom-Nijenhuis operator on $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$.

Remark 5.1. Hom-Nijenhuis operators are useful to study linear deformations of a Hom-Lie algebra. If N is a Hom-Nijenhuis operator on a Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$, then there is a deformed Hom-Lie bracket on $\mathfrak{g}$ given by

$$[x, y]_{\mathfrak{g},N} = [N\alpha^{-1}(x), y]_{\mathfrak{g}} + [x, N\alpha^{-1}(y)]_{\mathfrak{g}} - N[\alpha^{-1}(x), \alpha^{-1}(y)]_{\mathfrak{g}}, \quad \text{for } x, y \in \mathfrak{g}. \tag{5.1}$$

In other words $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},N}, \alpha)$ is a Hom-Lie algebra.

Theorem 5.2. *With the above notations, the quadruple* $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{g},N}, \alpha)$ *is a compatible Hom-Lie algebra.*

Proof. Let $((\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha), N)$ be a Hom-Nijenhuis Hom-Lie algebra. Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},N}, \alpha)$ is a Hom-Lie algebra and for any $x, y, z \in \mathfrak{g}$, we have

$$\begin{aligned}
&[[x, y]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g},N} + c.p.(x, y, z) + [[x, y]_{\mathfrak{g},N}, \alpha(z)]_{\mathfrak{g}} + c.p.(x, y, z) \\
=&[N\alpha^{-1}[x, y]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, N\alpha^{-1}(\alpha(z))]_{\mathfrak{g}} - N[\alpha^{-1}[x, y]_{\mathfrak{g}}, \alpha^{-1}(\alpha(z))]_{\mathfrak{g}} + c.p. \\
&+ [[N\alpha^{-1}(x), y]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + [[x, N\alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} - [N[\alpha^{-1}(x), \alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + c.p. \\
=&[N[\alpha^{-1}(x), \alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + [[x, y]_{\mathfrak{g}}, \alpha(N\alpha^{-1}(z))]_{\mathfrak{g}} - N[[\alpha^{-1}(x), \alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(\alpha^{-1}(z))]_{\mathfrak{g}} + c.p. \\
&+ [[N\alpha^{-1}(x), y]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + [[x, N\alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} - [N[\alpha^{-1}(x), \alpha^{-1}(y)]_{\mathfrak{g}}, \alpha(z)]_{\mathfrak{g}} + c.p. \\
=&0.
\end{aligned}$$

Then the condition (2.13) in Definition 2.6 holds, which implies that $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, [\cdot,\cdot]_{\mathfrak{g},N}, \alpha)$ is a compatible Hom-Lie algebra. This finishes the proof. □

Definition 5.2. A Hom-Nijenhuis Hom-Lie coalgebra is a tuple $(\mathfrak{g}, \Delta, \alpha, Q)$ consisting of a regular Hom-Lie coalgebra $(\mathfrak{g}, \Delta, \alpha)$ and a Hom-Nijenhuis operator Q on $\mathfrak{g}$, i.e., a linear map $Q : \mathfrak{g} \to \mathfrak{g}$ satisfying $\alpha \circ Q = Q \circ \alpha$ and

$$(Q \otimes Q) \circ \Delta = \big((Q\alpha^{-1} \otimes id) \circ \Delta\big) \circ Q + \big((id \otimes Q\alpha^{-1}) \circ \Delta\big) \circ Q - ((\alpha^{-1} \otimes \alpha^{-1}) \circ \Delta) \circ Q^2. \tag{5.2}$$

Remark 5.3. If Q is a Hom-Nijenhuis operator on a Hom-Lie coalgebra $(\mathfrak{g}, \Delta, \alpha)$, then there is a deformed Hom-Lie cobracket on $\mathfrak{g}$ given by

$$\Delta_Q = (Q\alpha^{-1} \otimes id) \circ \Delta + (id \otimes Q\alpha^{-1}) \circ \Delta - ((\alpha^{-1} \otimes \alpha^{-1}) \circ \Delta) \circ Q. \tag{5.3}$$

Then $(\mathfrak{g}, \Delta_Q, \alpha)$ is a Hom-Lie coalgebra.

