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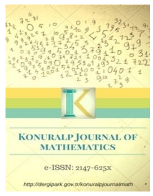

On Statistical Star-Compactness Restricted up to Order α

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Abstract

Expanding the concept of star compactness for a countable case, we confine the star statistical compactness up to order α, and the value of α is between 0 and 1. We compare this idea with other topological aspects pertaining to star statistical compactness of order α. This work illustrates the properties of this topological attribute and its sub-spaces under various conditions, particularly under open continuous surjection. It is obtained that star statistical compactness of order α can be characterized by a countable family of closed sets with some added set theoretic attributes.

Keywords: *Asymptotic Density; Open cover; Compactness; Star operator; Open continuous surjection*
2010 Mathematics Subject Classification: *Primary: 54D30; Secondary: 54B05, 54A05*

1. Introduction

The notion of density from number theory is extended to a more general setting in the context of topological spaces. An intuitive approach to thinking about natural density in topological spaces is as a method to expand the idea of how "large" a subset is inside a space. The natural density for a subset $A \subseteq \mathbb{N}$ is given by:

$$d(A) = \lim_{n\to\infty} \frac{1}{n}|\{k \leq n : k \in A\}|,$$

if such a limit exists. The notion of statistical convergence was first presented by H. Fast [14] and I. J. Schoenberg [26], and they utilized it in summability theory and analysis. In addition, J. A. Fridy [15] made substantial advances to statistical convergence theory by offering a framework that extends the ideas of traditional convergence. This idea expands on the conventional definition of convergence for real number sequences by taking into account the density of the set of indices for which the sequence elements are approaching the limit. Bhunia et al. [7] first introduced the concept of statistical convergence of order $\alpha(0 < \alpha < 1)$, which is a generalization of statistical convergence where the natural density is changed by elevating the denominator to the power α. This concept introduces a parameter α to provide a finer analysis of the convergence properties of sequences [9, 13, 16, 17, 18, 19, 23]. In a topological space X, a sequence $\{a_n : n \in \mathbb{N}\}$ is said to be statistically convergence of order α $(0 < \alpha < 1)$ to $a \in X$, if for every neighbourhood U of a, $\delta^\alpha(\{n \in \mathbb{N} : a_n \notin U\}) = 0$.
However, P. Bal has utilized the notion of natural density of order α in [10] to define and investigate the statistical compactness of order α. A topological space (X,τ) is called a statistical compact space of order alpha (in short s^α- compact) if for every countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathscr{V} = \{U_{m_k} : k \in \mathbb{N}\}$ such that $\delta^\alpha(\{m_k : U_{m_k} \in \mathscr{V}\}) = 0$.
Also, a concept called "star" operator, which is introduced by Eric K. van Douwen [11] is a concept in topology that deals with certain properties of open covers. Consider a topological space X and an open cover $\mathscr{U}$ of X; then, the star of a set $A \subseteq X$ with respect to $\mathscr{U}$, denoted by $St(A,\mathscr{U})$, is defined as follows:

$$St(A,\mathscr{U}) = \bigcup\{U \in \mathscr{U} : U \cap A \neq \emptyset\}.$$

This indicates that the union of all sets in $\mathscr{U}$ that intersect A is the star of A with respect to $\mathscr{U}$. A topological space X is deemed star-compact if each open cover $\mathscr{U}$ of X has a finite sub-cover whose stars encompass X. By formal definition, X is star-compact if there exists a finite sub-collection $\mathscr{V} \subseteq \mathscr{U}$ for any open cover $\mathscr{U}$ of X, such that $X = \bigcup_{V\in\mathscr{V}} St(V,\mathscr{U})$. Numerous mathematicians studied the variation of compactness and Lindelöfness under the star operator [22, 24, 26], whereas Kočinac [20], Bal [1, 3] concentrated on the variation of selection principles under that operator. You can find some recent developments in [2, 4, 8, 20, 21, 27].
St-compactness [11] explains the nature on an infinite cover in terms of finite covering and St-s-compactness is an stronger version of St compactness. To explain the nature of infinite covers at each phase from St-compactness to St-s-compactness, we use the parameter α which lies between 0 and 1. Again in St-s^α-compactness, $\alpha = 0$ will represent St-compactness; $\alpha = 1$ will represent St-s-compactness and other values of α will explain the intermediate cases.