By a straightforward computation, we have the following result

Theorem 5.4. *With the above notations, the quadruple* $(\mathfrak{g}, \Delta, \Delta_Q, \alpha)$ *is a compatible Hom-Lie coalgebra.*

Now, we introduce the notion of a representation of Hom-Nijenhuis Hom-Lie algebras.

Definition 5.3. A representation of a Hom-Nijenhuis Hom-Lie algebra $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha, N)$ is a quadruple (V, ρ, β, η), where (V, ρ, β) is a representation of $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha)$ and $\eta : V \to V$ is a linear map such that for all $x \in \mathfrak{g}$, $v \in V$

$$\rho(N(x))(\eta(v)) + \eta^2(\rho(\alpha^{-1}(x))(\beta^{-1}(v))) = \eta(\rho(N\alpha^{-1}(x))v) + \eta(\rho(x)(\eta(\beta^{-1}(v)))). \tag{5.4}$$

Proposition 5.5. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ be a Hom-Nijenhuis Hom-Lie algebra, (V,ρ,β) be a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ and $\eta: V\to V$ be a linear map. Define a linear map on $\mathfrak{g}\oplus V$ by*

$$N_{\mathfrak{g}\oplus V}:\mathfrak{g}\oplus V\to\mathfrak{g}\oplus V,\quad N_{\mathfrak{g}\oplus V}(x+v)=N(x)+\eta(u). \tag{5.5}$$

Then together with the semi-direct product defined by Eqs. (2.6) and (2.7), $(\mathfrak{g}\oplus V,[\cdot,\cdot]_{\mathfrak{g}\oplus V},\alpha\oplus\beta,N_{\mathfrak{g}\oplus V})$ is a Hom-Nijenhuis Hom-Lie algebra if and only if (V,ρ,β,η) is a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$. The resulting Hom-Nijenhuis Hom-Lie algebra is called the semi-direct product of $((\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha),N)$ by its representation (V,ρ,β,η).

Lemma 5.6. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ be a Hom-Nijenhuis Hom-Lie algebra, (V,ρ,β) be a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ and $\eta: V\to V$ be a linear map. Then the quadruple $(V^*,\rho^\star,(\beta^{-1})^*,\eta^*)$ is a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ if and only if for all $x\in\mathfrak{g}$, $v\in V$, the linear map η satisfies*

$$\rho(N\alpha^{-1}(x))(\eta(v))+\eta\rho(\alpha^{-1}(x))(\eta(\beta^{-1}(v))-\eta(\rho(Nx)(v))-(\rho(\alpha^{-2}(x))(\eta^2(\beta^{-1}(v)))=0. \tag{5.6}$$

Proof. Since $(V^*,\rho^\star,(\beta^{-1})^*)$ is a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$, we just need to determine when η satisfies the equation (5.4) where ρ,β and η are replaced by $\rho^\star,(\beta^{-1})^*$ and η^* respectively. For all $x\in\mathfrak{g}$, $v\in V$ and $v^*\in V^*$, we have

$$\begin{aligned}
&\langle\rho^\star(N(x))(\eta^*(V^*))+(\eta^*)^2(\rho^\star(\alpha^{-1}(x))(\beta^{-1})(V^*))\\
&-(\eta^*)(\rho^\star(N\alpha^{-1}(x))(V^*))-(\eta^*)(\rho^\star(x)(\eta^*((\beta^{-1})(V^*))),v\rangle\\
\overset{(2.18)+(1.2)}{=}\;&\langle v^*,-\eta(\rho(N\alpha^{-1}(x))(\beta^{-2}(v)))-\beta^{-1}(\rho(\alpha^{-2}(x))(\beta^{-2}(\eta^2(v))))\\
&+\rho(N\alpha^{-2}(x))(\beta^{-2}(\eta(v)))+\beta^{-1}(\eta(\rho(\alpha^{-1}(x))(\beta^{-2}(\eta(v)))))\rangle\\
\overset{(2.4)}{=}\;&\langle v^*,\rho(N\alpha^{-1}(x))(\eta(v))+\eta\rho(\alpha^{-1}(x))(\eta(\beta^{-1}(v))-\eta(\rho(Nx)(v))\\
&-(\rho(\alpha^{-2}(x))(\eta^2(\beta^{-1}(v)))\rangle.
\end{aligned}$$