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We are attempting to extend the works of Bal et. al. by merging the notion of a star operator with the statistical compactness of order α. This new covering property will be called star statistical compactness of order α. At the beginning, we give some example-based explanation to establish its intermediate nature. Then, the preservation of *St-s*-compactness under subspace and continuous mapping are explained. At the end of the paper, a finite intersection like attribute is given for *St-s*-compactness.

2. Preliminaries

We utilize the same standard terminology, symbols, and words as [12]. No separation axioms have been enforced or otherwise acknowledged in the work. For the convenience of the reader, we have provided a few essential ideas and expressions in this section.
A collection $\mathscr{U} \subseteq \tau$ in a topological space (X,τ) is referred to as an open cover of X if $\cup\mathscr{U} = X$. In a topological space, the space is referred to as compact if each open cover has a finite sub-cover [12].

Definition 2.1. *[12] Let $A \subseteq X$ in a topological space (X,τ). Then, the collection $\tau_A = \{U \cap A : U \in \mathscr{U}\}$ forms a topology on A, and the space (A,τ_A) is called a sub-space of (X,τ).*

Theorem 2.2. *[12] A closed sub-space of a compact space is compact.*

Definition 2.3. *[25] A topological space (X,τ) is called a statistical compact (in short, s-compact) space if for every countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathscr{V} = \{U_{m_k} : k \in \mathbb{N}\}$ such that $\delta(\{m_k : U_{m_k} \in \mathscr{V}\}) = 0$.*

Definition 2.4. *[11] In a topological space (X,τ), if for every open cover $\mathscr{U}$ there exists a finite subset $F \subset X$ such that $St(F,\mathscr{U}) = X$, then the space is called a star-compact space.*

Theorem 2.5. *[10] If a sub-space (A,τ_A) of a topological space (X,τ) is s^α-compact, then for every family $\{U_n\}_{n\in\mathbb{N}}$ of open subsets of X such that $A \subseteq \bigcup_{n\in\mathbb{N}} U_n$, there exists a subset $S \subseteq \mathbb{N}$ with $\delta^\alpha(S) = 0$ such that $A \subseteq \bigcup_{n\in S} U_n$.*

Definition 2.6. *[10] A topological space (X,τ) is called a statistical compact space of order alpha (in short s^α- compact) if for every countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a sub-cover $\mathscr{V} = \{U_{m_k} : k \in \mathbb{N}\}$ such that $\delta^\alpha(\{m_k : U_{m_k} \in \mathscr{V}\}) = 0$.*

Definition 2.7. *[6] A topological space (X,τ) is called a star statistical compact space (in short St-s-compact) if for every countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a subset $\mathscr{U}' = \{U_{m_k} : k \in \mathbb{N}\}$ with $\delta(\{m_k : U_{m_k} \in \mathscr{U}'\}) = 0$ and $St(\bigcup\mathscr{U}',\mathscr{U}) = X$.*

Definition 2.8. *[12] A function $f : (X,\tau) \to (Y,\sigma)$ is called a*

1. *continuous function if $f^{-1}(G) \in \tau$ for all $G \in \sigma$.*
2. *open function if $f(A) \in \sigma$ for all $A \in \tau$.*
3. *surjection if $f(X) = Y$.*

3. On Star Statistical Compactness of Order α

Compactness strictly restricts the sub-cover up to a finite number of elements, and statistical compactness directly allows the sub-cover to reach countable infinite keeping natural density 0. With the introduction of a parameter α, we try to control the star-statistical compactness by controlling the natural density.

Definition 3.1. *A topological space (X,τ) is said to be a star statistical compact space of order α (in short St-s^α-compact) if for each countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exists a subset $\mathscr{V} = \{U_{m_k} : k \in \mathbb{N}\}$ with $\delta^\alpha(\{m_k : U_{m_k} \in \mathscr{V}\}) = 0$ and $St(\bigcup\mathscr{V},\mathscr{U}) = X$.*

Theorem 3.2. *Every s^α-compact space is a St-s^α-compact space.*

Proof. Consider (X,τ) be a s^α-compact space and $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countably infinite open cover of (X,τ). So, there exists a sub-cover $\mathscr{A} = \{U_{n_k} : k \in \mathbb{N}\}$ with $\delta^\alpha(\{n_k : U_{n_k} \in \mathscr{A}\}) = 0$. Thus, $\bigcup\mathscr{A} = \bigcup_{k\in\mathbb{N}} U_{n_k} = X$. Moreover, $(\bigcup\mathscr{A})\bigcap U_n \neq \emptyset$ for all $n \in \mathbb{N}$ implies that $St(\cup\mathscr{A},\mathscr{U}) = X$. Therefore, (X,τ) is a *St*-s^α-compact space. □

Example 3.3. *A St-s^α-compact space may not be a s^α-compact space.*
Let us consider a topological space (X,τ), where $X = \mathbb{N}\cup\{a\}$. Now, we take a base $\mathscr{B} = \{\{a\},\emptyset\}\bigcup\{\{a,n\} : n \in \mathbb{N}\}$ for a suitable topology τ on X and let τ represents the topology formed by $\mathscr{B}$. Let's take $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be an arbitrary countable open cover of the space (X,τ). Now, if we consider $\mathscr{U}' = \{U_m\}$ a subset of $\mathscr{U}$, then $\delta^\alpha(\{m\}) = 0$ and $St(\bigcup\mathscr{U}',\mathscr{U}) = St(U_m,\mathscr{U}) \supseteq St(\{a\},\mathscr{U}), $[Because $a \in U_n$ for all $n \in \mathbb{N}$ so, $a \in U_m$.]$ = X$. Therefore, (X,τ) is St-s^α-compact.
Suppose $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be countable open cover of (X,τ), where $U_n = \{a,n\}$. We now take a sub-cover $\mathscr{U}' = \{U_{n_k} : k \in \mathbb{N}\}$ of $\mathscr{U}$ such that $\delta^\alpha(\{n_k : U_{n_k} \in \mathscr{U}'\}) = 0$. Then, at least one U_p exists such that $U_p \in \mathscr{U}$ but $U_p \notin \mathscr{U}'$. So, $U_p \notin \cup\mathscr{U}'$, which contradicts the fact that $\mathscr{U}'$ is a sub-cover of $\mathscr{U}$. Therefore, (X,τ) is not s^α-compact.

Theorem 3.4. *Every star s^α-compact space is star s-compact space and every star compact space is a star s^α-compact space.*

Proof. Let us consider (X,τ) be Star s^α-compact space and $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of space X. Then, there exists a subset $\mathscr{W} = \{U_{m_k} : k \in \mathbb{N}\}$ with $\delta^\alpha(\{m_k : U_{m_k} \in \mathscr{W}\}) = 0$ and $St(\bigcup\mathscr{W},\mathscr{U}) = X$. Thus, $\delta(\{m_k : U_{m_k} \in \mathscr{W}\}) = 0$. Therefore, (X,τ) is star *s*-compact space.
Let (X,τ) be star compact space and $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of space X. Then, there exist a finite subset $\{x_1,x_2,x_3,\ldots x_r\} = F \subseteq X$ so that $St(F,\mathscr{U}) = X$. So, there exists a $U_{n_k} \in \mathscr{U}$ such that $x_k \in U_{n_k}$ for all $i = 1,2,3,\ldots r$. So, $St\left(\cup_{k=1}^{r} U_{n_k},\mathscr{U}\right) \supseteq St(F,\mathscr{U}) = X$. Therefore, $St\left(\cup_{k=1}^{r} U_{n_k},\mathscr{U}\right) = X$.
Also, the collection $\{U_{n_k} : k = 1,2,3,\ldots r\}$ is finite set which implies that $\delta^\alpha(\{U_{n_k} : k = 1,2,3,\ldots r\}) = 0$. Hence, (X,τ) is a star s^α compact space. □