This finishes the proof. □

Definition 5.4. Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ be a Hom-Nijenhuis Hom-Lie algebra, (V,ρ,β) be a representation of $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha)$ and $\eta: V\to V$ be a linear map. If the condition (5.7) is satisfied, we say that η is admissible to the Hom-Nijenhuis Hom-Lie algebra $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ on (V,ρ,β).

Lemma 5.7. *Let $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha,N)$ be a Hom-Nijenhuis Hom-Lie algebra. A linear operator Q on $\mathfrak{g}$ is admissible to $(\mathfrak{g},[\cdot,\cdot],\alpha,N)$ if and only if the following condition holds, for all $x,y\in\mathfrak{g}$,*

$$[N\alpha^{-1}(x),Q(y)]_{\mathfrak{g}}+Q([\alpha^{-1}(x),Q\alpha^{-1}(y)]_{\mathfrak{g}})-Q([N(x),y]_{\mathfrak{g}})-[\alpha^{-2}(x),Q^2\alpha^{-1}(y)]_{\mathfrak{g}}=0. \tag{5.7}$$

Proof. We just replace η, ρ and β by Q, ad and α respectively in equation (5.7). □

Lemma 5.8. *Let $((\mathfrak{g},\Delta,\alpha),Q)$ be a Hom-Nijenhuis Hom-Lie coalgebra. A linear operator N on $\mathfrak{g}$ is admissible to $((\mathfrak{g},\Delta,\alpha),Q)$ if and only if the following condition holds*

$$(Q\alpha^{-1}\otimes N)\circ\Delta+((\alpha^{-1}\otimes N\alpha^{-1})\circ\Delta)\circ N-((Q\otimes id)\circ\Delta)\circ N-(\alpha^{-2}\otimes N^2\alpha^{-1})\circ\Delta=0. \tag{5.8}$$

Definition 5.5. The tuple $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\Delta,\alpha,N,Q)$ consisting of a vector space $\mathfrak{g}$ and linear maps

$$[\cdot,\cdot]_{\mathfrak{g}}:\wedge^2\mathfrak{g}\to\mathfrak{g},\quad \Delta:\mathfrak{g}\to\mathfrak{g}\otimes\mathfrak{g},\quad \alpha,Q,N:\mathfrak{g}\to\mathfrak{g},$$

is called a Hom-Nijenhuis Hom-Lie bialgebra if:

(1) the quadruple $(\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\Delta,\alpha)$ is a Hom-Lie bialgebras,
(2) Q is admissible to the Hom-Nijenhuis Hom-Lie algebra $((\mathfrak{g},[\cdot,\cdot]_{\mathfrak{g}},\alpha),N)$, that is the condition (5.7) holds,

(3) N is admissible to the Hom-Nijenhuis Hom-Lie coalgebra $((\mathfrak{g}, \Delta, \alpha^{-1}), Q)$, that is, we have

$$(Q\alpha \otimes N) \circ \Delta + ((\alpha \otimes N\alpha) \circ \Delta) \circ N - ((Q \otimes id) \circ \Delta) \circ N - (\alpha^2 \otimes N^2\alpha) \circ \Delta = 0. \quad (5.9)$$

Further the definition of a Hom-Nijenhuis Hom-Lie bialgebra can be rephrased as

Proposition 5.9. *The quadruple $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \Delta, \alpha, N, Q)$ is a Hom-Nijenhuis Hom-Lie bialgebra if and only if it satisfies the conditions $(1)-(2)$ in Definition 5.5 and that Q and N^* are admissible to Hom-Nijenhuis Hom-Lie algebras $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha, N)$ and $(\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*}, (\alpha^{-1})^*, Q^*)$ respectively.*