Theorem 3.5. *Every countably compact space is star s^{α}-compact space.*

Proof. Let (X,τ) be countably compact space. Then, for each countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of X, there exits a finite sub-cover $\mathscr{V} = \{U_{m_p} : p = 1,2,3\ldots k\}$. Since $\{m_p : U_{m_p} \in \mathscr{V}\}$ is finite, therefore $\delta^{\alpha}\{m_p : U_{m_p} \in \mathscr{V}\} = 0$. So, $\cup\mathscr{V} = \cup_{p=1}^{k} U_{m_p} = X$. Also, $(\cup\mathscr{V}) \cap U_n \neq \emptyset$ for all $n \in \mathbb{N}$. Thus, $St(\cup\mathscr{V},\mathscr{U}) = X$.
Therefore, (X,τ) is St-s^{α}-compact space. □

Example 3.6. *There exists a St-s^{α}-compact space that is not a countable compact space.*
Consider a topological space (X,τ), where $X = \mathbb{N} \cup \{x_1,x_2\}$. Now, we take a base $\mathscr{B} = \{\{x_1,x_2\},\emptyset\} \bigcup \{\{x_1,x_2,n\} : n \in \mathbb{N}\}$ for a suitable topology τ on X and let τ represents the topology formed by $\mathscr{B}$. Let us consider an arbitrary countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of the space (X,τ). Now, if we choose $\mathscr{V} = \{U_i\}$ a subset of $\mathscr{U}$, then $\delta^{\alpha}(\{i\}) = 0$ and $St(\bigcup\mathscr{V},\mathscr{U}) = St(U_i,\mathscr{U}) \supseteq St(\{x_1,x_2\},\mathscr{U})$ [Since $\{x_1,x_2\} \in U_n$ for all $n \in \mathbb{N}$ so, $\{x_1,x_2\} \in U_i$] $= X$. Therefore, (X,τ) is St-s^{α}-compact.
Suppose $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be countable open cover of (X,τ), where $U_n = \{x_1,x_2,n\}$. Now, take a finite sub-cover $\mathscr{V} = \{U_{n_k} : k = 1,2,3\ldots p\}$ of $\mathscr{U}$. Then, there must exist one U_m such that $U_m \in \mathscr{U}$ but $U_m \notin \mathscr{V}$. So, $U_m \notin \cup\mathscr{V}$, which contradicts the fact that $\mathscr{V}$ is a sub-cover of $\mathscr{U}$. Therefore, (X,τ) is not a countable compact.

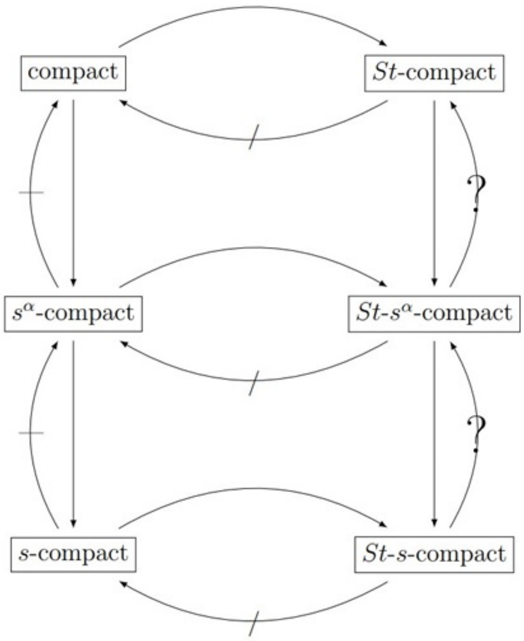

Figure 3.1: Relation Chart

Theorem 3.7. *Every clopen sub-space of St-s^{α}-compact space is a St-s^{α}-compact space.*

Proof. Consider (X,τ) be a St-s^{α}-compact space and (B,τ_B) be an clopen sub-space of (X,τ). Now, choose countable open cover $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ of (X,τ) and $\mathscr{W} = \{W_n : n \in \mathbb{N}\}$ of (B,τ_B). Thus, $B = \bigcup_{n\in\mathbb{N}} W_n = \bigcup\mathscr{W}$. Now, for each $n \in \mathbb{N}$, there exists a $U_n \in \tau$ such that $W_n = B \cap U_n \in \tau$. [Since $B \in \tau$ and $U_n \in \tau$]. Let us suppose a countable cover $\mathscr{V} = \{V_n : n \in \mathbb{N}\}$ where,

$$V_n = \begin{cases} X \setminus B, & \text{if } n = 1 \\ W_{n-1}, & \text{otherwise} \end{cases}$$

of (X,τ). But (X,τ) is St-s^{α}-compact space. Therefore, there exists a subset $\mathscr{V}' = \{V_{n_k} : k \in \mathbb{N}\}$ with $\delta^{\alpha}(\{n_k : V_{n_k} \in \mathscr{V}'\}) = 0$ be such that $St(\bigcup V_{n_k},\mathscr{V}) = X$. Now, $\mathscr{W}' = \{B \bigcap V_{n_k} : k \in \mathbb{N}\}$ is a subset of $\mathscr{W}$. In all cases $V_1 \in \mathscr{V}'$ then, $0 = \delta(\{n_k : U_{n_k} \in \mathscr{V}'\}) \geq \delta(\{n_k : B \cap V_{n_k} \in \mathscr{W}'\})$.

$$\text{Also, } St((\bigcup\mathscr{W}'),\mathscr{W}) = B \text{ and } St((X \setminus B),\mathscr{W}) = X \setminus B.$$

Therefore, (B,τ_B) is a St-s^{α}-compact space. □

Example 3.8. *Closed sub-space of St-s^{α}-compact may or may not be St-s^{α}-compact space.*
Let $X = \mathbb{N} \cup \{a\}$ and $\mathscr{B} = \{\{a\},\emptyset\} \bigcup \{\{a,n\}\} : n \in \mathbb{N}\}$ and τ be the topology generated by $\mathscr{B}$. By Example 3.3, (X,τ) is St-s^{α}-compact. Now, choose the closed subset $A = \mathbb{N}$ of (X,τ) where $\tau_A = \{\{k\} : k \in \mathbb{N}\} \bigcup \{\emptyset,\mathbb{N}\}$. Then, (A,τ_A) is a sub-space of (X,τ). We assume that (A,τ_A) is St-s^{α}-compact and consider the countably infinite open cover $\mathscr{W} = \{W_n = \{n\} : n \in \mathbb{N}\}$ of (A,τ_A).
Since (A,τ_A) is St-s^{α}-compact, so there exists a $S \subseteq \mathbb{N}$ with $\delta^{\alpha}(S) = 0$ such that $St(\bigcup_{n\in S} W_n,\mathscr{W}) = A$ as because $\delta^{\alpha}(S) < \delta^{\alpha}(\mathbb{N})$, therefore there exist $q \in \mathbb{N} \setminus S$. So, $q \notin \bigcup_{n\in S} W_n$. Thus, $(\cup_{n\in S} W_n) \cap W_q = \emptyset$ i.e., $q \notin st(\bigcup_{n\in S} W_n,\mathscr{W})$ which implies that $St(\bigcup_{n\in S} W_n,\mathscr{W}) \neq A$, which is a contradiction.
Therefore, (A,τ_A) is not a St-s^{α}-compact sub-space of (X,τ).

Theorem 3.9. *If $\{(X_n,\tau_n) : n = 1,2,\ldots,p\}$ is a finite collection of St-s^α-compact sub-spaces of X such that $X = \bigcup_{n=1}^{p} X_n$, then (X,τ) is St-s^α-compact space.*