Proof. By Lemma 5.7, the admissibility of Q to the Hom-Nijenhuis Hom-Lie algebra $((\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \alpha), N)$ determine the equation (5.7), Furthermore, for any $x \in \mathfrak{g}$ and $\xi, \nu \in \mathfrak{g}^*$, we have

$$\langle (Q\alpha \otimes N) \circ \Delta(x) + ((\alpha \otimes N\alpha) \circ \Delta) \circ N(x) - ((Q \otimes id) \circ \Delta) \circ N(x) - (\alpha^2 \otimes N^2\alpha) \circ \Delta(x), \xi \otimes \nu \rangle$$
$$= \langle x, [Q^*\alpha^*(\xi), N^*(\nu)]_{\mathfrak{g}^*} + N^*([\alpha^*(\xi), N^*\alpha^*(\nu)]_{\mathfrak{g}^*}) - N^*([Q^*(\xi), \nu]_{\mathfrak{g}^*}) - [(\alpha^*)^2(\xi), (N^*)^2\alpha^*(\nu)]_{\mathfrak{g}^*} \rangle.$$

Therefore, (5.9) leads to

$$[Q^*\alpha^*(\xi), N^*(\nu)]_{\mathfrak{g}^*} + N^*([\alpha^*(\xi), N^*\alpha^*(\nu)]_{\mathfrak{g}^*}) - N^*([Q^*(\xi), \nu]_{\mathfrak{g}^*}) - [(\alpha^*)^2(\xi), (N^*)^2\alpha^*(\nu)]_{\mathfrak{g}^*} = 0,$$

which means that N^* is admissible to the Hom-Nijenhuis Hom-Lie algebras $((\mathfrak{g}^*, [\cdot,\cdot]_{\mathfrak{g}^*}, (\alpha^{-1})^*, Q^*)$. This ends the proof. □

Combining Theorem 5.2 and Proposition 5.9, we have the following result:

Theorem 5.10. *Let $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g}}, \Delta, \alpha, N, Q)$ be a Hom-Nijenhuis Hom-Lie bialgebra. Then $(\mathfrak{g}, [\cdot,\cdot]_{\mathfrak{g},N}, \mathfrak{g}, \Delta_Q, \alpha)$ is a Hom-Lie bialgebra, where $[\cdot,\cdot]_{\mathfrak{g},N}$ and Δ_Q is defined in (5.1) and (5.3) respectively. Moreover, the tow Hom-Lie bialgebras are complatible.*

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References

[1] F. Ammar, Z. Ejbehi and A. Makhlouf, *Cohomology and deformations of Hom-algebras*, J. Lie Theory **21**(4), 813–836, 2011.

[2] C. Bai, Y. Tao and L. Guo, *Another approach to Hom-Lie bialgebras via Manin triples*, Commun. Algebra **48**(7), 3109–3132, 2020.

[3] S. Benayadi and A. Makhlouf, *Hom-Lie algebras with symmetric invariant nondegenerate bilinear forms*, J. Geom. Phys. **76**, 38–60, 2014.

[4] M. Bordemann, O. Elchinger and A. Makhlouf, *Twisting Poisson algebras, coPoisson algebras and quantization*, Trav. Math. **20**, 83–119, 2012.

[5] L. Cai and Y. Sheng, *Hom-big brackets: Theory and applications*, SIGMA **12**, 014, 2016.

[6] L. Cai and Y. Sheng, *Hom-Lie bialgebras*, Sci. China Math. **61**(9), 1553–1566, 2018.

[7] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.

[8] Y. Chen, Y. Wang and L. Zhang, *The construction of Hom-Lie bialgebras*, J. Lie Theory **20**, 767–783, 2010.

[9] T. Chtioui, A. Das and S. Mabrouk, *(Co)homology of compatible associative algebras*, Commun. Algebra, **52**(2), 582-603 2024.