Proof. Let (X_m,τ_m) for $m = 1,2,\ldots p$ are *St*-s^α-compact sub-spaces of (X,τ) such that $X = \bigcup_{m=1}^{p} X_m$. Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be a countable open cover of (X,τ). Thus, $\mathscr{U}_m = \{U_n^m = X_m \cap U_n : n \in \mathbb{N}\} \setminus \{\emptyset\}$ is a countable open cover of $(X_m,\tau_m\ )$ where $m = 1,2,\ldots,p$.
But (X_m,τ_m) are *St*-s^α-compact sub-spaces of (X,τ) for each $m = 1,2,3,\ \ldots,p$. Then, there exists a subset $\mathscr{U}'_m = \left\{U_{n_k}^m = X_m \cap U_{n_k} : k \in \mathbb{N}\right\}$ (for each $m = 1,2,\ldots,p$) of $\mathscr{U}_m$ of (X_m,τ_m) such that $\delta^\alpha\left(\{n_k : X_m \cap U_{n_k} \in \mathscr{U}'_m\}\right) = 0$, for $m = 1,2,\ldots,p$ and $St\left(\cup \mathscr{U}'_m, \mathscr{U}_m\right) = X_m$. Now, $\delta^\alpha(\bigcup_{m=1}^{p}\{n_k : X_m \cap U_{n_k} \in \mathscr{U}'_m\}) \leq \sum_{m=1}^{p} \delta^\alpha(\{n_k : X_m \cap U_{n_k} \in \mathscr{U}'_m\}) = 0$. Also, $X = \bigcup_{m=1}^{p} X_m = \bigcup_{m=1}^{p} St(\bigcup \mathscr{U}'_m, \mathscr{U}_m) \subseteq \bigcup_{m=1}^{p} St(\bigcup \mathscr{U}'_m, \mathscr{U})$ $\subseteq St(\bigcup_{m=1}^{p}(\bigcup \mathscr{U}'_m), \mathscr{U}) = St(\bigcup_{m=1}^{p}(\bigcup(U_{n_k}^m)), \mathscr{U}) \subseteq St\left(\bigcup_{m=1}^{p}(\bigcup U_{n_k}), \mathscr{U}\right)$. Therefore, $\mathscr{U}' = \left\{U_{n_k} : U_{n_k}^m \in \mathscr{U}'_m \text{ and } m = 1,2,\ldots p\right\}$ is a subset of $\mathscr{U}$ such that

$$\delta^\alpha\left(\{n_k : U_{n_k} \in \mathscr{U}'\}\right) = 0 \quad \text{and} \quad St\left(\bigcup \mathscr{U}', \mathscr{U}\right) = X.$$

Hence, (X,τ) is a *St*-s^α-compact space. □

Example 3.10. *Arbitrary union of St-s^α-compact sub-space may not be St-s^α-compact space.*
Let, $X = [1,\infty)$, $\mathscr{B} = \{[n,n+1) : n \in \mathbb{N}\} \bigcup \{\emptyset\}$ and τ be the topology generated by $\mathscr{B}$. Suppose (X,τ) is St-s^α-compact space and consider the countable open cover $\mathscr{W} = \{W_n = [n,n+1) : n \in \mathbb{N}\}$ of X. So, there exists a $S \subseteq N$ with $\delta^\alpha(S) = 0$ such that $St\left(\bigcup_{n\in S} W_n, \mathscr{W}\right) = X$. But there exist $q \in \mathbb{N}\setminus S$. Since $\delta^\alpha(S) < \delta^\alpha(\mathbb{N})$, so, $(\bigcup_{n\in S} W_n) \cap W_q = \emptyset$. i.e., $W_q \cap St\left(\bigcup_{n\in S} W_n, \mathscr{W}\right) = \emptyset$. Since W_q remains uncovered by $St\left(\bigcup_{n\in S} W_n, \mathscr{W}\right)$ so, $St(\bigcup_{n\in S} W_n, \mathscr{W}) \neq A$, which is a contradiction. Therefore, (X,τ) is not a St-s^α-compact.
Now, $X = \bigcup_{n\in\mathbb{N}}[n,n+1)$. Consider $X_n = [n,n+1)$ equipped with the indiscrete topology on X_n. For every topological space (X_n,τ_n), then each one is s^α-compact space and hence St-s^α-compact space as each s^α-compact space is St-s^α-compact space. Now, the next step is to determine whether their union (X,τ) is St-s^α-compact space or not. Let $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be the countable open cover of X. Then, there is a subset $\mathscr{V} = \{U_{n_k} : k \in \mathbb{N}\}$ such that $\delta^\alpha\{n_k : U_{n_k} \in \mathscr{V}\} = 0$. So, there must exist an open set which belong to $\mathscr{U}$ and does not belongs to $\mathscr{V}$.

$$\textit{So, } St(\bigcup \mathscr{V}, \mathscr{U}) \neq X.$$

Therefore, arbitrary union of St-s^α-compact sub-space may not be St-s^α-compact space.

Theorem 3.11. *An open continuous surjection of a St-s^α-compact space is also a St-s^α-compact space.*

Proof. Let (X,τ) be a *St*-s^α-compact space and $f : (X,\tau) \to (Y,\sigma)$ be an open continuous surjection.
Consider $\mathscr{V} = \{G_n : n \in \mathbb{N}\}$ be a countable open cover for Y. So, $\bigcup\{G_n : n \in \mathbb{N}\} = Y$ [by surjection]. So, $f^{-1}\{\bigcup\{G_n : n \in \mathbb{N}\}\} = f^{-1}(Y)$. i.e., $\bigcup\{f^{-1}(G_n) : n \in \mathbb{N}\} = X$. Also, f is continuous, and $\{G_n : n \in \mathbb{N}\}$ is open. Therefore, $\mathscr{U} = \{f^{-1}(G_n) : n \in \mathbb{N}\}$ is an open cover in X. Since X is *St*-s^α-compact space, then there exists a subset $\{f^{-1}(G_{n_1}), f^{-1}(G_{n_2}), \ldots\} = \mathscr{U}'$ (say) of the cover $\mathscr{U}$ where $\delta^\alpha(\{n_k : U_{n_k} \in \mathscr{U}'\} = 0$ with $St(\cup \mathscr{U}', \mathscr{U}) = X$. Let $\mathscr{U}_\alpha \subseteq \mathscr{U}$ be the collection of all those $U \in \mathscr{U}$ such that $(\cup \mathscr{U}') \cap U \neq \emptyset$ for all $U \in \mathscr{U}_\alpha$. Therefore, $\bigcup \mathscr{U}_\alpha = X$. Now, $\forall U \in \mathscr{U}_\alpha$,

$$\left(\bigcup \mathscr{U}'\right) \bigcap U \neq \emptyset.$$

Also, $\bigcup_{U\in\mathscr{U}_\alpha} f(U) = Y$. Moreover $\mathscr{U}_\alpha \subseteq \mathscr{U}$ implies that $Y = \bigcup_{U\in\mathscr{U}_\alpha} f(U) \subseteq \bigcup_{U\in\mathscr{U}} f(U) = \bigcup_{n\in\mathbb{N}} f(f^{-1}(G_n)) = \bigcup_{n\in\mathbb{N}} G_n = Y$ implies that $\bigcup_{U\in\mathscr{U}_\alpha} = Y$. Thus,

$$\mathscr{V}' = \{f(U) : U \in \mathscr{U}_\alpha\}$$

is a sub-cover of $\mathscr{V}$ which is covering Y. Suppose $\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'}(f(f^{-1}(G_{n_k}))) \bigcap f(U) = \emptyset$, for some $U \in \mathscr{U}_\alpha$. So, $(\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'} G_{n_k}) \cap$ $f(f^{-1}(G_n)) = \emptyset$, for $f^{-1}(G_n) \in \mathscr{U}_\alpha$. Thus, $(\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'} G_{n_k}) \cap G_n = \emptyset$, which contradicts the fact in equation $(\bigcup \mathscr{U}') \bigcap U \neq \emptyset$. Therefore, $(\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'} G_{n_k}) \cap f(U) \neq \emptyset$, for all $f(U) \in \mathscr{V}'$. So, $St((\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'} G_{n_k}), \mathscr{V}') = \bigcup \mathscr{V}' = Y$. i.e., $St(\bigcup_{f^{-1}(G_{n_k})\in\mathscr{U}'} G_{n_k}, \mathscr{V}) = Y$. Now, $\{G_{n_k} : f^{-1}(G_{n_k}) \in \mathscr{U}'\}$. So, $\{G_{n_k} : f^{-1}(G_{n_k}) \in \{f^{-1}(G_{n_1}), f^{-1}(G_{n_2}), \ldots\}\}$ that is $\{G_{n_k} : f(f^{-1}(G_{n_k})) \in \{f(f^{-1}(G_{n_1}), f(f^{-1}(G_{n_2}), \ldots\}\}$. which implies that $\{G_{n_k} : G_{n_k} \in \{G_{n_1}, G_{n_2}, \ \ldots\}\}$.
Therefore, $\mathscr{W}' = \{G_{n_1}, G_{n_2}, \ldots\} \subseteq \{G_n : n \in \mathbb{N}\}$ and $\delta^\alpha(\{n_k : n_k \in \mathscr{W}' \text{and} k \in \mathbb{N}\}) = 0$. Also, $St(\cup \mathscr{W}', \mathscr{V}) = Y$.
Hence, the theorem has been proven. □