[10] T. Chtioui and R. Saha, *On deformation cohomology of compatible Hom-associative algebras*, Afrika Matematika **36**(1), 31, 2025.

[11] A. Das, *Poisson-Nijenhuis groupoids*, Rep. Math. Phys. **84**(3),303–331, 2019.
[12] A. Das, *Cohomology and deformations of compatible Hom-Lie algebras*, J. Geom. Phys. **193**, 104951, 2023.
[13] V. Dotsenko and A.S. Khoroshkin, *Character formulas for the operads of two compatible brackets and for the bi-Hamiltonian operad*, Funct. Anal. Appl. **41**, 1–17, 2007.
[14] V. Drinfeld, *Hamiltonian structure on the Lie groups, Lie bialgebras and the geometric sense of the classical Yang-Baxter equations*, Sov. Math. Dokl. **27**, 68–71, 1983.
[15] I.Z. Golubchik and V.V. Sokolov, *Compatible Lie brackets and the Yang-Baxter equation*, Theor. Math. Phys. **146**, 159–169, 2006.
[16] J. Hartwig, D. Larsson and S. Silvestrov, *Deformations of Lie algebras using σ-derivations*, J. Algebra **295**, 314–361, 2006.
[17] Y. Kosmann-Schwarzbach and F. Magri, *Poisson-Nijenhuis structures*, Ann. Inst. Henri Poincaré A **53**, 35–81, 1990.
[18] I. Laraiedh and S. Silvestrov, *Hom-Leibniz bialgebras and BiHom-Leibniz dendriform algebras*, Afrika Mat. **34**(2), 28, 2023.
[19] D. Larsson and S. Silvestrov, *Quasi-hom-Lie algebras, central extensions and 2-cocycle-like identities*, J. Algebra **288**(2), 321–344, 2005.
[20] D. Larsson and S. Silvestrov, *Quasi-Lie algebras*, Contemp. Math. **391**, 241–248, 2005.
[21] X. Li, *Representations of 3-dimensional simple multiplicative Hom-Lie algebras*, Adv. Math. Phys. **2013**, 812789, 2013.
[22] F. Magri and C. Morosi, *A geometrical characterization of integrable Hamiltonian systems through the theory of Poisson-Nijenhuis manifolds*, Quaderno S/19, Milan 1981.
[23] S. Majid, *Matched pairs of Lie groups associated to solutions of the Yang-Baxter equations*, Pac. J. Math. **141**(2), 311–332, 1990.
[24] A. Makhlouf and R. Saha, *On compatible Leibniz algebras*, J. Algebra Appl. **24**(4), 2550105, 2025.
[25] A. Makhlouf and S. Silvestrov, *Hom-algebra structures*, J. Gen. Lie Theory Appl. **2**(2), 51–64, 2008.
[26] A. Makhlouf and S. Silvestrov, *Notes on 1-parameter formal deformations of Hom-associative and Hom-Lie algebras*, Forum Math. **22**, 715–739, 2010.
[27] A. Odesskii and V. Sokolov, *Pairs of compatible associative algebras, classical Yang-Baxter equation and quiver representations*, Commun. Math. Phys. **278**, 83–99, 2008.
[28] A. Panasyuk, *Compatible Lie brackets: Towards a classification*, J. Lie Theory **24**, 561–623, 2014.
[29] H. Strohmayer, *Operads of compatible structures and weighted partitions*, J. Pure Appl. Algebra **212**, 2522–2534, 2008.
[30] M. Wu and C. Bai, *Compatible Lie bialgebras*, Commun. Theor. Phys. **63**, 653–664, 2015.
[31] D. Yau, *Hom-algebras and homology*, J. Lie Theory **19**, 409–421, 2009.
[32] D. Yau, *Hom-Maltsev, Hom-alternative, and Hom-Jordan algebras*, Int. Electron. J. Algebra **11**, 177–217, 2012.
[33] D. Yau, *The classical Hom-Yang-Baxter equation and Hom-Lie bialgebras*, Int. Electron. J. Algebra **17**, 11–45, 2015.