Theorem 3.12. *In a particular topological space (X,τ), if for every countable family $\mathscr{F} = \{F_n : n \in \mathbb{N}\}$ of closed sets there exists a subfamily $\mathscr{E} = \{F_{n_k} : k \in \mathbb{N}\}$ with $\delta^\alpha(\{k \in \mathbb{N} : F_{n_k} \in \mathscr{E})\} = 0$ such that $\bigcap\{F_n : (\cap_{k\in\mathbb{N}} F_{n_k}) \cup F_n \neq X\} = \emptyset$, then (X,τ) is a Star s^α-compact space.*

Proof. Let (X,τ) be a topological space and $\mathscr{U} = \{U_n : n \in \mathbb{N}\}$ be an arbitrary countable open cover of X. So, $\mathscr{F} = \{F_n = X \setminus U_n : n \in \mathbb{N}\}$ is a countable family of closed subsets of X. Then, based on the given condition, there exists a sub-family $\mathscr{E} = \{F_{n_k} : k \in \mathbb{N}\}$ with $\delta^\alpha(\{k \in \mathbb{N} : F_{n_k} \in \mathscr{E})\} = 0$ and $\bigcap\{F_n : (\cap_{k\in\mathbb{N}} F_{n_k}) \cup F_n \neq X\} = \emptyset$. So, $\cap\{X \setminus U_n : (\cap_{k\in\mathbb{N}}(X \setminus U_{n_k})) \cup (X \setminus U_n) \neq X\} = \emptyset$ which implies that $X \setminus \bigcup\{U_n : (X \setminus \cup_{k\in\mathbb{N}} U_{n_k}) \cup (X \setminus U_n) \neq X\} = \emptyset$ i.e., $X \setminus \bigcup\{U_n : X \setminus (\cup_{k\in\mathbb{N}} U_{n_k}) \cap U_n) \neq X\} = \emptyset$. i.e., $X \setminus \bigcup\{U_n : (\cup_{k\in\mathbb{N}} U_{n_k}) \cap U_n) \neq \emptyset\} = \emptyset$. i.e., $\bigcup\{U_n : (\cup_{k\in\mathbb{N}} U_{n_k}) \cap U_n) \neq \emptyset\} = X$. Therefore, $St((\cup_{k\in\mathbb{N}} U_{n_k}), \mathscr{U}) = X$. Moreover, if $\mathscr{V} = \{U_{n_k} = X \setminus F_{n_k} : F_{n_k} \in \mathscr{E}\}$, then

$$\delta^\alpha(\{k \in \mathbb{N} : U_{n_k} \in \mathscr{V}\}) = \delta^\alpha(\{k \in \mathbb{N} : F_{n_k} \in \mathscr{E})\} = 0.$$

Hence, (X,τ) is a Star s^α-compact space. □

Open Problem: There exists a *St*-*s*-compact space that is not a *St*-s^α-compact space. Also, there exists a *St*-s^α-compact space, which is not a star compact space.

4. Conclusion

St-s^{α}-compact space provide a robust and flexible framework for understanding compactness and star covering properties in topological spaces. This particular kind of compact space is weaker than *St*-*s*-compact and stronger than *St*-compact space. While *St*-s^{α}-compact space can be preserved under open continuous surjection, it cannot be preserved under closed sub-space. The family of closed sets with some additional set theoretic qualities can be used for further description of *St*-s^{α}-compact space. This concept can further be extended for the study of sequential covering properties like Rothberger and menger spaces.

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